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Sets of Statistical Cluster Points and $\mathcal{I}$-Cluster Points

Abstract

Let $\mathcal{I}$ be an admissible (i. e. proper and containing all finite subsets of $\mathbb{N}^+$) ideal of subsets of the set $\mathbb{N}^+$ of positive integers. The concept of $\mathcal{I}$-convergence of sequences in metric spaces generalizes the concept of statistical convergence and also the usual concept of convergence of sequences.

In this paper we investigate some problems concerning the sets of $\mathcal{I}$-cluster points and, in particular, the sets of statistical cluster points of sequences in metric spaces which are known to be closed sets.

In the first part of the paper we give a sufficient condition on a sequence $x = (x_n)_{n=1}^{\infty}$ in a boundedly compact metric space which ensures the connectedness of the set of all statistical cluster points of x.

If $\Gamma_x(\mathcal{I})$ denotes the set of all $\mathcal{I}$-cluster points of the sequence x and M is a set of sequences in a metric space X such that, for each $x \in M$, $\Gamma_x(\mathcal{I}) \neq \emptyset$, then the assignment $x \mapsto \Gamma_x(\mathcal{I})$ gives a map $M \to \mathcal{F}$ where $\mathcal{F}$ is the set of all non-empty closed subsets of the space X or $M \to \mathcal{C}$ where $\mathcal{C}$ is a suitable subset of $\mathcal{F}$.

In the second part of the paper we study the continuity of this map with respect to the sup-metric on M and some standard hypertopologies on $\mathcal{C}$ (the Vietoris topology, the Fell topology, the proximal topology and the topology given by the Hausdorff metric). We obtain some positive results in the case of locally compact and, particularly, boundedly compact metric spaces.

Introduction

Let (X, ϱ) be a metric space. The notion of a statistically convergent sequence of points of a metric space can be defined using the asymptotic density of

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subsets of the set of positive integers $\mathbb{N}^+ = \{1, 2, \dots\}$ (cf. [7]). For any $A \subseteq \mathbb{N}^+$ and $n \in \mathbb{N}^+$ we denote

$$A(n) := \operatorname{card} A \cap \{1, 2, \dots, n\}$$

and we define lower and upper asymptotic density of the set A by the formulas

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{A(n)}{n}, \quad \overline{d}(A) := \limsup_{n\to\infty} \frac{A(n)}{n}.$$

If $\underline{d}(A) = \overline{d}(A) =: d(A)$, then the common value $d(A)$ is called the asymptotic density of the set A and

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}.$$

Obviously all three densities $\underline{d}(A)$, $\overline{d}(A)$ and $d(A)$ (if they exist) lie in the unit interval $[0, 1]$.

The notion of statistical convergence was originally defined for sequences of numbers in the paper [4] and also in [11]. The idea of statistical convergence can be easily generalized to sequences of points of a metric space (see [7]).

We say that a sequence $x = (x_n)_1^\infty$ of points of a metric space (X, ϱ) *statistically converges* to a point $\xi \in X$ if for each $\varepsilon > 0$ we have

$$d(A(\varepsilon)) = 0,$$

where

$$A(\varepsilon) := \{n \in \mathbb{N}^+ : \varrho(x_n, \xi) \geq \varepsilon\}$$

and in such situation we will write $\xi = \lim\operatorname{stat} x$ or in more detail $\xi = \lim\operatorname{stat} x_n$.

The notion of statistical convergence was further generalized in the paper [8] using the notion of an ideal of subsets of the set $\mathbb{N}^+$. We say that a non-empty family of sets $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N}^+)$ is an *ideal* on $\mathbb{N}^+$ if $\mathcal{I}$ is hereditary (i.e. $B \subseteq A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}^+$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N}^+)$ is called a *proper* ideal. A proper ideal $\mathcal{I}$ is called *admissible* if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}^+$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal.

Recall the generalization of statistical convergence from [8].

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}^+$ and $x = (x_n)_1^\infty$ be a sequence of points in a metric space (X, ϱ). We say that the sequence x is *$\mathcal{I}$-convergent* (or *$\mathcal{I}$-converges*) to a point $\xi \in X$, and we denote it by $\mathcal{I}\text{-}\lim x = \xi$, if for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{n \in \mathbb{N}^+ : \varrho(x_n, \xi) \geq \varepsilon\} \in \mathcal{I}.$$

This generalizes the notion of usual convergence, which can be obtained when we take for $\mathcal{I}$ the ideal $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}^+$. A sequence is statistically convergent if and only if it is $\mathcal{I}_d$-convergent, where $\mathcal{I}_d := \{A \subseteq \mathbb{N}^+ : d(A) = 0\}$ is the admissible ideal of the sets of zero asymptotic density.

Many other types of $\mathcal{I}$-convergence are obtained by specifying the ideal. Here we mention just two other types of $\mathcal{I}$-convergence introduced in the paper [9]. Let $g\colon \mathbb{N}^+ \to \mathbb{R}^+$; and $\sum\limits_{n=1}^{\infty} g(n) = +\infty$. Then it is easy to verify that

$$\mathcal{I}^g := \{A \subseteq \mathbb{N}^+ : \sum_{n\in A} g(n) < +\infty\}$$

is an ideal on $\mathbb{N}^+$. For a special case when $g(n) = \frac{1}{n}$ for $n = 1, 2, \ldots$ the ideal $\mathcal{I}^g$ is denoted $\mathcal{I}_c$ in [8].

