Discrete Mathematics I. – Homework 12

Problem sessions in the week of December 16, 2013

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1. Denote $0 = |\emptyset|$ and $1 = |\{\emptyset\}|$. Show that for every cardinal number p the following holds:
a) $p + 0 = p \cdot 1 = p$,
b) $p^0 = 0^0 = 1$,
c) $p^1 = p$.
(recall that equalities between cardinal numbers are represented by some bijections, their summation corresponds to a disjoint union, their product to Cartesian product of sets, and powers are represented by sets of mappings)

2. Show that for any cardinal numbers p, q and r we have:
a) $p \leq q \Rightarrow pr \leq qr$,
b) $p \leq q \Rightarrow p^r \leq q^r$,
c) if $r \neq 0$, then $p \leq q \Rightarrow r^p \leq r^q$. What happens when $r = 0$?

3. Do the statements from problem 2 remain true when we replace inequalities by strict inequalities?

The arithmetics of infinite cardinal numbers is slightly different than expected:
For example $\aleph_0 + \aleph_0 = \aleph_0$, because $\mathbb{N}_0 \cup -\mathbb{N} = \mathbb{Z}$. Similarly, $\aleph_0 \cdot \aleph_0 = \aleph_0$, because $|\mathbb{N}||\mathbb{Z}| = |\mathbb{Q}|$.

4. Denote $\aleph_0 = |\mathbb{N}|$ and $c = |\mathbb{R}| = 2^{\aleph_0}$. By using inequalities from problem 2 and relations from the lecture show that for any integer $n \in \mathbb{N}$, $n \geq 2$ holds:
a) $c \leq nc \leq \aleph_0 c \leq cc \leq c^n \leq c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = c$,
b) $2^{\aleph_0} = n^{\aleph_0} = \aleph_0^{\aleph_0} = c^{\aleph_0} = c$,
c) $2^c = n^c = \aleph_0^c = c^c$.

5. Show that the set of all irrational numbers is uncountable.

Bonus problems

6. Let A be some uncountable set of real numbers and a map $f : A \to \mathbb{R}$ be injective. Show that there exists an irrational number $x \in A$ such that $f(x)$ is irrational as well. Deduce that there exists a pair of irrational numbers a, b such that a^b is rational. Find a concrete example of such a pair.

7. Does there exists uncountable system $\mathcal{B}$ of subsets of $\mathbb{N}$ such that for different $A, B \in \mathcal{B}$ is the intersection $A \cap B$ finite?