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HEREDITY, HEREDITARY COREFLECTIVE HULLS AND OTHER PROPERTIES OF COREFLECTIVE SUBCATEGORIES OF CATEGORIES OF TOPOLOGICAL SPACES

Dedičnosť, dedičné koreflektívne obaly a ďalšie vlastnosti koreflektívnych podkategorií kategorií topologických priestorov

Dizertačná práca

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Abstract

This thesis deals mainly with hereditary coreflective subcategories of the category **Top** of topological spaces. After preparing the basic tools used in the rest of thesis we start by a question which coreflective subcategories of **Top** have the property S**A** = **Top** (i.e., every topological space can be embedded in a space from **A**). We characterize such classes by finding generators of the smallest coreflective subcategory of **Top** with this property. In particular we show that this holds for the category **PsRad** of the pseudoradial spaces and by modification of the proof of this result we get that the pseudoradial spaces have similar property also in the category $\mathbf{Top}_1$ of T_1-spaces. This answers a question posed in [AIT].

As the next topic we study hereditary coreflective hull SCH(A) of a prime space A. We succeed to construct a space which generates SCH(A) as a coreflective subcategory of **Top**.

We also provide some results concerning the coreflective subcategories of **Top** with the property HCK(**C**) = **FG** (here HCK(**C**) denotes the hereditary coreflective kernel of **C** and **FG** is the subcategory of finitely generated spaces). We show that the lattice of such subcategories is closed under arbitrary intersection and finite joins, but not under countable joins.

In the last two chapters we study a generalization of the original problem. We take an epireflective subcategory **A** of **Top** as the base category and we study subcategories of **A** which are hereditary, additive and divisible in **A** (i.e., closed under the formation of subspaces, topological sums and quotients with codomain in **A**; such classes are called more briefly HAD-classes). We show that (under some reasonable conditions on **A** and **B**) an additive and divisible class **B** in **A** is hereditary if and only if it is closed under the formation of prime factors.

Keywords: category of topological spaces, coreflective subcategory, epireflective subcategory, hereditary coreflective subcategory

Introduction

A brief history of (co)reflections

Categorial topology is one of many branches of general topology which appeared in the course of the last century. It studies the topological spaces (and related constructs) from the viewpoint of category theory. Both these mathematical disciplines, general topology as well as category theory, have profitted from their interconnection. Category theory brought to general topology new insights, new questions and directions of research and also more appropriate language for some situations. On the other hand, the category-theorist were interested in the features which make the category **Top** of topological spaces interesting and useful. From this arose the notion of the topological category and some related concepts, as well as the search for "convenient categories for topology" – various generalizations of topological spaces which have all "good" features of **Top** and are even better in some aspects.

One of the typical features of the category theory is the effort to capture similarities between various mathematical objects and constructions and to find a unifying approach to them. One of very successful notions has been the notion of universal property (later called universal morphism or universal arrow). This concept applied to the category **Top** leads to the concept of reflective subcategory (and the dual concept of coreflective subcategory).

In the context of the category **Top**, epireflections proved to be more useful than reflections in general, since they have a nice characterization and the latter can behave very oddly. Many epireflective subcategories of **Top** (and of other categories of topological spaces) were studied by topologists, in this connection we can mention the Stone-Čech compactification, which seems to be one of the most important constructions in general topology. Many important classes of spaces, like k-spaces or sequential spaces, are coreflective in **Top**. The works of S. P. Franklin [F1] and [F2] on sequential spaces contain many results which were later generalized for other coreflective subcategories of **Top**. Another example of coreflection, which had been studied earlier than the notion of coreflection was introduced, is the locally connected refinement [Gl], [Y]. A detailed account on how the ideas of reflection or coreflection appeared in general topology can be found in [HS4].

The striking similarities between various constructions in topology (and also

in other areas) lead to the concepts of reflection and coreflection, which were studied by H. Herrlich [He3], G. Strecker [HS1], [HS2], [HS3], M. Hušek [Hu2], [HH2], [BH], [Hu3], V. Kannan [K1], [K2], [K4] and many others. The monograph [He2] was the first book systematically studying the epireflective and coreflective subcategories of **Top** and was a great impulse for this area of research.

The recognition of importance of these notions for study of topological spaces deeply influenced this area of research. Many questions could be very naturally formulated after the categorical descriptions of various topological properties were at hand. Among other questions the following were studied:

- Cartesian closedness of convenient subcategories of **Top** was studied, since the category **Top** lacks this useful property. For an overview on cartesian closedness in topological categories see e.g. [HH1].

- Another often considered problem was the simplicity of epireflective and coreflective subcategories of **Top** (=the question whether there exists a single space which generates the given subcategory). This often leads to very nice characterization of spaces from a given subcategory, as the spaces obtained from the generator applying some simple operations. Tychonoff's theorem can be seen as a prototype of results of this type. We will obtain a generator for some hereditary coreflective subcategories in Chapter 3. Since the generator of a coreflective subcategory is intended to give a more elegant description of this subcategory, it is worthwhile to try to find a generator which is as nice as possible. We show, that the generator we construct has several useful properties.

- Various generalizations of reflectivity and coreflectivity, including the subcategories which are simultaneously reflective and coreflective in **Top**. It was shown in [K2] that no such proper subcategories of **Top** exist. This leads to the natural question whether there are some subcategories of **Top** which are closed only under some of the basic four topological operations (sum, quotient, subspace and product). Questions of this type were studied e.g. in [DW], [Hu3]. The most of this thesis is devoted to this type of problem – we will study the hereditary coreflective subcategories of **Top**.

- The large lattice of all epireflective resp. coreflective subcategories of **Top** was studied (e.g. [Ma], [He3], [Hu2]). It is a convenient way to describe the relations between epireflective or coreflective subcategories of **Top**. Some interesting subcategories can be characterized as minimal or maximal elements of this lattice with some given property. We include such description of some coreflective subcategories of **Top** and **ZD** (the category of all zero-dimensional spaces) in the end of Chapter 5.

- All above mentioned problems are interesting not only in the category **Top** but also in various subconstructs and superconstructs of **Top**. A well-known example is the subcategory of all compact T_2-spaces which

is epireflective in **Haus**. Sober spaces, which form an epireflective subcategory of $\mathbf{Top}_0$, give another interesting example. Although mostly we will deal only with the subcategories of **Top**, in Chapters 5 and 6 we will show that some of our results can be extended to many epireflective subcategories of **Top**.

Goals and main results of this dissertation

We have already mentioned the study of coreflective subcategories of **Top** which have some additional properties as one type of questions studied in the connection with the coreflective subcategories of **Top**.

In [HH2, Problem 7] H. Herrlich and M. Hušek suggested to study the hereditary coreflective subcategories, in the other words, the classes of topological spaces closed under the formation of topological sums, quotients and subspaces. Among some interesting directions of research, they suggested also studying hereditary coreflective hulls and kernels of coreflective subcategories of **Top** and the extremal cases - the coreflective subcategories having the hereditary coreflective kernel as small as possible and having the whole **Top** as the hereditary coreflective hull.

The hereditary coreflective subcategories of **Top** (resp. of epireflective subcategories of **Top** in Chapters 5 and 6) and the above mentioned questions were the main topic of our research. In this section we would like to give an overview of known results and also of our contributions to this area.

State of the art

The results concerning the hereditary coreflective subcategories as such can be found mostly in works of J. Činčura and V. Kannan. So we start with a summary of these results. But there are also many interesting results for particular hereditary coreflective subcategories, which can be often adapted to the general situation. We will include also several such results, namely the results connected with the subspaces of sequential and pseudoradial spaces.

V. Kannan deals in his monograph [K4] and his thesis [K1] mostly with the coreflective subcategories of **Top** in general, but we can find here also several results on hereditary coreflective subcategories. For instance, he included here the basic observation that from a coreflective subcategory $\mathbf{C}$ we obtain its hereditary coreflective hull simply by taking the class $\mathrm{S}\mathbf{C}$ of all subspaces of spaces from $\mathbf{C}$. He also provided a few conditions for a class $\mathbf{B}$ which imply that its coreflective hull $\mathrm{CH}(\mathbf{B})$ is hereditary.

In the paper [K2] V. Kannan proved that there is no proper subcategory of **Top** which is simultaneously reflective and coreflective. Consequently, if a hereditary coreflective subcategory of **Top** is productive, then it is equal to **Top**.

