Discrete mathematics I. – Homework 9

Problem sessions in the week of November 25, 2013

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1. Let $A = \{a, b, c\}$ and $B = \{x, y\}$. Find following mappings (if they exist):
a) all injective maps from A to B, and all injective maps from B to A.
b) all surjective maps from A to B, and all surjective maps from B to A.
c) all bijections from A to B, and all bijections from B to A.

2. Does the injectivity of f follow from the fact that the composed mapping $f \circ g$ is injective? How about the injectivity of g? What happens when we replace words "injectivity" with "surjectivity" in previous questions? (composed map is defined as $(f \circ g)(x) = f(g(x))$)

3. Let A be a finite set. Show:
a) If $f : A \to A$ is injective, then it is surjective as well.
b) If $f : A \to A$ is surjective, then it is injective as well.
Do such claims hold for an infinite set (e.g. $\mathbb{N}$) as well?

4. Let $f : A \to B$ be a mapping. Show that for every $X, Y \subseteq A$:

$$f(X \cup Y) = f(X) \cup f(Y),$$
$$f(X \cap Y) \subseteq f(X) \cap f(Y).$$

Here, $f(X)$ denotes the set $\{y \in B \mid (\exists x \in X)\, y = f(x)\}$ – the *image* of the set X, which can be expressed also as $\{f(x) \mid x \in X\}$.

Find a map f and sets X, Y so that we do not have equality in the latter inclusion.

5. Find a bijection between $\langle 0, 1)$ and $\{[x, y] \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$.

6. Let $f : A \to B$ be a map. Prove that for any $U, V \subseteq B$:

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V),$$
$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V).$$

Here, $f^{-1}(Y)$ denotes the set $\{x \in A \mid f(x) \in Y\}$ – the *preimage* of the set Y.

7. Find a bijection between intervals: $(0, 1)$, $(0, \infty)$.

Bonus problems

8. Find a bijection between intervals: $(0, 1]$, $(0, 1)$.

9. Find an injective map from $\mathbb{R}^2$ to $\mathbb{R}$.