Final exam, Discrete Mathematics I., 12.1.2010

1. *(5 pt)* Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $\dots$ be elements of the Fibonacci sequence. Show that:

$$nF_1 + (n-1)F_2 + \cdots + 2F_{n-1} + F_n = F_{n+4} - (n+3).$$

2. *(6 pt)* Let T be a satisfiable set of propositions. Show that the set of propositions $T \cup \{\neg A\}$ is unsatisfiable if and only if the proposition A is a logical (tautological) consequence of the set of propositions T (i.e. $T \models A$).

3. a) *(5 pt)* Show that the following formula with quantifiers is logically true

$$((\exists x)(A(x) \Rightarrow B(x))) \Rightarrow (((\forall x)A(x)) \Rightarrow ((\exists x)B(x))).$$

b) *(5 pt)* Let $A, B \subseteq \mathcal{U}$ and A^c denote the complement of the set A with respect to $\mathcal{U}$. Show that:

$$(A^c \cap B) = (B^c \cap A)^c \Leftrightarrow B = A^c.$$

4. *(5 pt)* Denote the set of all finite subsets of $\mathbb{N}$ as $\mathcal{F}$. Define a relation $\sim$ on $\mathcal{F}$ by: for $A, B \in \mathcal{F}$ is $A \sim B$ whenever $|A \cup B| = \max(|A|, |B|)$. Is $\sim$ an equivalence relation?

5. *(6 pt)* For the following two statements decide whether they are true or false and justify your answer.

a) If the sets A and B have the same cardinality and $\phi : A \to B$ is injective then ϕ is surjective as well.

b) Intersection of two equivalence relations on A (as subsets of $A \times A$) is also an equivalence relation.

6. *(6 pt)* Decide whether the cardinality of the set of all infinite subsets of $\mathbb{N}$ is countable. Justify your answer.

7. *(7 pt)* Let $f : X \to Y$ be a map and sets $A, B \subseteq Y$ be disjoint, i.e. $A \cap B = \emptyset$. Decide whether $f^{-1}(A) \cap f^{-1}(B) = \emptyset$.

8. *(5 pt)* Using general rules for computing with cardinal numbers decide which one of the following cardinal numbers is larger or whether they are equal: $2^{(\aleph_0{}^{c})}$ and $2^{(c^{\aleph_0})}$.