The hereditary coreflective subcategories were studied more thoroughly in [Č3]. In this paper J. Činčura characterized them as precisely those coreflective subcategories of **Top** that are closed under the formation of prime factors. (The

only exception is the subcategory **Ind** of indiscrete spaces, which is hereditary without being closed under the formation of prime factors.) This result shows that in the study of hereditary coreflective subcategories the prime factors and prime spaces will play an important rôle.

He also provided a characterization of coreflective subcategories of **Top** having the subcategory **FG** of all finitely generated spaces as their hereditary coreflective kernel and gives several examples of such subcategories.

The characterization of hereditary coreflective subcategories via closedness under prime factors was generalized in [Č4]. Here the author studied hereditary coreflective subcategories in epireflective subcategories of **Top** rather then in **Top**.

The same author showed in [Č2] that, apart from the trivial cases $\mathbf{C} \subseteq \mathbf{FG}$, no hereditary coreflective subcategory **C** of **Top** is cartesian closed.

Another interesting fact which can be observed in the paper [Č3] is that often the methods which were used for the subcategory **Seq** of sequential spaces can be used also when dealing with a coreflective hull of some class of prime spaces. Namely, J. Činčura generalized here some results from [FR]. (Some other generalizations of the results and methods used for the subsequential spaces can be found throughout this thesis.)

S. P. Franklin and M. Rajagopalan studied in [FR] the subsequential spaces, i.e., the subspaces of sequential spaces. They showed that this class of spaces is a coreflective subcategory of **Top** and that there exists a countable generator of this subcategory. The notion of prime factor was introduced in this paper. The description of subsequential spaces using prime factors served as the motivation for the more general results of J. Činčura mentioned above. Other topics studied in this paper are sequential and subsequential order of spaces. (Similar ordinal invariants were studied before in [AF] and [K4].)

Another hereditary coreflective subcategory of **Top** which attracted the attention of topologists was the class of all subspaces of pseudoradial spaces (possibly with an additional separation axiom). J. Zhou showed in [Z] under an additional set-theoretical assumption that every countably tight space can be embedded into a pseudoradial space. As an auxiliary result he has shown that every prime T_1-space (i.e., a T_1-space with unique non-isolated point) can be embedded into a Hausdorff pseudoradial space. In the paper [DZ] it was shown that the question whether every prime space of the form $\mathbb{N} \cup \{p\}$, where $p \in \beta\mathbb{N} \setminus \mathbb{N}$, can be embedded into a regular pseudoradial space is ZFC-independent.

Our main results

The results mentioned above served as the starting point of our research of hereditary coreflective subcategories. Here we describe the results on this topic obtained in this thesis.

Using the results of [Č3] and a generalization of some methods from [FR] and [AF] we were able to obtain in [Sl3] some results on the hereditary coreflective hull SCH(A) of a prime space A. We have constructed a prime space $(A_\omega)_a$ which generates SCH(A) as a coreflective subcategory of **Top** and moreover, it

has the same cardinality as the original space A. This yields, in particular, a new countable generator of the subcategory **SSeq** of all subspaces of sequential spaces. (Another such generator was constructed in [FR] using maximal almost disjoint families on a countable set. The advantage of our generator is that it is constructive and simpler.) These results can be found in Chapter 3 of this thesis.

We have also studied the questions regarding the hereditary coreflective hulls and kernels. In [Sl2] (resp. Chapter 2) we characterized the coreflective subcategories of **Top** such that $\mathrm{S}\mathbf{A} = \mathbf{Top}$, i.e., the hereditary coreflective hull of $\mathbf{A}$ is **Top**. Among other results, we have shown that the subcategory **PsRad** of the pseudoradial spaces has this property. This characterization helped us to show in [Sl3] that the lattice of all coreflective subcategories fulfilling $\mathrm{S}\mathbf{A} = \mathbf{Top}$ and $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$ (where $\mathrm{HCK}(\mathbf{A})$ stands for the hereditary coreflective kernel of $\mathbf{A}$ and **FG** is the subcategory of all finitely generated spaces) is closed under arbitrary intersections and finite joins (but not under countable joins).

As we have already noticed, the category **Top** is not the only subcategory which we can use as the "base category" when studying some topological questions. Frequently it is replaced by some epireflective subcategory of **Top**. In [Č4] J. Činčura generalized the mentioned result about hereditary coreflective subcategories and prime factors also to the situation when dealing with the hereditary coreflective subcategories of $\mathbf{A}$, where $\mathbf{A}$ is an epireflective subcategory of **Top** which does not contain the 2-point indiscrete space.

In Chapter 5 we have studied another similar generalization - the subclasses of $\mathbf{A}$ which are hereditary, additive and divisible in $\mathbf{A}$. (By this we mean the closedness under subspaces, sums and topological quotients with the codomain in $\mathbf{A}$. Each coreflective subcategory of $\mathbf{A}$ is additive and divisible in $\mathbf{A}$, therefore this is a very natural generalization. We call additive and divisible classes briefly AD-classes. Hereditary AD-class is called HAD-class.) Although we were not able to show this in full generality, we have shown that under some mild conditions on $\mathbf{A}$ or on an additive and divisible subcategory $\mathbf{B}$ of $\mathbf{A}$, that the subcategory $\mathbf{B}$ is hereditary if and only if it is closed under prime factors.

Finally, in Chapter 6 we have shown how we can avoid the condition that $\mathbf{A}$ does not contain the 2-point indiscrete space. (We will work only with the epireflective subcategories fulfilling this condition in Chapter 5, since precisely these epireflective subcategories are closed under the formation of prime factors.) Thus at last we can deal with arbitrary epireflective subcategory of **Top**. The results of Chapters 5 and 6 are contained in the paper [Sl4].

Methods used in this thesis

Here we want to describe our main results more closely. This gives us also an opportunity to mention the methods which we used to obtain our results and various results which where only auxiliary or were obtained as byproducts, even though they were not our main interest.

The first chapter of this thesis is devoted to overview of the basic facts on

epireflective and coreflective subcategories of **Top**. Of course, this thesis is not intended to be self-contained textbook-like work, nevertheless, it is useful to gather somewhere the basic results just to refresh the memory of the reader.

In Chapter 2 we start to study some problems connected with the hereditary coreflective subcategories of **Top**. As we have already mentioned, one of the most important tools is the observation of J. Činčura that for a coreflective subcategory of **Top** heredity is equivalent to closedness under the formation of prime factors.

We first answer the question posed in [AIT] by showing that every (T_1)-space can be embedded into a pseudoradial (T_1)-space. The proof is not complicated at all – it is in fact only simple transfinite induction. What is perhaps interesting is the fact, that looking at this problem from the categorical viewpoint can help to understand better the rôle of spaces S^α in the proof of this results – we use the fact that these spaces are absolute retracts in **Top**. This leads us to a simple description of coreflective subcategories of **Top** such that $\mathrm{S}\mathbf{A} = \mathbf{Top}$, i.e., every topological space can be embedded into some space from **A**.

We have already mentioned, that very often the same methods which were used for some particular coreflective subcategory of **Top** can be later reused for a more general class of (sometimes for all) coreflective subcategories of **Top**. Chapter 3 is a typical example of this phenomena. S. P. Franklin and M. Rajagopalan studied in [FR] the subcategory **SSeq** of all subspaces of sequential spaces. We wanted to study the hereditary coreflective hull of a class (or a set) of prime spaces. By introducing a closure operator which can serve in the results about such subcategories as a substitute for the sequential closure used in [FR] we were able to find appropriate generalization for most of the results of this paper. In particular, if we deal with the hereditary coreflective hull of a single prime space A, there exists a generator of this subcategory which has the same cardinality as the space A.

In Chapter 4 dealing with the hereditary coreflective kernels we have used some results obtained in the foregoing chapter to show some basic facts concerning the conglomerate of all coreflective subcategories of **Top** having **FG** as their hereditary coreflective kernel.

In Chapter 5 we tried to continue the research started by J. Činčura in his paper [Č4] concerning the hereditary coreflective subcategories of an epireflective subcategory **A** of **Top**. Similarly as in that paper we restrict ourselves to epireflective subcategories fulfilling the condition $I_2 \notin \mathbf{A}$, where I_2 denotes the 2-point indiscrete spaces. The reason for this restriction is the fact, that these are precisely the epireflective subcategories which are closed under prime factors. But instead of the coreflective subcategories of **Top**, which were studied in [Č4], we study the AD-classes in **A**, which are slightly more general.