Let us consider an infinite sequence $e\colon \mathbb{N}^+ \to \mathbb{R}^+$ with $\sum\limits_{n=1}^{\infty} e(n) = +\infty$ and

$$\lim_{n\to\infty} \frac{e(n)}{\sum\limits_{k<n} e(k)} = 0.$$

Then we define the Erdös-Ulam ideal by

$$\mathcal{I}_e = \{A \subseteq \mathbb{N}^+ : \lim_{n\to\infty} \frac{\sum\{e(k) : k < n, k \in A\}}{\sum\limits_{k<n} e(k)} = 0\}.$$

Statistical convergence and its generalization $\mathcal{I}$-convergence, enable us to introduce notions of statistical limit point and a statistical cluster point of a sequence, or their generalizations $\mathcal{I}$-limit point and $\mathcal{I}$-cluster point, respectively (cf. [6], [8], [9], [10]).

Since statistical convergence is in fact $\mathcal{I}_d$-convergence we recall those notions for $\mathcal{I}$-convergence only.

Definition. Let (X, ϱ) be a metric space and x be a sequence in X.

(a) A point $\xi \in X$ is called *$\mathcal{I}$-limit point* of a sequence x if there is a set $K = \{k_1 < k_2 < \ldots\} \in \mathcal{P}(\mathbb{N}^+) \setminus \mathcal{I}$ such that $\lim\limits_{n\to\infty} x_{k_n} = \xi$. The set of all $\mathcal{I}$-limit points of a sequence x will be denoted $\Lambda_x(\mathcal{I})$ or just Λ_x for statistical convergence.

(b) A point $\xi \in X$ is called *$\mathcal{I}$-cluster point* of a sequence x if for any $\varepsilon > 0$, $\{n \in \mathbb{N}^+ : \varrho(x_n, \xi) < \varepsilon\} \notin \mathcal{I}$. The set of all $\mathcal{I}$-cluster points of x will be denoted $\Gamma_x(\mathcal{I})$ or just Γ_x for the statistical convergence.

In the papers [6], [7], [8] some basic properties of the sets Λ_x and Γ_x were studied. It was proved in [8] that the set $\Gamma_x(\mathcal{I})$ is closed in (X,ϱ) for any admissible ideal $\mathcal{I}$. Note that if $\mathcal{I}$ is an admissible ideal, then any convergent sequence in (X,ϱ) is $\mathcal{I}$-convergent. In this paper we will study some topological properties of the sets $\Gamma_x(\mathcal{I})$, in particular, we will study connectedness of the set $\Gamma_x(\mathcal{I})$ and also some topological properties of the mapping $x \mapsto \Gamma_x(\mathcal{I})$.

1 Basic topological properties of the sets $\Gamma_x(\mathcal{I})$

Let (X,ϱ) be a metric space and $x=(x_n)_1^\infty$ be a sequence in X. We denote by $s(X)$ the set of all sequences in X and $bs(X)$ is the set of all bounded sequences from $s(X)$. For any $x\in s(X)$ let L_x denote the set of all limit points ξ (accumulation points) of the sequence x, i.e. $\xi\in L_x$ if there exists an infinite set $K=\{k_1<k_2<\dots\}\subseteq\mathbb{N}^+$ such that $\lim\limits_{n\to\infty} x_{k_n}=\xi$ or equivalently $\lim\limits_{n\to\infty}\varrho(x_{k_n},\xi)=0$. When $\mathcal{I}$ is an admissible ideal then obviously we have

$$\Lambda_x(\mathcal{I})\subseteq\Gamma_x(\mathcal{I})\subseteq L_x. \tag{1}$$

The following examples show that the inclusions in (1) can be strict.

Example 1.1. a) Let $X=\mathbb{R}$ (with the usual euclidean metric). We decompose the set $\mathbb{N}^+$ into countably many disjoint sets

$$N_p := \{2^{p-1}(2k-1) : k\in\mathbb{N}^+\}, \qquad (p=1,2,\dots).$$

It is obvious that $\mathbb{N}^+=\bigcup\limits_{p=1}^{\infty} N_p$ and it can be easily shown that the asymptotic density of N_p is

$$d(N_p)=\frac{1}{2^p} \qquad (p=1,2,\dots).$$

Let us define the sequence $x=(x_n)_1^\infty$ with general term $x_n=\frac{1}{p}$ if $n\in N_p$ $(p=1,2,\dots)$. This is well-defined since the sets N_p are pairwise disjoint. One can show easily that $L_x=\Gamma_x(\mathcal{I}_d)=\{0,1,\frac12,\dots\}$ and $\Lambda_x(\mathcal{I}_d)=\{1,\frac12,\dots\}$. Therefore $\Lambda_x(\mathcal{I})\subsetneq\Gamma_x(\mathcal{I})$.

b) Let us take $X=\mathbb{R}$ and $x_n=(-1)^{n-1}$ $(n=1,2,\dots)$. Then one can show easily that $L_x=\Gamma_x(\mathcal{I}_d)=\Lambda_x(\mathcal{I}_d)=\{-1,1\}$.

c) Let $X=\mathbb{R}$ and define $x_n=1$ if $n=k^2$ for some $k\in\mathbb{N}^+$ and $x_n=0$ otherwise. Then $\Lambda_x(\mathcal{I}_d)=\Gamma_x(\mathcal{I}_d)=\{0\}$ and $L_x=\{0,1\}$ and so we have $\Gamma_x(\mathcal{I}_d)\subsetneq L_x$.

d) The sets $\Lambda_x(\mathcal{I})$, $\Gamma_x(\mathcal{I})$ can be empty when $\operatorname{diam} X=\sup\limits_{\alpha,\beta\in X}\varrho(\alpha,\beta)=+\infty$. In such spaces we can inductively define the sequence $(x_n)_1^\infty$ with

$\varrho(x_i, x_j) \geq 1$ for $i \neq j$. Then $L_x = \emptyset$ and consequently due to (1) we get $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = \emptyset$.

e) Let $(r_1, r_2, \ldots)$ be an injective sequence of all rationals of $[0, 1]$ and let $X = \mathbb{R}$. Using the partition of $\mathbb{N}^+ = \bigcup_{p=1}^{\infty} N_p$ from a) we can define a sequence $(x_n)_1^\infty$ taking $x_n = r_p$ if $n \in N_p$ $(n = 1, 2, \ldots)$. Then $L_x = [0, 1] = \Gamma_x(\mathcal{I}_d)$ and $\Lambda_x(\mathcal{I}_d) = \{r_1, r_2, \ldots\}$.

