Discrete mathematics I. – Homework 11

Problem sessions in the week of December 9, 2013

---

1. Show that the set of all finite subsets of the set of natural numbers $\mathbb{N}$ is countable. What would go wrong if we tried to use the diagonal principle to show that it is uncountable?

2. Show that we can not have uncountably many mutually disjoint open disks in a plane. (An open disk with the center at $[x_0, y_0]$ and the radius $\varepsilon > 0$ is $D_\varepsilon[x_0, y_0] = \{[x, y] \in \mathbb{R}^2;\ (x-x_0)^2+(y-y_0)^2 < \varepsilon\}$). What would change if we replaced 'open disks' with 'circles'.

Problems 3 and 4 are meant for better understanding of the construction of bijective map in the proof of Cantor–Bernstein theorem. Students should try to work out this problem.

3. Let $A = (0, 1)$ and $B = (0, 1]$ be intervals. Then the maps $f : A \to B$ (given by $x \mapsto x$) and $g : B \to A$ (given by $g : x \mapsto x/2$) are injective. Hence we can apply Cantor–Bernstein theorem and there exists a bijection h between A and B. Closely follow the construction from the proof of the theorem and determine the sets A_1, A_2, B_1 and B_2 as well as the map h.

4. Do the same as in problem 3 for the intervals $A = (0, 1)$ and $B = [0, 1]$, an injection $f : A \to B$ (given by $x \mapsto x$) and an injection $g : B \to A$ (pick your own).

5. Show that there exist injective map from $\mathbb{R}^2$ to $\mathbb{R}$. Does there exists injection from the set of all infinite real sequences to $\mathbb{R}$? Could you construct it?

6. We say that a sequence $f : \mathbb{N} \to \mathbb{N}$ is *nondecreasing* if $f(n + 1) \geq f(n)$ for all n and *nonincreasing* if $f(n + 1) \leq f(n)$ for all n. Is the set of all nondecreasing sequences countable or uncountable? How about the set of nonincreasing sequences?

Bonus problems

7. Consider the partition of the interval $(0, 1)$ corresponding to the equivalence relation $\sim\, : x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. Show that it has uncountably many equivalence classes. Is there a bijection between the set of classes and $(0, 1)$?

8. Construct a function $f : \langle 0, 1\rangle \to \langle 0, 1\rangle$, such that it attains all possible values on every open interval. In other words, for all $0 \leq a < b \leq 1$ and for all $c \in \langle 0, 1\rangle$ there exists x such that $a < x < b$ and $f(x) = c$.

9. Let $\mathcal{S}$ be a system of subsets of $\mathbb{N}$ such that for every $A, B \in \mathcal{S}$ we have $A \subseteq B$ or $B \subseteq A$. Could the set $\mathcal{S}$ be uncountable?