We use several constructions of new topological spaces like $X \triangle_b Y$ and $X \bigtriangledown_b Y$ (introduced in [Č4]) and the wedge sum $\bigvee_I X$. We are interested in closedness under prime factors and these constructions are useful because from such spaces we can obtain prime factors. Therefore we study also the closedness of the epireflective subcategory **A** under these subcategories. Using some results on strongly rigid spaces we were able to construct examples showing that not

every epireflective subcategory of **Top** is closed under $\triangle$ and $\bigvee$.

Some of results we have obtained lead to new interesting facts on the lattice of all coreflective subcategories of **Top** or of the category $\mathbf{ZD_0}$ of all zero-dimensional T_2-spaces.

Chapter 6 is devoted to the extension of results from Chapter 5 to the epireflective subcategories with $I_2 \in \mathbf{A}$. The correspondence between bireflective subcategories of **Top** and all other epireflective subcategories of **Top** appears to be a useful tool in this case. We describe this correspondence in detail. In order to obtain some properties of this correspondence we introduce also some useful properties of the T_0-reflection.

Suggestions for further research

There are several problems, which are left open and could be worth studying in the connection with the topic of this thesis.

First of all, we were able neither to show nor to disprove that, if a hereditary, additive and divisible class **B** in an epireflective subcategory **A** of **Top** (with $I_2 \notin \mathbf{A}$) contains a non-discrete space, then it must contain a prime space. If this is true then the characterization of HAD-classes among all AD-classes using prime factors holds (with the exception of classes consisting of sums of indiscrete spaces only).

Also we have no characterization of the hereditary coreflective kernel of a given subcategory and no satisfactory characterization of the coreflective subcategories with $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$.

One of promising areas is also the study of cartesian closedness, e.g. in coreflective subcategories of **Top** and in various superconstructs of **Top**. Papers [Č1] and [Č2] suggest that these questions can be studied also in connection with the (hereditary) coreflective subcategories of **Top**.

Chapter 1

Preliminaries

The aim of this chapter is to give an overview of the basic results needed throughout this thesis. With the exception of subsection 1.5.2, which contains some original results published in [Sl3], all these results are well-known.

1.1 Category of topological spaces

Categorical topology employs many notions from category theory in the study of topological spaces (and often more general structures as well). The category theory serves not only as a language and a unifying principle, but many deep categorical results are applicable in the realm of general topology. For example, reflective and coreflective subcategories, which are central notions in this thesis, can be viewed as a special cases of adjoint situation.

In this work we are interested in the category **Top** of all topological spaces with continuous mappings as morphisms. This category is very useful as various categorical constructions (limits, colimits, and some kinds of special morphisms and special objects) correspond to frequently used topological constructions.

We assume that the reader is familiar with the basic facts about this category, so it is not necessary to give lengthy introduction describing topological constructions and categorical notions corresponding to them. But we would like to point out some notions and facts, which are not quite common or for which our terminology is not standard.

In general topology two spaces are considered the same if they are homeomorphic. Therefore it is natural that we assume all subcategories of **Top** to be isomorphism-closed. Moreover, all subcategories we study will be full. This assumptions means, that they are completely determined by the class of objects, therefore a class of topological spaces and a (full and isomorphic-closed) subcategory of **Top** are essentially the same notions. It enables us to use categorical language when dealing with various classes of topological spaces. To avoid some trivial cases we will require all subcategories of **Top** to contain at least one non-empty space.

Convention: All subcategories of **Top** are supposed to be full, isomorphism-closed and to contain a non-empty space.

We will use the notation $\langle f_i \rangle$ in accordance with [AHS] to denote the unique map $f\colon X \to \prod Y_i$ determined by the maps $f_i\colon X \to Y_i$. In the dual situation we use the notation $[f_i]\colon \coprod Y_i \to X$. Sometimes we use, for better readability, $\sqcup$ instead of (usual) $\coprod$ to denote the topological sum.

1.1.1 Initial topology

We recall here some basic facts about initial topology and initial sources.

Definition 1.1.1. We say that a topology $\mathcal{T}$ on X is *initial* with respect to the family of maps $f_i\colon X \to X_i$, $i \in I$ (where I is a class), if $\mathcal{T}$ is the coarsest topology on X which makes all f_i's continuous.

If a topology is initial with respect to a family consisting of a single map $f\colon X \to Y$, we say that f is an *initial map.*

The initial topology is characterized by the universal property that a map $g\colon Y \to X$ is continuous if and only if every $f_i \circ g\colon Y \to X_i$ is continuous.

The family of sets $\mathcal{S} = \{f_i^{-1}(U) : U \text{ is open in } X_i\}$ form a subbase for the initial topology, their finite intersections form a base. In the case of initial map $f\colon X \to Y$ the set $\mathcal{T} = \{f^{-1}(U) : U \text{ is open in } Y\}$ is the topology of X.

Well-known examples of the initial topology are the product topology (which is initial w.r.t. the projections) and the subspace topology (initial w.r.t the embedding).

The initial topology is sometimes called topology generated by a family of mappings [E], weak topology [Wi] or projective topology.

From the viewpoint of category theory, a topology is initial with respect to some family of mappings if and only if this family of mappings is an initial source. In the topological categories, initial topologies correspond to initial structures with respect to some family of mappings (see [P]).

The dual notions are final sources, final structures and the *final topology.* For a single map, this is the usual notion of the quotient topology.

We include here also a few facts about monosources and initial sources, since they will be useful to us later.

Let A and B_i (for $i \in I$) be topological spaces and let $f_i\colon A \to B_i$, $i \in I$, be some family of maps. This family is said to *separate points* if for every two different points $x \neq y$ of the space A there exists an $i \in I$ such that $f_i(x) \neq f_i(y)$.

For the future reference we formulate the following easy observation into a lemma.

Lemma 1.1.2. *A family $f_i\colon A \to B_i$, $i \in I$, of continuous maps between topological spaces is a monosource in* **Top** *if and only if this family separates points.*

We can also mention here that $f_i\colon A \to B_i$, $i \in I$, is a monosource if and only if $\langle f_i \rangle\colon A \to \prod_{i \in I} B_i$ is a monomorphism ([AHS, Proposition 10.26]).

In the above situation we say that a family of maps *separates points and closed set* if for any $a \in A$ and a closed subset $V \subseteq A$ such that $a \notin V$ there exists some $i \in I$ such that $f_i(a) \notin \overline{f_i[V]}$.

If A is T_0, then every family which separates points and closed sets separates points as well.

Proposition 1.1.3. *If a family $f_i \colon A \to B_i$, $i \in I$, of continuous maps separates points and closed sets then it is an initial source.*

Proof. Let us denote the topology of A by $\mathcal{T}$ and the initial topology w.r.t. the family f_i by $\mathcal{T}'$. Since all f_i's are continuous with respect to $\mathcal{T}$, we have $\mathcal{T}' \prec \mathcal{T}$.

$\boxed{\mathcal{T} \prec \mathcal{T}'}$ Let V be subset of A closed in $\mathcal{T}$. For each $a \notin V$ there exists $i \in I$ such that $f_i(a) \notin \overline{f_i[V]}$. Let us denote $V_a := f_i^{-1}(\overline{f_i[V]})$. The set V_a fulfills the conditions $a \notin V_a$ and $V_a \supseteq V$. This set is moreover closed in $\mathcal{T}'$.

Thus we get $V = \bigcap_{a \in X \setminus V} V_a$ and V is closed in $\mathcal{T}'$. □

Lemma 1.1.4. *If a map $f \colon X \to Y$ is initial, then the family, consisting of this map only, separates points and closed set.*

Proof. Recall that if f is initial then the topology of X consists precisely of the sets $\{f^{-1}(U); U \text{ is open in } Y\}$. If $a \notin V$ for some set V closed in the initial topology, then there exists a neighborhood $f^{-1}(U)$ containing a such that $f^{-1}(U) \cap V = \emptyset$. Then $f(a) \notin \overline{f[V]}$, since $f(a) \in U$ and $U \cap f[V] = \emptyset$. □

Proposition 1.1.3 cannot be reversed. To show this we can use the notion of strongly rigid space.

Definition 1.1.5. A topological space X is called *strongly rigid* if any continuous map $f \colon X \to X$ is either constant or $f = id_X$.

For more information on strongly rigid spaces see e.g. [KR]. It should be noted that such spaces are called rigid by some authors. Apart from [KR], examples of Hausdorff strongly rigid spaces can be found also in [C] or [dG].