Considering the previous examples the following question arises: When one can claim the set $\Gamma_x(\mathcal{I})$ is a connected set? We know that the set $\Gamma_x(\mathcal{I})$ is always a closed set, but the foregoing examples show that for some sequences it can be a connected set and for other sequences it is not a connected set. The problem of connectedness of L_x was studied in [1] and the following theorem was proved:

Theorem A. *Let (X, ϱ) be a metric space and $x = (x_n)_1^\infty \in s(X)$ be a sequence satisfying the condition*

$$\lim_{n \to \infty} \varrho(x_n, x_{n+1}) = 0. \tag{2}$$

Then the set L_x is a connected set in X.

It seems that the set $\Gamma_x(\mathcal{I})$ has a more complicated structure, since the condition (2) above is not sufficient to ensure the connectedness of the set $\Gamma_x(\mathcal{I})$ as the following example shows.

Example 1.2. Let us take $X = \mathbb{R}$, $\mathcal{I} = \mathcal{I}_d$. Fridy showed in [5] the following Cauchy-type characterization of statistically convergent sequences:

A sequence $x = (x_n)_1^\infty$ is statistically convergent if and only if

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}^+ : d(\{n : |x_n - x_{N_\varepsilon}| < \varepsilon\}) = 1 \tag{C}$$

So if (C) is satisfied then x statistically converges to a point ξ and therefore $\Gamma_x(\mathcal{I}_d) = \{\xi\}$ which is a connected set. We are going to construct a sequence satisfying condition (2) and also a condition weaker than (C) namely

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}^+ : \overline{d}(\{n : |x_n - x_{N_\varepsilon}| < \varepsilon\}) = 1 \tag{C*}$$

and nevertheless the set $\Gamma_x(\mathcal{I}_d)$ is disconnected.

Let us define a set $A := \bigcup_{k=2}^{\infty} A_k$ with $A_k = \{k^{k^2} + 1, k^{k^2} + 2, \ldots, k^{(k+1)^2} - (k+1)\}$ $(k = 2, 3, \ldots)$. It can be seen easily that $d(A) = 1$.

Let us define also the sets

$$B_1 := \{1, 2, 3, \ldots, 2^{2^2}\},$$
$$B_k := \{k^{(k+1)^2} - k, k^{(k+1)^2} - (k-1), \ldots k^{(k+1)^2} - 1, k^{(k+1)^2}\}$$

$(k = 2, 3, \dots)$. The blocks A_k, B_k taken in the natural order of $\mathbb{N}^+$ form the sequence

$$B_1, A_2, B_2, A_3, B_3, \dots$$

Since $d(A) = 1$ we have at once that $d(B) = 0$ with $B := \bigcup_{k=1}^{\infty} B_k$. Let us take

$$A_1^* := \bigcup_{k=2}^{\infty} A_{2k-1}, \qquad A_2^* := \bigcup_{k=1}^{\infty} A_{2k}.$$

Then it is easy to see that

$$\overline{d}(A_1^*) = \overline{d}(A_2^*) = 1$$

Now we can define a sequence $x := (x_n)_1^\infty$ as follows

$$x_n = \begin{cases} 0, & \text{if } n \in B_1 \cup A_2^* \\ 1, & \text{if } n \in A_1^* \end{cases}$$

Between the blocks A_{2k}, A_{2k+1} there is the block B_{2k} with $2k+1$ elements. The values of x are elements of the interval $[0, 1]$ and we make the equidistant partition of the interval $[0, 1]$ into the intervals: $[0, \frac{1}{2k+2}], [\frac{1}{2k+2}, \frac{2}{2k+2}], \dots, [\frac{2k+1}{2k+2}, 1]$. If $n \in B_{2k}$ then $n = (2k)^{(2k+1)^2} - r$ for some $0 \le r \le 2k$ and we put $x_n := \frac{2k-r+1}{2k+2}$. We continue the same way from $\frac{2k+2}{2k+3}$ to $\frac{1}{2k+3}$ when defining x_n for $n \in B_{2k+1}$. If $k \ge 2$ and $k^{(k+1)^2} < n < (k+1)^{(k+1)^2} + 1$, then $x_n = x_l$ where $l = (k+1)^{(k+1)^2} + 1$.

The sequence $(x_n)_1^\infty$ defined in that way satisfies the condition (2). We show that it satisfies also the condition (C*). For an arbitrarily chosen $\varepsilon > 0$ let us choose N_ε such that $N_\varepsilon \in A_1^*$ or $N_\varepsilon \in A_2^*$. In both cases we have

$$\overline{d}(\{n : |x_n - x_{N_\varepsilon}| < \varepsilon\}) = 1$$

since if $N_\varepsilon \in A_1^*$ then $|x_n - x_{N_\varepsilon}| < \varepsilon$ is satisfied for all elements of A_1^* and similarly when $N_\varepsilon \in A_2^*$.

Obviously $\Gamma_x(\mathcal{I}_d) = \{0, 1\}$ and this is a disconnected set.

The previous example shows that the condition (2) must be strengthened to ensure the connectedness of $\Gamma_x(\mathcal{I})$.

The following theorem was inspired by the papers [1] and [12].

