Discrete Mathematics I. – Holiday homework set

Preparation for the exam or problems for those who still have not got enough

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1. Write down negations of the following statements:
(i) n is even or m is multiple of 3,
(ii) every $x \in A$ is an element of $A \cap B$,
(iii) if it does not rain today then frogs are not falling from the sky.

2. Translate the meaning of the following symbolic statement into a short English sentence (it can be done in 6 words). Write its negation in symbolic language as well. (In this problem the variables m, n, a, b are understood as positive integers, i.e. $\forall m$ means $\forall m \in \mathbb{N}^+$.)

$$(\forall m)(\exists n)(\forall a)(\forall b)[[n \geq m] \wedge [(a = 1) \vee (b = 1) \vee ((ab \neq n) \wedge (ab + 2 \neq n))]].$$

3. Let the binary operation $*$ on positive integers satisfy:
(i) $1 * n = n - 1 \quad$ for all $n \in \mathbb{N}$,
(ii) $m * 1 = (m-1) * 2 \quad$ for all $m \in \mathbb{N}$, $m > 1$,
(iii) $m * n = (m-1) * (m * (n-1)) \quad$ for $m, n \in \mathbb{N}$, $m, n > 1$.
Find the value of $5 * 5$.

4. The *symmetric difference* $A \triangle B$ of sets A and B is defined as $(A - B) \cup (B - A)$. Using the truth table for the propositions $x \in A$, $x \in B$, $x \in C$ show that the operation $\triangle$ is associative. Show that x belongs to $A \triangle (B \triangle C)$ if and only if x belongs to an odd number of sets A, B, C. Use this observation to construct another proof of associativity of $\triangle$.

5. Define a binary operation $*$ on $\mathbb{Z}^2$ by: $(a, b) * (c, d) = (ac, ad + bc)$. Show that the operation $*$ is commutative and associative. Find a formula for

$$(a_1, b_1) * (a_2, b_2) * \cdots * (a_k, b_k).$$

6. Let f be a map from real numbers to real numbers. We say that f is *strictly increasing* if for $x < y$ we have $f(x) < f(y)$ as well. Show that if f is strictly increasing then it is injective. Does it have to be surjective? Suppose that f is bijective and $f(0) = 0$, $f(1) = 1$. Does it follow that f is strictly increasing? What if f is continuous?

7. Let R be an rectangle which can be subdivided into smaller rectangles, each of them having at least one side of integer length. Show that R has at least one side of integer length.

8. Let n be an even number and $\mathcal{A}$ is a system of subsets of the set $\{1, 2, \ldots, n\}$ with the property that for every $A, B \in \mathcal{A}$ is the number of elements of $A \cap B$ even (this holds for the pair $A = B$ as well). How many sets could $\mathcal{A}$ contain? How does the answer change if the sets in $\mathcal{A}$ have an even number of elements but for $A \neq B$ is the cardinality of the intersection $A \cap B$ odd?

9. Can we partition the closed interval $[0, 1]$ into countable infinite union of disjoint nonempty closed intervals?

10. There are n points in a plane not lying on a single line. Show that there exists a line containing exactly two of those points.

11. For $i \in \mathbb{N}$ let $[a_i, b_i]$ be a closed interval of positive real numbers. Assume that $\sum_i |b_i - a_i| < \infty$. Does it follow that there exists a real number x such that nx does not belong to any interval $[a_i, b_i]$ for any n?

12. Countably infinitely many dons are standing in a circle. Every don wears red or blue hat. Each don can see hats of all his colleagues but not his own. In a certain moment dons have to shout the colour of their hats. Is it possible to give dons such instructions that only finitely many would guess colour of their hats incorrectly?