Example 1.1.6. If X is a strongly rigid space, then the non-constant continuous maps from X^2 to X are only the projections (see [He2, Satz 20.1.3], or [He4, Theorem 2.4]). Thus the family $C(X^2, X)$ does not separate points and closed sets (for any non-isolated point $a \in X$ and a neighborhood U of a such that $U \neq X$, the point (a, a) cannot be separated by the projections from the closed sets $X \times X \setminus U \times U$). But this family of maps is an initial monosource (it is moreover a limit).

Lemma 1.1.7. *A source $(f_i \colon X \to Y_i)_{i \in I}$ is initial if and only if the map $\langle f_i \rangle \colon X \to \prod_{i \in I} Y_i$ is initial.*

Proof. Observe that for any map $g\colon Z \to X$ the condition that $\langle f_i \rangle \circ g$ is continuous is equivalent to continuity of all maps $p_i \circ \langle f_i \rangle \circ g = f_i \circ g$.

$$\begin{tikzcd} Z \arrow[r, "g"] & X \arrow[r, "\langle f_i \rangle"] \arrow[dr, "f_i"'] & \prod_{i \in I} Y_i \arrow[d, "p_i"] \\ & & Y_i \end{tikzcd}$$

□

Corollary 1.1.8. *A family $f_i\colon A \to B_i$, $i \in I$, of continuous maps is an initial source in* **Top** *if and only if the map $\langle f_i \rangle$ separates points and closed sets.*

Corollary 1.1.9. *If X is a T_0-space then any initial source with domain X is a monosource.*

Proof. If $(f_i)_{i \in I}$ is an initial source, then $\langle f_i \rangle$ separates points and closed sets. Since X is T_0, this implies that $\langle f_i \rangle$ is a monomorphism and therefore $(f_i)_{i \in I}$ is a monosource. □

1.2 Prime spaces

We will use prime spaces frequently in this work. Therefore we gathered here the basic definitions.

The notions of prime space and prime factor were adopted from [FR]. Both of them are well-known and often used, although sometimes not under these names but simply described as the topological spaces having unique accumulation point. For example Lemma 1.2.5 was used in [Č3], [CGT, 6.5], [DW, Lemma 2.4], [FR, Section 5], to list only a few examples. The prime spaces per se were studied e.g in [BM].

Definition 1.2.1. A topological space is called a *prime space,* if it has precisely one accumulation point.

There is a one-to-one correspondence between the prime spaces on the set P with the accumulation point a and the filters on $P \setminus \{a\}$ (see [Č3]). The filter corresponding to the prime space consists of all sets of the form $U \setminus \{a\}$ where U is an open neighborhood of a.

A non-discrete T_2-space X is a prime space if and only if each subset of X is open or closed (such spaces are called door spaces).

Definition 1.2.2. We say that a subspace B of a prime space A is a *prime subspace* if it is a prime space as well. This is equivalent to saying that B is not discrete, or, B contains the accumulation point a of A and $a \in \overline{B \setminus \{a\}}$.

We will frequently use the following lemma. The easy proof is omitted.

Lemma 1.2.3. *If B is a prime subspace of a prime space A with the accumulation point a, then the mapping $q\colon A \to B$ given by $q(x) = x$ for $x \in B$ and $q(x) = a$ for $x \notin B$ is a retraction. Consequently, q is hereditary quotient.*

Recall that a quotient map $q\colon X \to Y$ is called *hereditary quotient* if every codomain-restriction $q|_{q^{-1}(A)}$, $A \subseteq Y$, is a quotient map.

This lemma will be useful, because we will often deal with classes of spaces which are closed under quotient spaces and the above result says that such a class of spaces contains, with each prime space, also all its prime subspaces. (Even more, it contains all subspaces of P, since the class **Disc** of all discrete spaces is the smallest coreflective subcategory of **Top**.)

Definition 1.2.4. Let X be a topological space and $a \in X$. The *prime factor* X_a of the space X at the point a is the topological space obtained by making all points of X other than a isolated and retaining the original neighborhoods of a.

The topology of the prime factor X_a can be described equivalently as follows: $U \subseteq X$ is open in X_a if and only if $a \notin X$ or there exists a set V such that $a \in V \subseteq U$ and V is open in X.

The prime factor X_a is either a discrete space or a prime space with the accumulation point a. All sets containing a are closed, all sets not containing this point are open in X_a.

Since the topology of X_a depends only on the neighborhoods of the single point a, prime factors can be understood as a tool to study local properties of the topological space X at the point a.

One could argue that the notation X_a could be confused with the usage of a as an index. (That is why V^a is used in the proof of Lemma 1.2.5.) But we tried to do our best to make this clear from the context.

Lemma 1.2.5. *Every topological space can be obtained as a hereditary quotient of a topological sum of its prime factors.*

Proof. We define the map $q\colon \coprod_{a\in X} X_a \to X$ simply by mapping each x in X_a to the same point in X.

If $q^{-1}(U)$ is open then U is open in each X_a. This means that for each $a \in U$ we have an open set V^a in X such that $a \in V^a \subseteq U$. Hence $U = \bigcup_{a\in U} V^a$ is a union of open sets and is itself open.

By the restriction to a subspace Y of X we obtain prime factors Y_a in place of X_a or (if $a \notin Y$) discrete spaces, so the map remains quotient for the same reasons as above. □

Some other useful facts on prime spaces will be given later, after we introduce the notion of coreflective subcategory.

1.3 Reflective and coreflective subcategories of Top

In this section we review the definition and the basic properties of the central notion of this thesis, reflective and coreflective subcategories.

1.3.1 Reflective subcategories of Top

Anybody who works in category theory would agree on the importance of the concept of the reflection, and of the dual concept, coreflection. For topologists, epireflective subcategories are the most interesting case, since they are nicely characterized in terms of products and subspaces. In the category **Top** (and its close relative, the category **Haus** of Hausdorff spaces and continuous maps), many important and widely studied classes of spaces are epireflective (T_i-spaces for $i = 0, 1, 2$ in **Top** or compact T_2-spaces in **Haus**) or coreflective (k-spaces, sequential spaces, countably tight spaces).

Here we introduce the notion of reflective subcategories in full generality and then we state some characterizations of epireflective and coreflective subcategories of **Top**.

Definition 1.3.1. A subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is said to be *reflective in* $\mathbf{B}$ if for each $\mathbf{B}$-object B there exists an $\mathbf{A}$-object A_B and a morphism $r_B \colon B \to A_B$ such that for each $\mathbf{B}$-morphism $f \colon B \to A$ there exists a unique $\mathbf{A}$-morphism $\overline{f} \colon A_B \to A$ with $\overline{f} \circ r_B = f$.

$$\begin{tikzcd} B \arrow[r, "r_B"] \arrow[dr, "f"'] & A_B \arrow[d, dashed, "\overline{f}"] \\ & A \end{tikzcd}$$

The pair (A_B, r_B) is called the $\mathbf{A}$-*reflection* of B. The morphism r_B we call $\mathbf{A}$-*reflection arrow.*

For the sake of brevity we will often use the name $\mathbf{A}$-reflection for the object A_B rather than for the pair (A_B, r_B).

$\mathbf{A}$-reflections are unique up to an isomorphism. That is why we speak about "the" $\mathbf{A}$-reflection of X rather than "an" $\mathbf{A}$-reflection.

The name reflection is self-explaining - it is the object in $\mathbf{A}$ which reflects the properties of $B \in \mathbf{B}$ as well as possible. Typical examples are the Stone-Čech compactification in the category **Haus** or free objects in various categories. After the announced characterization of the epireflective and coreflective subcategories in **Top** we will present many examples in the following sections.

A subcategory $\mathbf{A}$ is reflective in $\mathbf{B}$ if and only if the embedding functor $E \colon \mathbf{A} \hookrightarrow \mathbf{B}$ is adjoint. The coadjoint functor $R \colon \mathbf{B} \to \mathbf{A}$ is called *reflector.* The map r_B is the unit of this adjunction.

The reflector assigns to B the $\mathbf{A}$-object A_B and Rf for a $\mathbf{B}$-morphism f is determined by the commuting diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r_X"'] & Y \arrow[d, "r_Y"] \\ RX \arrow[r, "Rf"'] & RY \end{tikzcd}$$

If all $\mathbf{A}$-reflection arrows are (extremal) epimorphisms, then the subcategory $\mathbf{A}$ is said to be (*extremal*) *epireflective,* similarly we say that it is *bireflective,* if all reflection arrows are bimorphisms. Since in **Top** the extremal epimorphisms are precisely the quotient maps, we will use the (more usual) name *quotient reflective* subcategory in this case.