Recall that a metric space (X, ϱ) is called *boundedly compact* if each closed bounded subset of (X, ϱ) is compact.

Theorem 1.1. *Let (X, ϱ) be a boundedly compact metric space. Let $x = (x_n)_1^\infty \in bs(X)$ satisfy the condition*

$$\exists c > 1 : \lim_{n\to\infty} M_n^{(c)} = 0 \tag{3}$$

where $M_n^{(c)} := \max\{\varrho(x_k, x_n) : n \leq k \leq [cn]\}$ and $[cn]$ is the integer part of cn. Then $\Gamma_x(\mathcal{I}_d)$ is a connected set.

Proof. The condition (3) obviously implies (2) and the condition (2) ensures by Theorem A the connectedness of L_x. So it suffices to prove that the assumptions of Theorem 1.1 imply

$$L_x = \Gamma_x(\mathcal{I}_d) \tag{4}$$

Taking into consideration (1) it is sufficient to prove

$$L_x \subseteq \Lambda_x(\mathcal{I}_d) \tag{4'}$$

To prove (4') let us choose $\alpha \in L_x$. Then there exists a set

$$K = \{k_1 < k_2 < \ldots\} \subseteq \mathbb{N}^+,$$

such that $\lim_{n\to\infty} x_{k_n} = \alpha$. If we consider the set

$$H(K) = \bigcup_{k\in K} \{k, k+1, \ldots, [ck]\} \subseteq \mathbb{N}^+$$

a simple estimate gives

$$\frac{H(K)([ck])}{[ck]} \geq \frac{[ck] - k}{[ck]} \to 1 - \frac{1}{c} > 0 \qquad (\text{if } k \to \infty)$$

Therefore

$$\overline{d}(H(K)) > 0.$$

To finish the proof of (4') it suffices to show that

$$\lim_{\substack{n\to\infty \\ n\in H(K)}} x_n = \alpha. \tag{5}$$

As $x_{k_n} \to \alpha$ there exists an $n_0 \in \mathbb{N}^+$ such that for every $n \geq n_0$ we have

$$\varrho(x_{k_n}, \alpha) < \frac{\varepsilon}{2}. \tag{6}$$

We can suppose n_0 to be so large that

$$\forall n \geq k_{n_0} : M_n^{(c)} < \frac{\varepsilon}{2}. \tag{7}$$

Let us choose $l \in H(K)$, $l \geq k_{n_0}$. Then there is $n \geq n_0$ such that

$$l \in \{k_n, k_n + 1, \dots, [ck_n]\}.$$

So we have $l \geq k_n \geq k_{n_0}$ and following (7) we get

$$\varrho(x_l, x_{k_n}) < \frac{\varepsilon}{2}. \tag{8}$$

Using the triangle inequality we get from (6) and (8)

$$\varrho(x_l, \alpha) \leq \varrho(x_l, x_{k_n}) + \varrho(x_{k_n}, \alpha) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\varrho(x_l, \alpha) < \varepsilon$ for $l \geq k_{n_0}$ and we have proved (5). □

Remark. Assumptions of Theorem 1.1 allowed us to prove the equality $L_x = \Lambda_x(\mathcal{I}_d)$ but this equality is neither necessary nor sufficient to ensure the connectedness of $\Gamma_x(\mathcal{I})$ as can be seen from the Example 1.1 b), e).

2 Properties of the mapping $\Gamma_{\mathcal{I}}\colon x \mapsto \Gamma_x(\mathcal{I})$

Let (X, ϱ) be a metric space, $s(X)$ be the set of all sequences in X and $\mathcal{I}$ be an admissible ideal on the set $\mathbb{N}^+$. If we assign to any sequence $x \in s(X)$ the set $\Gamma_x(\mathcal{I})$ of all its $\mathcal{I}$-cluster points, then we obtain a mapping of $s(X)$ to $\mathcal{P}(X)$. In [8] it is proved that for any admissible ideal $\mathcal{I}$ on $\mathbb{N}^+$ the set $\Gamma_x(\mathcal{I})$ is closed in (X, ϱ) for each $x \in s(X)$. Denote by $cs(X)$ the set of all $x \in s(X)$ with $\Gamma_x(\mathcal{I}) \neq \emptyset$ and by $\mathcal{F}$ the set of all non-empty closed subset of (X, ϱ). Then for any non-empty subset $M \subseteq cs(X)$ the assignment $x \mapsto \Gamma_x(\mathcal{I})$ defines a mapping $\Gamma_{\mathcal{I}}\colon M \to \mathcal{F}$. In this section we want to investigate the continuity of the mapping $\Gamma_{\mathcal{I}}$ with respect to suitable topologies on M and on $\mathcal{F}$. It seems to be natural to endow the set M with the *sup-metric* σ defined by

$$\sigma(x, y) = \min\{\sup_{n\geq 1} \varrho(x_n, y_n), 1\}$$

or, equivalently,

$$\sigma(x, y) = \sup_{n\geq 1} \varrho(x_n, y_n)$$

provided that all sequences in M are bounded. The set $\mathcal{F}$ can be endowed with some standard hyperspace topology, e.g. Vietoris topology, Fell topology or proximal topology. If all $\Gamma_x(\mathcal{I})$ for $x \in M$ are bounded, then it is natural to use the Hausdorff metric.

In what follows by $\mathcal{F}$ we always denote the set of all non-empty closed sets of a given metric space (X, ϱ). Recall that $\mathcal{I}$ denotes an admissible ideal on $\mathbb{N}^+$.

First we need to know for which sequences $x \in s(X)$ the set $\Gamma_x(\mathcal{I})$ is non-empty.

