Discrete mathematics I. – Homework 5

Problem sessions in the week of October 28, 2013

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1. Decide whether are the following formulae with quantifiers and arbitrary predicates Φ, Ψ and A logically true. If so, prove it; if not, find a counterexample:
a) $(\forall x)(\Phi(x) \Rightarrow \Psi(x)) \Rightarrow ((\forall x)\,\Phi(x) \Rightarrow (\forall x)\,\Psi(x))$,
b) $(\forall x)(\Phi(x) \wedge \Psi(x)) \Leftrightarrow ((\forall x)\,\Phi(x) \wedge (\forall x)\,\Psi(x))$,
c) $((\exists x)\,\Phi(x) \wedge (\exists x)\,\Psi(x)) \Rightarrow (\exists x)(\Phi(x) \wedge \Psi(x))$,
d) $(\exists x)(\Phi(x) \vee \Psi(x)) \Leftrightarrow ((\exists x)\,\Phi(x) \vee (\exists x)\,\Psi(x))$,
e) $(\forall x)(\Phi(x) \vee \Psi(x)) \Rightarrow ((\forall x)\Phi(x) \vee (\forall x)\Psi(x))$,
f) $(\forall x)(\Phi(x) \Rightarrow \Psi(x)) \Rightarrow (\exists x)\Phi(x)$,
g) $(\exists x)(\forall y)\,A(x, y) \Rightarrow (\forall y)(\exists x)\,A(x, y)$,
h) $(\forall y)(\exists x)\,A(x, y) \Rightarrow (\exists x)(\forall y)\,A(x, y)$,
i) $(\forall x)(\forall y)\,A(x, y) \Rightarrow (\forall x)\,A(x, x)$,
j) $(\exists x)(\exists y)\,A(x, y) \Rightarrow (\exists x)\,A(x, x)$.
Note: $\Phi(x)$, $\Psi(x)$ and $A(x, y)$ are arbitrary predicates, for example binary predicate A assigns to a couple x, y some truth value. Familiar examples are equality "=" and inequality ">", etc. The problem is to decide whether are given formulae always true without specifying exact meaning of Φ, Ψ or A, or specifying which objects variables x and y represent.

For example, in 1a) we can choose $\Phi(x) =$ "the car x is red" and $\Psi(x) =$"the car x has four wheels". Then formula $(\forall x)(\Phi(x) \Rightarrow \Psi(x)) \Rightarrow ((\forall x)\,\Phi(x) \Rightarrow (\forall x)\,\Psi(x))$ says:
"If for every car holds: it is red $\Rightarrow$ has four wheels, then if all cars are red, all cars have four wheels."
This sounds true, but that doesn't prove anything. One example does not prove that something holds always.
To prove that such formula is logically true, without considering specific meaning of the predicates Φ and Ψ, we have to do something else. We can try to investigate whether such formula could be evaluated as false and find some contradiction (or construct a counterexample).
In this case we would have:

$$v((\forall x)(\Phi(x) \Rightarrow \Psi(x))) \equiv 1, \qquad \text{and} \qquad v(((\forall x)\,\Phi(x) \Rightarrow (\forall x)\,\Psi(x))) \equiv 0,$$

hence

$$v((\forall x)\,\Phi(x)) \equiv 1 \qquad \text{and} \qquad v((\forall x)\,\Psi(x)) \equiv 0.$$

The former gives $v((\exists x)\,\neg\Psi(x)) \equiv 1$, hence for some x_0 we have $v(\Psi(x_0)) \equiv 0$. On the other hand we know (why?), that $v(\Phi(x_0)) \equiv 1$, and so $v(\Phi(x_0) \Rightarrow \Psi(x_0))) \equiv 0$. But that contradicts $v((\forall x)(\Phi(x) \Rightarrow \Psi(x))) \equiv 1$.

2. We say that formula A is in *prenex normal form*, if it is written as $(Q_1x_1)(Q_2x_2)\dots(Q_nx_n)B$, where Q_i's are quantifiers ($\exists$ or $\forall$), $x_1, x_2, \dots, x_n$ are distinct variables and part B does not contain any quantifiers (is open). In other words, formula in prenex normal form has all its quantifiers placed at its beginning.
In problem 1b) we proved that formula $(\forall x)\,\Phi(x) \wedge (\forall x)\,\Psi(x)$ is logically equivalent to formula $(\forall x)(\Phi(x) \wedge \Psi(x))$, which is in prenex normal form. (think through the details).
Find formulae in prenex normal form, which are logically equivalent to the following formulae and explain why they are equivalent:
a) $(\forall x)\,\Phi(x) \vee (\forall x)\,\Psi(x)$,
b) $(\exists x)\,\Phi(x) \wedge (\forall x)\,\Psi(x)$,
c) $(\exists x)\,\Phi(x) \vee (\exists x)\,\Psi(x)$,
d) $(\exists x)\,\Phi(x) \Rightarrow (\exists x)\,\Psi(x)$.