All these notions are special case of the common generalization – *E-reflective subcategory,* where E is a class of morphisms. The *E-reflective hull* of a class $\mathbf{A}$ of spaces is defined as the smallest E-reflective subcategory containing $\mathbf{A}$.

The *epireflective hull* of $\mathbf{A}$ is denoted by $\mathrm{EH}(\mathbf{A})$, the *bireflective hull* by $\mathrm{BH}(\mathbf{A})$ and the *quotient reflective hull* by $\mathrm{QH}(\mathbf{A})$.

The following characterizations of various kinds of E-reflective subcategories and E-reflective hulls in **Top** can be deduced from the more general result [AHS, Theorem 16.8].

Theorem 1.3.2. *Let $\mathbf{A}$ be a subcategory of* **Top**. *Then the following conditions are equivalent.*

(i) $\mathbf{A}$ *is epireflective* (*bireflective/quotient reflective*) *in* **Top**.

(ii) $\mathbf{A}$ *is closed under the formation of initial monosources* (*initial sources/monosources*).

(iii) $\mathbf{A}$ *is closed under the formation of products and subspaces* (*products and initial topologies/products, subspaces and spaces with finer topologies*).

Corollary 1.3.3. *An epireflective subcategory of* **Top** *is quotient reflective if and only if it is closed under the formation of spaces with finer topologies.*

The above theorem allows us to obtain a characterization of E-reflective hulls in all three cases.

By $C(Y, \mathbf{A})$ we denote the class of all continuous maps with domain Y and codomain in $\mathbf{A}$. (The symbol $C(Y, X)$ will be used in a similar meaning – for the case $\mathbf{A} = \{X\}$.)

Proposition 1.3.4. *Let $\mathbf{A}$ be a class of topological spaces and Y be a topological space. The following conditions are equivalent:*

(i) $Y \in \mathrm{EH}(\mathbf{A})$,

(ii) Y *is a subspace of some product of spaces from* $\mathbf{A}$,

(iii) *there exists an extremal* (*initial*) *monosource with domain* Y *and codomain* $(A_i)_{i\in I}$ *with* $A_i \in \mathbf{A}$,

(iv) $C(Y, \mathbf{A})$ *is an extremal* (*initial*) *monosource,*

Proposition 1.3.5. *Let* $\mathbf{A}$ *be a class of topological spaces and* Y *be a topological space. The following conditions are equivalent:*

(i) $Y \in \mathrm{BH}(\mathbf{A})$,

(ii) *there exists an initial source with domain* Y *and codomain* $(A_i)_{i\in I}$ *with* $A_i \in \mathbf{A}$,

(iii) Y *exist an initial map* $f\colon Y \to \prod_{i\in I} A_i$, *where* $A_i \in \mathbf{A}$ *for each* $i \in I$,

(iv) $C(Y, \mathbf{A})$ *is an initial source,*

The following lemma shows the relationship between the epireflective hull and the bireflective hull.

Lemma 1.3.6. *For any* $\mathbf{A} \subseteq \mathbf{Top}_0$ *the equality* $\mathbf{Top}_0 \cap \mathrm{BH}(\mathbf{A}) = \mathrm{EH}(\mathbf{A})$ *holds.*

Proof. If $X \in \mathbf{Top}_0 \cap \mathrm{BH}(\mathbf{A})$, then there is an initial source from X to $\mathbf{A}$. This source is a monosource as well (since X is T_0, see Corollary 1.1.9), thus $X \in \mathrm{EH}(\mathbf{A})$. □

Also the following easy criterion for bireflective subcategories is often useful.

Let I_2 denotes the 2-point indiscrete space. Since I_2 is a coseparator (cogenerator) in **Top** we obtain from the known result on reflective categories in general (the dual of Proposition 1.3.10) that

Proposition 1.3.7. *An epireflective subcategory* $\mathbf{A}$ *in* **Top** *is bireflective if and only if* $I_2 \in \mathbf{A}$. *Consequently,* $\mathrm{BH}(\mathbf{A}) = \mathrm{EH}(\mathbf{A}, \mathrm{I}_2)$.

From Proposition 1.1.3 and Lemma 1.1.7 we get the following results.

Lemma 1.3.8. *If* $\mathbf{A}$ *is a class of topological spaces which is closed under the formation of finite products, then* $X \in \mathrm{BH}(\mathbf{A})$ ($X \in \mathrm{EH}(\mathbf{A})$) *if and only if* $C(X, \mathbf{A})$ *separates points and closed sets* (*and separates points*).

The condition that $C(X, \mathbf{A})$ separates points is superfluous if X is a T_0-space.

Apart from the general theorems we mentioned, the characterizations of various kinds of E-reflective hulls can be found e.g. in [Ma] (for topological categories) and [K3] (bireflective subcategories of **Top**, this paper contains also the characterization of the bireflective hull of a finitely productive class of spaces formulated in Lemma 1.3.8).

1.3.2 Coreflective subcategories of Top

The notion of the *coreflective subcategory* is dual to the notion of reflective subcategory.

Definition 1.3.9. A subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is *coreflective* in $\mathbf{B}$ if for each $\mathbf{B}$-object B there exists an $\mathbf{A}$-object A_B and a morphism $c_B\colon A_B \to B$ such that for each $\mathbf{B}$-morphism $f\colon A \to B$ there exists a unique $\mathbf{A}$-morphism $\overline{f}\colon A \to A_B$ such that $c_B \circ \overline{f} = f$.

$$\begin{tikzcd} B & A_B \arrow[l, "c_B"'] \\ & A \arrow[u, dashed, "\overline{f}"'] \arrow[ul, "f"] \end{tikzcd}$$

The notions of *coreflection* and *coreflection arrow* are used in the dual meaning to the reflection and reflection arrow.

In this case, the embedding functor is coadjoint and its adjoint is called the *coreflector*.

The following results show, why in the case of coreflective subcategories of **Top** the terms monocoreflective or bicoreflective subcategory are not used.

Proposition 1.3.10 ([AHS, Proposition 16.4]). *If* $\mathbf{A}$ *is a coreflective* (*full and isomorphism-closed*) *subcategory of* $\mathbf{B}$ *and* $\mathbf{A}$ *contains a* $\mathbf{B}$*-separator, then* $\mathbf{A}$ *is bicoreflective in* $\mathbf{B}$.

Since every non-empty topological space is a separator in **Top** and we deal only with subcategories containing at least one non-empty space, all coreflection arrows are automatically bijective.

The following results correspond to Theorem 1.3.2 and Proposition 1.3.4.

Theorem 1.3.11. *A subcategory* $\mathbf{C}$ *of* **Top** *is coreflective if and only if it is closed under the formation of topological sums and quotient spaces.*

Similarly as in the case of epireflective subcategories, this result implies that any intersection of coreflective subcategories of **Top** is again coreflective, and, consequently, every class of topological spaces has a coreflective hull.

Definition 1.3.12. The *coreflective hull* $\mathrm{CH}(\mathbf{A})$ of a class $\mathbf{A}$ of topological spaces is the smallest subcategory of **Top** which contains the class $\mathbf{A}$.

Proposition 1.3.13. *Let* $\mathbf{A} \subseteq \mathbf{Top}$ *and* X *be a topological space. The following are equivalent*

(i) $X \in \mathrm{CH}(\mathbf{A})$,

(ii) X *is a quotient of a topological sum of spaces from* $\mathbf{A}$,

(iii) *there exists a final episink with domain in* $\mathbf{A}$ *and codomain* X.

Apart from the coreflective subcategories of **Top**, we will deal also with the (mono)coreflective subcategories of $\mathbf{A}$, where $\mathbf{A}$ is an epireflective subcategory of **Top**. We postpone the characterization of such subcategories to Chapter 5.

1.4 Examples of epireflective subcategories of Top

The best reference, where many examples of epireflective and coreflective subcategories of **Top** can be found, is the monograph [He2]. In the following two sections we try to point out some special examples, which we will use later or which we find for some reasons interesting.

Most of the facts presented in this sections can be found in [He2, §18].

Zero-dimensional spaces A topological space is said to be *zero-dimensional,* if it has a base consisting of clopen sets. Some authors require zero-dimensional spaces to be T_2, e.g. [E]. We do not include this condition into the definition of a zero-dimensional space.

Both, the category **ZD** of zero-dimensional spaces and the category $\mathbf{ZD_0}$ of T_2 zero-dimensional spaces are epireflective subcategories of **Top**. (To explain the notation – for the zero-dimensional spaces the axioms T_2 and T_0 are equivalent.)