Lemma 2.1. Let (X, ϱ) be a metric space and K be a compact subset of X. Then for every $x \in s(X)$ with $\{n \in \mathbb{N}^+ : x_n \in K\} \notin \mathcal{I}$ we have $K \cap \Gamma_x(\mathcal{I}) \neq \emptyset$.

Proof. Suppose that $K \cap \Gamma_x(\mathcal{I}) = \emptyset$. Then any point $p \in K$ does not belong to $\Gamma_x(\mathcal{I})$ and therefore for any $p \in K$ there exists $\varepsilon_p > 0$ such that

$$A_p = \{n \in \mathbb{N}^+ : \varrho(x_n, p) < \varepsilon_p\} \in \mathcal{I}.$$

Since K is compact there exists a finite number of points $p_1, \dots, p_m \in K$ such that $K \subseteq \bigcup_{i=1}^{m} B_{\varepsilon_{p_i}}(p_i)$ where $B_{\varepsilon_{p_i}}(p_i) = \{t \in X; \varrho(p_i, t) < \varepsilon_{p_i}\}$. Obviously, we have $A = \{n \in \mathbb{N}^+ : x_n \in K\} \subseteq \bigcup_{i=1}^{m} A_{p_i}$.

Since $\bigcup_{i=1}^{m} A_{p_i} \in \mathcal{I}$ we obtain that $A \in \mathcal{I}$. ☐

We start by considering the case of boundedly compact metric spaces (X, ϱ) (the euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}^+$, are of such type). Denote by $bs(X)$ the set of all bounded sequences in (X, ϱ). Obviously, for every $\varepsilon > 0$ and $p \in X$ the closed ball $D_\varepsilon(p) = \{t \in X : \varrho(p, t) \leq \varepsilon\}$ is a compact set in (X, ϱ). Consequently, according to Lemma 2.1, for every sequence $x \in bs(X)$ we have $\Gamma_x(\mathcal{I}) \neq \emptyset$ (i.e. $bs(X) \subseteq cs(X)$) and $\Gamma_x(\mathcal{I})$ is compact (it is closed and bounded). Hence, the assignment $x \mapsto \Gamma_x(\mathcal{I})$ defines a mapping $\Gamma_{\mathcal{I}}$ of the set $bs(X)$ to the set $K(X)$ of all non-empty compact subsets of (X, ϱ). If we endow the set $bs(X)$ with the sup-metric $\sigma(x, y) = \sup_{n \geq 1} \varrho(x_n, y_n)$ and the set $K(X)$ with the *Hausdorff metric* ϱ_H defined by

$$\varrho_H(A, B) = \max\{\sup_{a \in A} \varrho(a, B), \sup_{b \in B} \varrho(b, A)\}$$

(where, as usual, $\varrho(a, B) = \inf_{d \in B} \varrho(a, d)$), then we obtain:

Theorem 2.1. *Let (X, ϱ) be a boundedly compact metric space. Then the mapping*

$$\Gamma_{\mathcal{I}} \colon (bs(X), \sigma) \to (K(X), \varrho_H), \qquad x \mapsto \Gamma_x(\mathcal{I})$$

is uniformly continuous (even lipschitzian).

Proof. Let $\delta > 0$, $x = (x_n)_1^\infty \in bs(X)$, $y = (y_n)_1^\infty \in bs(X)$ and $\sigma(x,y) < \delta$. We want to prove that this implies

$$\varrho_H(\Gamma_x(\mathcal{I}), \Gamma_y(\mathcal{I})) \leq 2\delta.$$

To show this it suffices to verify that for each $t \in \Gamma_x(\mathcal{I})$ we have $\varrho(t, \Gamma_y(\mathcal{I})) \leq 2\delta$ and for each $u \in \Gamma_y(\mathcal{I})$ we obtain $\varrho(u, \Gamma_x(\mathcal{I})) \leq 2\delta$. Let $t \in \Gamma_x(\mathcal{I})$. Then

$$D_{2\delta}(t) = \{v \in X : \varrho(t,v) \leq 2\delta\}$$

is a compact subset of X. Since $t \in \Gamma_x(\mathcal{I})$ we have

$$A = \{n \in \mathbb{N}^+ : \varrho(t, x_n) < \delta\} \notin \mathcal{I}.$$

By the triangle inequality we get for $n \in A$:

$$\varrho(y_n, t) \leq \varrho(y_n, x_n) + \varrho(x_n, t) < \delta + \delta = 2\delta.$$

But then

$$B = \{n \in \mathbb{N}^+ : y_n \in D_{2\delta}(t)\} \supseteq A$$

and therefore $B \notin \mathcal{I}$.

Using Lemma 2.1 we obtain that there exists a point

$$t' \in D_{2\delta}(t) \cap \Gamma_y(\mathcal{I})$$

and this yields $\varrho(t, \Gamma_y(\mathcal{I})) \leq 2\delta$.

Similarly we can show that $\varrho(u, \Gamma_x(\mathcal{I})) \leq 2\delta$. Hence,

$$\varrho_H(\Gamma_x(\mathcal{I}), \Gamma_y(\mathcal{I})) \leq 2\delta.$$

$\square$

Remark. One can ask a natural question whether the mapping $\Gamma_{\mathcal{I}}$ from Theorem 2.1 is surjective. We show that this is true for suitable choice of $\mathcal{I}$, e.g. for $\mathcal{I} = \mathcal{I}_d$.

Let $K \subseteq X$ be a non-empty compact subset in (X, ϱ). Then $(K, \varrho|_{K\times K})$ is a separable metric space and hence there exists a countable set $M \subseteq K$ dense in K, i.e. $\overline{M} = K$. Let $M = \{\alpha_1, \alpha_2, \ldots, \alpha_p, \ldots\}$ and $N_p := \{2^{p-1}(2k-1) : k \in \mathbb{N}^+\}$. Then $\mathbb{N}^+ = \bigcup_{p=1}^{\infty} N_p$ (see Example 1.1) and we can define a sequence $x = (x_n)_{n\geq 1}$ by

$$x_n = \alpha_p \text{ if } n \in N_p \qquad (n = 1, 2, \ldots)$$

Then $x \in bs(X)$ and $\Gamma_x(\mathcal{I}_d) = K$.

