Discrete mathematics I. – Homework 8

Problem sessions in the week of November 18, 2013

We say that the relation R is:
reflexive if $(\forall x)\,[x,x] \in R$,
irreflexie if $(\forall x)\,[x,x] \notin R$,
symmetric if $[x,y] \in R \Rightarrow [y,x] \in R$,
asymmetric if $[x,y] \in R \Rightarrow [y,x] \notin R$,
antisymmetric if $([x,y] \in R \wedge [y,x] \in R) \Rightarrow x = y$,
transitive if $([x,y] \in R \wedge [y,z] \in R) \Rightarrow [x,z] \in R$,
dichotomous if $(\forall x,y)([x,y] \in R \vee [y,x] \in R)$.

1. Find a relation which is symmetric and transitive but is not reflexive. Or show that such a relation can not exist.

2. How many different equivalence relations there are on a set of four elements?

3. On the set of positive integers $\mathbb{N}$ we define a relation R as: aRb if and only if a divides b or b divides a. Is R an equivalence?

4. Sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are sets of integers, rational and real numbers. Let us define the relations:

$$T = \{[x,y] \in \mathbb{R} \times \mathbb{R} \,|\, x - y \in \mathbb{Z}\},$$
$$U = \{[x,y] \in \mathbb{R} \times \mathbb{R} \,|\, x - y \in \mathbb{Q}\}.$$

Show that T and U are equivalence relations. Describe partitions of the set $\mathbb{R}$ given by these equivalences.

5. Denote the set of all continuous real-valued functions defined on the interval $[0,1]$ as $C(0,1)$. Define:

$$[f,g] \in T \Leftrightarrow (\forall x)\, f(x) \leq g(x) \wedge (\exists y)\, f(y) < g(y).$$

Show that T is a partial (strict) order on $C(0,1)$ (i.e. T is not reflexive since $[f,f] \notin T$, but T is transitive and asymmetric). Find two incomparable elements.

In a partially ordered set A with an order $\leq$ an element a is called *least* if $(\forall x \in A)\, a \leq x$. An element b is called *minimal* if $(\forall x \in A)(x \leq b \Rightarrow x = b)$. Similarly, an element c is *greatest* if $(\forall x \in A)\, x \leq c$ and an element d is *maximal* if $(\forall x \in A)(d \leq x \Rightarrow x = d)$. Notions of minimal and least element describe two different things, especially for partially ordered sets (i.e. not totally ordered).

6. Find a partially ordered set, which has exactly one maximal element but does not have the greatest element.

7. Prove that in a totally ordered set is the minimal element also the least.

8. Prove that if a partially ordered set has a greatest element, then it is its only maximal element.

Bonus problem

9. Let $\mathbb{Q}[x]$ be the set of polynomials in a variable x with rational coefficients. Let $A[x] = \mathbb{Q}[x] - \mathbb{Q}$ (the set difference), i.e. $A[x]$ is the set of all rational polynomials in a variable X of degree at least 1. Define:

$$[f(x), g(x)] \in D \Leftrightarrow g(x) = f(x) \cdot q(x) \quad \text{for some} \quad q(x) \in A[x].$$

Show that D is a partial (strict) order. Find two incomparable elements in $A[x]$.
Note: Relation D is analogous to the relation of divisibility $|$ on the set of natural numbers $\mathbb{N}$.