$\mathbf{ZD}_0 = \mathrm{EH}(\mathrm{D}_2)$, where D_2 is the 2-point discrete space (see [E, Theorem 6.2.16]). Similarly, $\mathbf{ZD} = \mathrm{EH}(\mathrm{I}_2, \mathrm{D}_2)$.

We will use the following observation several times:

Lemma 1.4.1. *Every prime T_2-space is zero-dimensional.*

Proof. Let P be a prime space and a be the accumulation point of P. Then a clopen base for P consists of all one-point sets $\{x\}$, $x \in P \setminus \{a\}$ and all open neighborhoods of a. The set $\{x\}$ is clopen, since there exists (by the T_2 axiom) an open set U with $a \in U$ and $x \notin U$. □

T_i-spaces Various separation axioms give rise to several interesting epireflective subcategories of **Top**.

In connection with T_0-spaces, we will often use the following result:

Theorem 1.4.2 ([E, Theorem 2.3.26]). *Every T_0-spaces X of weight $w(X) = \alpha$ can be embedded into the power S^α of the Sierpiński space S.*

Recall that the *weight* $w(X)$ of a topological space X is the smallest cardinality of a base for X. In the case that X has a finite base, we put $w(X) = \aleph_0$ by definition.

The spaces S^α are called *Alexandroff cubes* in [E].

The Sierpiński space S is the 2-point space with only one point isolated.

From Theorem 1.4.2 we can see, that the category $\mathbf{Top}_0$ of T_0-spaces is precisely the epireflective hull of the Sierpiński space S. Moreover, T_0-spaces are closed under creating spaces with finer topologies, thus by Corollary 1.3.3 $\mathbf{Top}_0$ is a quotient reflective subcategory of **Top**.

For the same reason the subcategories $\mathbf{Top}_1$ of all T_1-spaces and **Haus** of all Hausdorff spaces are quotient reflective in **Top**. But this is not the case for the epireflective subcategories **Reg** (regular spaces) and **CReg** (completely regular spaces).

Totally disconnected spaces A topological space is X said to be *totally disconnected* if the components of X are one-point sets ([E, Notes after 6.2], [Wi, Definition 29.1]).

The subcategory **TD** of all totally disconnected spaces is a quotient reflective subcategory of **Top**. The reflection arrow is the quotient map which maps all components to one-point sets.

Totally separated spaces If for every pair of points a and b in X there exists a clopen set U such that $a \in U$, $b \notin U$, we say that X is *totally separated*. An equivalent condition says that the quasicomponents of X consist of single points.

The subcategory **TS** is again a quotient reflective subcategory of **Top** and the reflection arrow is obtained by mapping each quasicomponent to one point. The subcategory **TS** is the quotient reflective hull of the space D_2.

Functionally Hausdorff spaces A topological space X is said to be *functionally Hausdorff* if for any two points $x \neq y$ of X there exists a function $f\colon X \to I$ with $f(x) \neq f(y)$. They are called Urysohn spaces by some authors, e.g. [SS].

We will denote the subcategory consisting of all functionally Hausdorff spaces by **FHaus**. Also the subcategory **FHaus** is a quotient reflective subcategory of **Top**.

1.4.1 The lattice of epireflective subcategories of Top

The epireflective subcategories of **Top** form a complete large lattice. (The word large means, that they form a proper conglomerate. By complete is meant in this context that every collection of elements of this lattice has a supremum.) Suprema in this lattice are obtained by the epireflective hull of the union of the given class of epireflective subcategories, infima are simply their intersections. In this section this lattice will be denoted by $\mathcal{B}$.

In the study of epireflective subcategories of **Top** it is useful to know the structure of this lattice.

The smallest element of $\mathcal{B}$ is the subcategory containing only the one-point spaces and the empty space. Clearly, every other epireflective subcategory of **Top** contains at least one two-point space.

Recall that I_2 denotes the 2-point indiscrete space and D_2 denotes the 2-point discrete space. Then $\mathrm{EH}(\mathrm{I}_2) = \mathbf{Ind}$ is the subcategory of all indiscrete spaces and $\mathrm{EH}(\mathrm{D}_2) = \mathbf{ZD_0}$ is the subcategory of all zero-dimensional T_2-spaces. These two subcategories are minimal elements of $\mathcal{B}$ containing at least one 2-point space.

On the other hand, $\mathrm{EH}(\mathrm{S}) = \mathbf{Top}_0$ is maximal element of $\mathcal{B}$ different from **Top** since $\mathrm{EH}(\mathrm{I}_2, \mathrm{S}) = \mathbf{Top}$ and $\mathrm{EH}(\mathrm{D}_2) \subseteq \mathrm{EH}(\mathrm{S})$.

1.5 Some coreflective subcategories of Top

1.5.1 Sequential spaces and pseudoradial spaces

Sequential spaces are the spaces in which convergent sequences suffices to determine their topology. Their study was initiated by S. P. Franklin in [F1] and [F2].

They arise very naturally in praxis, since every metric space is sequential. (In fact, the coreflective hull of metric spaces coincides with the class of sequential spaces.)

Definition 1.5.1. A topological space X is *sequential* if $V \subseteq X$ is closed whenever every limit of a sequence $(v_n)_{n=1}^{\infty}$ of elements of V belongs to V.

We denote the subcategory of **Top** consisting of all sequential spaces by **Seq**.

For a subset V of a topological space X we denote by V_1 the set of all limits of sequences of points from V. Clearly, $V \subseteq V_1 \subseteq \overline{V}$. Using this notation, we can inductively define V_α for any ordinal α, by putting:
$V_0 = V$,
$V_{\alpha+1} = (V_\alpha)_1$ and
$V_\alpha = \bigcup\limits_{\beta<\alpha} V_\beta$, if α is a limit ordinal.

Then we put $\widetilde{V} = \bigcup\limits_{\alpha\in\mathrm{On}} V_\alpha$.

One can see that $\widetilde{V}$ is the smallest subset of X containing V and closed under limits of sequences. Therefore a topological space is sequential if and only if $\overline{V} = \widetilde{V}$ for every $V \subseteq X$.

The sequential closure would be a reasonable name for the set $\widetilde{V}$ (and it is used in this sense in some papers), but this name is used also for the set V_1 by some authors, so there is no standard terminology.

The set $\widetilde{V}$ is a prototype for an analogous construction which we will introduce later in a more general situation.

We will see in Proposition 1.5.12 that $\widetilde{V} = V_{\mathfrak{c}}$ for sequential spaces.

Pseudoradial spaces were defined by H. Herrlich in [He1] and were later extensively studied by A. V. Arhangel'skiĭ and many others. They are a natural generalization of sequential spaces – we simply use arbitrary transfinite sequences instead of sequences.

The definition of convergence of transfinite sequences is analogous to the convergence of sequences. Transfinite sequences are a special case of nets – the nets on well-ordered sets. We will briefly say *α-sequence* for a transfinite sequence $(x_\xi)_{\xi<\alpha}$ of the length α.

Definition 1.5.2. We say that a transfinite sequence $(x_\xi)_{\xi<\alpha}$ of points of a topological space X converges to $x \in X$ if for each open U with $x \in U$ there exists $\eta < \alpha$ such that $x_\xi \in U$ whenever $\xi > \eta$.

We write this as $x \in \lim x_\xi$, or $x_\xi \to x$.

Using the above definition we can define pseudoradial spaces in a similar manner as we defined sequential spaces.

Definition 1.5.3. A topological space X is *pseudoradial* if $V \subseteq X$ is closed whenever for every convergent transfinite sequence $(v_\xi)_{\xi<\alpha}$ of elements of V every limit v of the α-sequence $(v_\xi)_{\xi<\alpha}$ belongs to V.

The subcategory of all pseudoradial spaces will be denoted **PsRad**.

For the pseudoradial spaces we can define closures V_1 and $\widetilde{V}$ in the same way as for sequential spaces:
V_1 is the set of all limits of transfinite sequences of points from V.
V_α's are defined inductively:
$V_0 = V$,
$V_{\alpha+1} = (V_\alpha)_1$ and
$V_\alpha = \bigcup\limits_{\beta<\alpha} V_\beta$, if α is a limit ordinal.
Finally, $\widetilde{V} = \bigcup\limits_{\alpha\in\mathrm{On}} V_\alpha$.

Again, X is pseudoradial if and only if $\overline{V} = \widetilde{V}$ for every $V \subseteq X$ and $\widetilde{V}$ is the smallest subset of X containing V and closed under limits of transfinite sequences. We will call the subset $\widetilde{V}$ the *pseudoradial closure* of V.