Similarly, if $\mathcal{I} = \mathcal{I}_f$ is the ideal of all finite subsets of $\mathbb{N}^+$, $K \in K(X)$ and $M = \{\alpha_1, \alpha_2, \dots, \alpha_p, \dots\}$ is a countable dense subset of K, then for the sequence $x \in bs(X)$ such that

$$x = (\alpha_1, \alpha_1, \alpha_2, \alpha_1, \alpha_2, \alpha_3, \dots)$$

we have $\Gamma_x(\mathcal{I}_f) = K$.

We next recall the definition of Vietoris topology on the set $\mathcal{F}$ of all non-empty closed subsets of a metric space (X, ϱ).

For any subset U of X put

$$U^- := \{A \in \mathcal{F} : A \cap U \neq \emptyset\} \text{ and}$$
$$U^+ := \{A \in \mathcal{F} : A \subseteq U\}.$$

Then the family $\mathcal{S}_V^- := \{U^- : U \text{ is open in } X\}$ is a subbase of the *lower Vietoris topology* on $\mathcal{F}$ denoted by τ_V^-, the family $\mathcal{S}_V^+ := \{U^+ : U \text{ is open in } X\}$ is a subbase of the *upper Vietoris topology* on $\mathcal{F}$ denoted by τ_V^+ and $\mathcal{S}_V = \mathcal{S}_V^- \cup \mathcal{S}_V^+$ is a subbase of the *Vietoris topology* on $\mathcal{F}$ which is denoted by τ_V.

It is well known (see e.g. [3, p. 371]) that the family $K(X)$ of all non-empty compact subsets of (X, ϱ) considered as a subspace of $(\mathcal{F}, \tau_V)$ is metrizable by the Hausdorff metric on $K(X)$. Thus applying Theorem 2.1 we obtain

Theorem 2.2. *If (X, ϱ) is a boundedly compact metric space, then the mapping*

$$\Gamma_{\mathcal{I}} \colon (bs(X), \sigma) \to (\mathcal{F}, \tau_V); \qquad x \mapsto \Gamma_x(\mathcal{I})$$

is continuous.

Boundedly compact metric spaces are locally compact and from Lemma 2.1 it easily follows that for locally compact metric spaces the following statement holds:

Proposition 2.1. *Let (X, ϱ) be a locally compact metric space and $x \in s(X)$. Then $\Gamma_x(\mathcal{I}) \neq \emptyset$ if and only if there exists a compact set $K \subseteq X$ with*

$$\{n \in \mathbb{N}^+ : x_n \in K\} \notin \mathcal{I}.$$

Consider the set $cs(X)$ of all sequences in (X, ϱ) with $\Gamma_x(\mathcal{I}) \neq \emptyset$ together with the sup-metric σ defined by $\sigma(x, y) = \min\{1, \sup\limits_{n \geq 1} \varrho(x_n, y_n)\}$. Then the following theorem completes in a certain sense Theorem 2.2.

Theorem 2.3. *Let (X, ϱ) be a locally compact metric space. Then*

$$\Gamma_{\mathcal{I}} \colon (cs(X), \sigma) \to (\mathcal{F}, \tau_V^-); \qquad x \mapsto \Gamma_x(\mathcal{I})$$

is a continuous mapping.

Proof. It suffices to prove that for every open subset U in (X,ϱ) the set ${\Gamma_{\mathcal{I}}}^{-1}(U^-)$ is open in $(cs(X),\sigma)$. Let $x\in{\Gamma_{\mathcal{I}}}^{-1}(U^-)$. Then $\Gamma_x(\mathcal{I})\cap U\neq\emptyset$ and we can choose a point $t\in\Gamma_x(\mathcal{I})\cap U$. Since (X,ϱ) is a locally compact space there exists $\varepsilon>0$ such that

$$D_\varepsilon(t)=\{v\in X:\varrho(v,t)\leq\varepsilon\}$$

is a compact set with $D_\varepsilon(t)\subseteq U$. Let $0<\delta\leq\frac{\varepsilon}{2}$, $\delta<1$ and $y\in cs(X)$ satisfying $\sigma(x,y)<\delta$. Since $t\in\Gamma_{\mathcal{I}}(x)$ we have

$$A=\{n\in\mathbb{N}^+:\varrho(x_n,t)<\delta\}\notin\mathcal{I}.$$

Using the triangle inequality we obtain $\varrho(y_n,t)\leq\varrho(y_n,x_n)+\varrho(x_n,t)<\delta+\delta\leq\varepsilon$ for any $n\in A$ and it follows that $A\subseteq B=\{n\in\mathbb{N}^+:y_n\in D_\varepsilon(t)\}\notin\mathcal{I}$.

Since $D_\varepsilon(t)$ is a compact set we get (using Lemma 2.1) $\Gamma_y(\mathcal{I})\cap D_\varepsilon(t)\neq\emptyset$ and because of $D_\varepsilon(t)\subseteq U$ also $\Gamma_y(\mathcal{I})\cap U\neq\emptyset$. Hence $y\in{\Gamma_{\mathcal{I}}}^{-1}(U^-)$ and, consequently, the open ball $B_\delta(x)$ is contained in ${\Gamma_{\mathcal{I}}}^{-1}(U^-)$. Hence ${\Gamma_{\mathcal{I}}}^{-1}(U^-)$ is open in $(cs(X),\sigma)$. □

Remark. Using the language of set-valued mappings Theorem 2.3 can be alternatively formulated as follows: If (X,ϱ) is a locally compact metric space, then the set-valued mapping of $s(X)$ to (X,ϱ) given by $x\mapsto\Gamma_x(\mathcal{I})$ is lower semicontinuous.