We next define a class of prime spaces $\{C(\alpha); \alpha \text{ is an infinite cardinal}\}$ which generates the subcategory **PsRad**. Moreover the space $C(\omega_0)$ generates **Seq**. Since the proof that these spaces are indeed generators can be used without any change for a more general case of hulls of prime spaces, we postpone this proof to the next section.

Definition 1.5.4. Let α be an infinite cardinal. The space $C(\alpha)$ is the space on the set of ordinals $\alpha \cup \{\alpha\}$ in which all points different from the point α are isolated and a local base at α is formed by sets

$$B_\beta = \{\xi \in \alpha \cup \{\alpha\}; \xi \geq \beta\}$$

for ordinals $0 \leq \beta < \alpha$.

Clearly, $C(\alpha)$ is a prime space with the non-isolated point α.

As the following lemma shows, these spaces are linked with the convergence of transfinite sequences.

Lemma 1.5.5. *A transfinite sequence $(x_\xi)_{\xi<\alpha}$ converges to x in a topological space X if and only if the map $f\colon C(\alpha) \to X$ defined by $f(\xi) = x_\xi$ for $\xi < \alpha$ and $f(\alpha) = x$ is continuous.*

Proof. By the definition of convergence of transfinite sequences $x_\xi \to \xi$ if and only if for each neighborhood U there exists $\eta < \alpha$ such that $B_\eta \subseteq f^{-1}(U)$. This is equivalent to the continuity of f. □

This connection between the continuity of maps from $C(\alpha)$ and the convergence of α-sequences is the basis for the proof that these spaces generate the category of pseudoradial spaces.

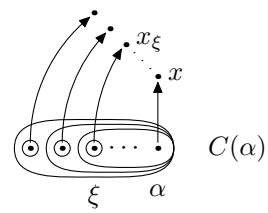


Figure 1.1: The map f from Lemma 1.5.5

1.5.2 Coreflective hull of a class of prime spaces

The subcategories **PsRad** and **Seq** motivate the generalization introduced in this section. We will see, that the results mentioned above for pseudoradial and sequential spaces can be formulated and proven for the coreflective hulls of classes of prime spaces as well.

We first try to define an analogue to the closure $\widetilde{V}$ defined for pseudoradial and sequential spaces.

Suppose we are given a class $\mathbf{A} = \{A_i; i \in I\}$ of prime spaces with a_i being the accumulation point of A_i for $i \in I$. Then for a topological space X and any $V \subseteq X$ we define the set V_1 as follows:
$V_1 = \{x \in X$: there exists a prime subspace B_i of some A_i, $i \in I$, and a continuous map $f\colon B_i \to X$ such that $f[B_i \setminus \{a_i\}] \subseteq V$ and $f(a_i) = x\}$.

The procedure of defining $\widetilde{V}$ from V_1 is now familiar. We put
$V_0 = V$,
$V_{\alpha+1} = (V_\alpha)_1$ and
$V_\alpha = \bigcup\limits_{\beta<\alpha} V_\beta$ for a limit ordinal α.
We define $\widetilde{V} = \bigcup\limits_{\alpha\in\mathrm{On}} V_\alpha$. (Although we used the same notation $\widetilde{V}$ in all three cases, it will be clear from the context which one we have in mind.)

We will show in Proposition 1.5.7 that the equality $\widetilde{V} = \overline{V}$ characterizes the spaces from the coreflective hull $\mathrm{CH}(\mathbf{A})$; similarly as it was in the cases of pseudoradial and sequential spaces. We first need to prove the following lemma.

Lemma 1.5.6. *Let $\{A_i; i \in I\}$ be a family of prime spaces and let $a_i \in A_i$ be the accumulation point of A_i for $i \in I$. A topological space X belongs to $\mathrm{CH}(\{A_i; i \in I\})$ if and only if for every non-closed subset M of X there exists $i \in I$, a prime subspace B of A_i and a continuous map $f\colon B \to X$ such that $f[B \setminus \{a_i\}] \subseteq M$ and $f(a_i) \notin M$.*

Proof. Let $\mathbf{B} \subseteq \mathbf{Top}$ be the class of all topological spaces satisfying the given condition. First we show that $\mathbf{B}$ is a coreflective subcategory of **Top**. It is evident that $\mathbf{B}$ is closed under the formation of topological sums. Now let $X \in \mathbf{B}$ and $q\colon X \to Y$ be a quotient map. Let M be a non-closed subset of Y. Then $q^{-1}(M)$ is a non-closed subset of X and $X \in \mathbf{B}$, so there exists $i \in I$, a prime subspace B of A_i and a continuous map $g\colon B \to X$ such that

$g[B \setminus \{a_i\}] \subseteq q^{-1}(M)$ and $g(a_i) \notin q^{-1}(M)$. Then for $f = q \circ g \colon B \to X$ we get $f[B \setminus \{a_i\}] \subseteq M$ and $f(a_i) \notin M$, hence $Y \in \mathbf{B}$.

We proved that $\mathbf{B}$ is coreflective. Since evidently $A_i \in \mathbf{B}$ for each $i \in I$, we have $\mathrm{CH}(\{A_i; i \in I\}) \subseteq \mathbf{B}$. To prove the opposite inclusion we construct a quotient map from a sum of subspaces of A_i to arbitrary space $X \in \mathbf{B}$. (Every subspace of A_i belongs to $\mathrm{CH}(A_i)$ by Lemma 1.2.3.)

Let $X \in \mathbf{B}$. Let $f_j \colon B_j \to X$, $j \in J$, be the family of all continuous maps such that B_j is a prime subspace of some A_i, $i \in I$. We claim that the topology of X is final w.r.t the maps f_j, $j \in J$. Suppose that $M \subseteq X$ is non closed in X. Then, by the definition of $\mathbf{B}$, there exists a map $f_j \colon B_j \to X$ for some $j \in J$, such that $f_j[B_j \setminus \{a\}] \subseteq M$ and $f(a) \notin M$, where a is an accumulation point of B_j. Obviously $f_j^{-1}(M)$ is not closed in B_j. □

The following proposition characterizes the spaces belonging to $\mathrm{CH}(\mathbf{A})$, where $\mathbf{A}$ is a class of prime spaces. It can be deduced also from [K4, Theorem 3.1.7] which deals with closure operators in general.

Proposition 1.5.7. *Let $\mathbf{A}$ be a class of prime spaces and $\widetilde{M}$ be defined as above. A topological space X belongs to $\mathrm{CH}(\mathbf{A})$ if and only if $\overline{M} = \widetilde{M}$ for any $M \subseteq X$.*

Proof. Let $X \in \mathrm{CH}(\mathbf{A})$ and $M \subseteq X$. Then $(\widetilde{M})_1 \setminus \widetilde{M} = \emptyset$, so by Lemma 1.5.6 $\widetilde{M}$ is closed and $\widetilde{M} = \overline{M}$.

Conversely, if $\overline{M} = \widetilde{M}$ for each $M \subseteq X$ and M is non-closed, then $M_1 \setminus M \neq \emptyset$ and there exists a prime subspace B_i of A_i and a continuous map $f \colon B_i \to X$ such that $f[B_i \setminus \{a_i\}] \subseteq M$ and $f(a_i) \notin M$. Hence according to Lemma 1.5.6 we get that $X \in \mathrm{CH}(\mathbf{A})$. □

Note that by Lemma 1.5.5 for the class $\mathbf{A} = \{C(\alpha); \alpha \in \mathrm{Cn}\}$ we get $\mathrm{CH}(\mathbf{A}) = \mathbf{PsRad}$. (By Cn we will denote the class of all infinite cardinals.) For the class $\mathbf{A} = \{C(\omega_0)\}$ we get $\mathrm{CH}(\mathbf{A}) = \mathbf{Seq}$. We will later show in Lemma 1.5.17 that we can restrict ourselves to regular cardinals in order to obtain generators of $\mathbf{PsRad}$. Since every prime subspace of $C(\alpha)$ for α being regular is homeomorphic to $C(\alpha)$, the description of the closure V_1 can be simplified in these cases.

One can immediately see, that everything we have proved here was inspired by the sequential spaces. (In that case it was not necessary to include the subspaces to the definition of $\widetilde{V}$, because every prime subspace of $C(\omega_0)$ is homeomorphic to $C(\omega_0)$.) On the basis of the foregoing considerations it seems expectable that many known results for sequential spaces will have analogues for coreflective hulls of prime spaces.