Problem. Let (X,ϱ) be a locally compact metric space. Is the map $\Gamma_{\mathcal{I}}\colon(cs(X),\sigma)\to(\mathcal{F},\tau_V^+)$; $x\mapsto\Gamma_x(\mathcal{I})$ continuous? (Equivalently, is the map $\Gamma_{\mathcal{I}}\colon(cs(X),\sigma)\to(\mathcal{F},\tau_V)$ continuous?)

We next study the continuity of $\Gamma_{\mathcal{I}}$ with respect to the Fell topology and the proximal topology on the set $\mathcal{F}$. We start with the definitions (see [2]).

Let (X,ϱ) be a metric space. Put

$$\mathcal{S}_F^+:=\{U^+:X\setminus U\text{ is a compact set in }(X,\varrho)\}.$$

Then the family $\mathcal{S}_F^+\cup\mathcal{S}_V^-$ is a subbase of the *Fell topology* on $\mathcal{F}$ denoted by τ_F. Since $\mathcal{S}_F^+\subseteq\mathcal{S}_V^+$ we have $\tau_F\subseteq\tau_V$.

The notion of proximal topology is defined as follows: For any $U\subseteq X$ put

$$U^{++}:=\{F\in\mathcal{F}:\text{ there exists }\varepsilon>0\text{ with }S_\varepsilon(F)\subseteq U\},$$

where $S_\varepsilon(F):=\{a\in X:\varrho(a,F)<\varepsilon\}$ is the open ε-neighbourhood of the set F. Put

$$\mathcal{S}_P^+:=\{U^{++}:U\text{ is open in }(X,\varrho)\}.$$

Then the family $\mathcal{S}_P^+ \cup \mathcal{S}_V^-$ is a subbase of the *proximal topology* on $\mathcal{F}$ denoted by τ_P.

Recall that metrics ϱ and ϱ' in the set X are called equivalent if they induce the same topology on X. It is well known that equivalent metrics on X can induce non-equivalent sup-metrics σ, σ' on $s(X)$ and different proximal topologies on $\mathcal{F}$. On the other hand, the Fell topology depends only on the topology of (X, ϱ).

The following property determines the class of metric spaces which contains the class of all boundedly compact spaces (and is contained in the class of all locally compact metric spaces).

A metric space is (X, ϱ) is said to have *nice closed balls* provided whenever D is a closed ball in X that is a proper subset of X, then D is compact. Clearly, every boundedly compact metric space has nice closed balls. The converse is not true, e.g. every infinite metric space with zero-one metric has nice closed balls without being boundedly compact. For locally compact spaces the following statement holds (see [2]):

Theorem B. *For any metric space (X, ϱ) the following assertions are equivalent:*

(a) (X, ϱ) is locally compact space.

(b) There is a metric ϱ' on X equivalent to ϱ such that (X, ϱ') has nice closed balls.

Proposition 2.2. *Let a metric space (X, ϱ) have nice closed balls, σ be the sup-metric on $cs(X)$ defined by $\sigma(x, y) = \min\{1, \sup_{n\geq 1} \varrho(x_n, y_n)\}$, U be an open set in (X, ϱ), and $x \in cs(X)$. If $\varrho(\Gamma_x(\mathcal{I}), X \setminus U) = r > 0$, then there exists $\delta > 0$ such that for any $y \in cs(X)$ with $\sigma(x, y) < \delta$ we have*

$$\varrho(\Gamma_y(\mathcal{I}), X \setminus U) \geq \frac{r}{2}.$$

Proof. Let $0 < \delta \leq \frac{r}{6}$, $\delta < 1$, $\sigma(x, y) < \delta$ and $t \in \Gamma_y(\mathcal{I})$. Then

$$A = \{n \in \mathbb{N}^+ : \varrho(t, y_n) < \frac{r}{6}\} \notin \mathcal{I}.$$

For every $n \in A$ we have

$$\varrho(t, x_n) \leq \varrho(t, y_n) + \varrho(y_n, x_n) < \frac{r}{6} + \frac{r}{6} = \frac{r}{3}$$

and therefore $A \subseteq B = \{n \in \mathbb{N}^+ : \varrho(t, x_n) \leq \frac{r}{2}\} \notin \mathcal{I}$.

Since $D_{\frac{r}{3}}(t) \neq X$ ($\operatorname{diam} X \geq r$) it is a compact set and this together with $B \notin \mathcal{I}$ yields that $D_{\frac{r}{3}}(t) \cap \Gamma_x(\mathcal{I}) \neq \emptyset$. Hence there exists $u \in \Gamma_x(\mathcal{I})$ with $\varrho(t,u) \leq \frac{r}{3}$. For any $v \in X \setminus U$ we get

$$r \leq \varrho(u,v) \leq \varrho(u,t) + \varrho(t,v).$$

Then

$$\varrho(t,v) \geq \varrho(u,v) - \varrho(t,u) \geq r - \frac{r}{3} = \frac{2}{3}r > \frac{r}{2}.$$

Consequently

$$\varrho(\Gamma_y(\mathcal{I}), X \setminus U) \geq \frac{r}{2}.$$

$\square$

Corollary. *Let a metric space (X,ϱ) have nice closed balls. Then, using the notation of Proposition 2.2 we obtain:*

(a) If $\Gamma_x(\mathcal{I}) \in U^{++}$, then there exists $\delta > 0$ such that for each $y \in cs(X)$ with $\sigma(x,y) < \delta$ we have $\Gamma_y(\mathcal{I}) \in U^{++}$.

(b) If $X \setminus U$ is compact in (X,ϱ) then $\Gamma_x(\mathcal{I}) \in U^+$ implies there exists $\delta > 0$ such that for each $y \in cs(X)$ with $\sigma(x,y) < \delta$ we have $\Gamma_y(\mathcal{I}) \in U^+$.

Combining Proposition 2.1, the above Corollary and Theorem 2.3 we obtain

Theorem 2.4. *Let (X,ϱ) be metric space which has nice closed balls and σ be the sup-metric on the set $cs(X)$ given by $\sigma(x,y) = \min\{1, \sup_{n\geq 1} \varrho(x_n,y_n)\}$. Then the mappings*

$$\Gamma_{\mathcal{I}}\colon (cs(X),\sigma) \to (\mathcal{F},\tau_F); \qquad x \mapsto \Gamma_x(\mathcal{I})$$
$$\Gamma_{\mathcal{I}}\colon (cs(X),\sigma) \to (\mathcal{F},\tau_P); \qquad x \mapsto \Gamma_x(\mathcal{I})$$

are continuous.

Theorem 2.4 and Theorem B yield:

Corollary. *Let (X,ϱ) be a locally compact metric space. Then there exists a metric ϱ' on X equivalent to ϱ such that the mappings*

$$\Gamma_{\mathcal{I}}\colon (cs(X),\sigma') \to (\mathcal{F},\tau_F); \qquad x \mapsto \Gamma_x(\mathcal{I})$$
$$\Gamma_{\mathcal{I}}\colon (cs(X),\sigma') \to (\mathcal{F},\tau_P); \qquad x \mapsto \Gamma_x(\mathcal{I})$$

where σ' is the sup-metric and τ_P the proximal topology corresponding to ϱ', are continuous.

Let (X,ϱ) be a metric space, $A \subseteq cs(X)$ and $B \subseteq \mathcal{F}$. We conclude this section with studying the continuity of the mapping $\Gamma_{\mathcal{I}}\colon A \to B$ where A is endowed by the Fréchet metric instead of the sup-metric and B is endowed by the Hausdorff metric or one of the above considered hypertopologies.

Recall that if (X,ϱ) is a metric space, $M \subseteq s(X)$, then the *Fréchet metric* φ on M is defined by

$$\varphi(x,y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, \varrho(x_n,y_n)\}.$$

We start with the case of the Hausdorff metric. Let (X,ϱ) be a metric space $cbs(X)$ be the set of all bounded sequences in (X,ϱ) with $\Gamma_x(\mathcal{I}) \neq \emptyset$, $b\mathcal{F}$ the set of all non-empty closed bounded subsets in (X,ϱ) and ϱ_H be the Hausdorff metric on $b\mathcal{F}$. Then we have

Theorem 2.5. *If a metric space (X,ϱ) contains at least two different points, M is an arbitrary subset of $cbs(X)$ containing all stationary sequences (i.e. constant up to finite number of terms), then the mapping*

$$\Gamma_{\mathcal{I}}\colon (M,\varphi) \to (b\mathcal{F},\varrho_H); \qquad x \mapsto \Gamma_x(\mathcal{I})$$

is discontinuous at any point $x \in M$.

Proof. Let $x \in M$, $t \in \Gamma_x(\mathcal{I})$ and $v \in X$ with $v \neq t$. Put $\varepsilon_0 = \varrho(v,t) > 0$ and choose an arbitrary $\delta > 0$. Then there exists $k \in \mathbb{N}^+$ such that $\sum\limits_{n\geq k} 2^{-n} < \delta$.

Define a sequence $y = (y_n)_1^\infty$ as follows:

$$y_j = \begin{cases} x_j & \text{if } j \leq k \\ v & \text{if } j > k \end{cases}$$

Then we have

$$\varphi(x,y) \leq \sum_{n>k} 2^{-n} < \delta$$

and, consequently, $y \in B_\delta(x)$. Obviously $\Gamma_y(\mathcal{I}) = \{v\}$ and

$$\varrho_H(\Gamma_x(\mathcal{I}),\Gamma_y(\mathcal{I})) = \varrho_H(\Gamma_x(\mathcal{I}),\{v\}) \geq \sup_{u\in\Gamma_x(\mathcal{I})} \varrho(u,\{v\}) \geq \varrho(t,v) = \varepsilon_0.$$

This yields that $\Gamma_{\mathcal{I}}$ is discontinuous at x. □

The method used in the proof of Theorem 2.5, namely the fact that for any $x \in M$, $v \in X$ and $\delta > 0$ there exists $y \in M$ such that $\varphi(x,y) < \delta$ and $\Gamma_y(\mathcal{I}) = \{v\}$, can be readily used to show that the following assertion holds:

Theorem 2.6. *Let (X, ϱ) be a metric space containing at least two different points, M be a subset of $cs(X)$ containing all stationary sequences and τ be any of topologies τ_V^-, τ_V, τ_F, τ_P on $\mathcal{F}$. Then the mapping*

$$\Gamma_{\mathcal{I}} \colon (M, \varphi) \to (\mathcal{F}, \tau); \qquad x \mapsto \Gamma_x(\mathcal{I})$$

is discontinuous at any $x \in M$.

In the case of τ_V^+ the mapping $\Gamma_{\mathcal{I}} \colon (M, \varphi) \to (\mathcal{F}, \tau_V^+)$; $x \mapsto \Gamma_x(\mathcal{I})$ is discontinuous at any $x \in M$ such that $\Gamma_x(\mathcal{I}) \neq X$.

Remark. Obviously, the discontinuity with respect to τ_V^- implies the discontinuity with respect to τ_V, τ_F, τ_P.

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