The above proposition is the main result of this section. We include here also some observations about coreflective hulls of prime spaces which will be useful later.

We will often use the operation described in the following definition.

Definition 1.5.8. Let X_i, $i \in I$, be a family of topological spaces with $x_i \in X_i$. Then the *wedge sum* $\bigvee_{i\in I}(X_i, x_i)$ is the space obtained by identifying all points x_i to one point, i.e., the space on the set $X := (\bigcup_{i\in I} \{i\} \times (X_i \setminus \{x_i\})) \cup \{0\}$ which is quotient w.r.t. the map $q\colon \coprod X_i \to X$ such that $q(x_i) = 0$ and $q(x) = (i, x)$ if $x \in X_i$, $x \neq x_i$.

We will denote the wedge sum of several copies of the same space X by $\bigvee_I (X, x)$ or, more briefly, by $\bigvee_I X$.

The Figure 1.2 represents the space $\bigvee_I (X, x)$ by depicting a typical base neighborhood of the point 0.

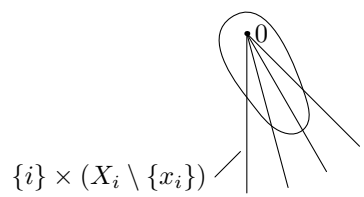


Figure 1.2: The wedge sum $\bigvee_I (X, x)$

The *wedge sum* is used in algebraic topology. The spaces $\bigvee_I (C(\omega_0), \omega_0)$ are sometimes called sequential fans or Fréchet fans, see e.g. [DW].

Lemma 1.5.9. *If $\{A_i, i \in I\}$, is a set of prime spaces and a_i is the accumulation point of A_i for each $i \in I$, then $\bigvee_{i\in I}(A_i, a_i)$ is also a prime space.*

Proposition 1.5.10. *If $\mathbf{A} = \{A_i; i \in I\}$ is a set of prime space then there exists a prime space A with $\mathrm{CH}(\mathbf{A}) = \mathrm{CH}(A)$. Moreover, $\operatorname{card} A = \sum_{i\in I} \operatorname{card} A_i$, whenever at least one space in $\mathbf{A}$ is infinite.*

Proof. Let a_i be the accumulation point of the prime space A_i. The desired space A is $\bigvee_{i\in I}(A_i, a_i)$.

The construction implies the claim on the cardinality and the inclusion $A \in \mathrm{CH}(\mathbf{A})$. On the other hand, each A_i is a subspace of A, thus by Lemma 1.2.3 $A_i \in \mathrm{CH}(A)$. □

In some cases we can bound the number of iterations of taking the closure M_1 which is needed to get $\widetilde{M}$.

Definition 1.5.11. The *tightness* $t(X)$ of a topological space X is defined by $t(X) := \sup\{t(x, X)\} + \aleph_0$, where $t(x, X)$ denotes the *tightness of X at the point* $x \in X$. The tightness of X at x is the smallest cardinal number α with the property that whenever $x \in \overline{V}$ in X then there exists a subset $W \subseteq V$ with $\operatorname{card} W \leq \alpha$ and $x \in \overline{W}$.

Proposition 1.5.12. *Let $\mathbf{A} = \{A_i; i \in I\}$ be a set of prime spaces and $X \in \mathrm{CH}(\mathbf{A})$. Denote $\alpha = \sup\{t(A_i); i \in I\}$. Then $M_{\alpha^+} = \overline{M}$ holds for every subset $M \subseteq X$.*

Proof. Let us denote by a_i the accumulation point of the prime space A_i.

If suffices to prove that $(M_{\alpha^+})_1 = M_{\alpha^+}$. Let $c \in (M_{\alpha^+})_1$. Then there exists $i \in I$, a prime subspace B of A_i and a continuous map $f\colon B \to X$ with $f(a_i) = c$ and $f[B \setminus \{a_i\}] \subset M_{\alpha^+}$. Since $t(A_i) \leq \alpha$ and $a_i \in \overline{B \setminus \{a_i\}}$, there exists $C \subseteq B \setminus \{a_i\}$ such that $a_i \in \overline{C}$ and $\operatorname{card} C \leq \alpha$. The subspace $B_1 = C \cup \{a_i\}$ of A_i is a prime subspace, $f|_{B_1}\colon B_1 \to X$ is continuous and $f|_{B_1}[C] \subseteq M_{\alpha^+}$.

For each $x \in C$ choose $\beta_x < \alpha^+$ such that $x \in M_{\beta_x}$. (Existence of such β_x follows from the fact that α^+ is a limit ordinal.) Since $\operatorname{card} C \leq \alpha < \alpha^+$ and α^+ is regular we obtain that $\gamma = \sup\{\beta_x, x \in C\} < \alpha^+$. Then $C \subseteq M_\gamma$ and, obviously, $f|_{B_1}(a_i) = f(a_i) = c \in M_{\gamma+1} \subseteq M_{\alpha^+}$. Thus $(M_{\alpha^+})_1 \subseteq M_{\alpha^+}$. □

We include also the simple (but useful) observation, that the coreflective hull of a class of prime spaces is closed hereditary and open hereditary.

Lemma 1.5.13 ([Č3, Corollary 3.4]). *Let* $\mathbf{A} = \{A_i; i \in I\}$ *be a class of prime spaces. Let* $X \in \mathrm{CH}(\mathbf{A})$*. Any closed subspace and any open subspace of* X *belongs to* $\mathbf{A}$*.*

Proof. There exists a quotient map q from a sum of prime spaces belonging to $\mathbf{A}$ onto X. For any closed (open) subspace A of X the restriction $q|_{q^{-1}(A)}\colon q^{-1}(A) \to A$ is quotient too. Lemma 1.2.3 implies, that any subspace of a sum of spaces from $\mathbf{A}$ belongs to $\mathrm{CH}(\mathbf{A})$ as well. So the subspace $q^{-1}(A)$ and its quotient $\mathbf{A}$ belong to $\mathrm{CH}(\mathbf{A})$. □

Later (in Chapter 5) we will deal also with the coreflective subcategories of an epireflective subcategory of **Top**. We will see that in this situation it is advantageous to use only prime T_2-spaces. The following lemma shows, that the prime T_2-spaces are often sufficient. Its proof is essentially contained in the proof of [Č4, Lemma 1].

Definition 1.5.14. A space X is *finitely generated* if the intersection of any system of open subsets of X is again open.

We will later show that the finitely generated spaces form a coreflective subcategory of **Top**.

Lemma 1.5.15. *If* P *is a prime space which is not finitely generated then there exists a subspace* P_1 *of* P *such that* P_1 *is a prime* T_2*-space and* $P \in \mathrm{CH}(P_1)$ *(i.e.,* P *is a quotient of a sum of copies of the space* P_1*).*

Proof. Let a be the accumulation point of P and U be the intersection of all open neighborhoods of a. Since P is not finitely generated, U is not open in P. This also implies $U \neq P$.

Let P_1 be the subspace of P on the set $(P \setminus U) \cup \{a\}$. (Obviously if P is T_2 then $P = P_1$.) Since any neighborhood of a contains a point from $P \setminus U$, P_1 is again a prime space. It is also easy to see that P_1 is Hausdorff.

We claim that P has the final topology with respect to the family of mappings $e\colon P_1 \hookrightarrow P$ and $f_b\colon S \to P$, $b \in U \setminus \{a\}$, where $S = \{0, 1\}$ is the Sierpiński space with the non-isolated point 0 and f_b is given by $f_b(0) = b$, $f_b(1) = a$.

To show this it is enough to compare the neighborhoods of a, since the other points are isolated in both topologies. Any set W open in the final topology, which contains a, must contain the whole U, otherwise some $f_b^{-1}(U)$ would not be open. Together with condition that $W \cap P_1$ is open in P_1 this implies that W is open in P. We have shown that the final topology is finer than the topology of P. The opposite inequality follows from the fact that all maps we used are continuous (as maps to the topological space P).

Since S is a quotient of P_1 (the quotient map can be obtained by mapping all isolated points to one point) we get $S \in \mathrm{CH}(P_1)$ and $P \in \mathrm{CH}(S, P_1) = \mathrm{CH}(P_1)$. □

Note that by Lemma 1.2.3 we have a quotient map (in fact a retraction) $P \to P_1$. So $\mathrm{CH}(P) = \mathrm{CH}(P_1)$ holds.

1.5.3 Generators of pseudoradial spaces

In this section we show the interpretation of the results on the coreflective hull of a class of prime spaces in the case of subclasses of $\{C(\alpha); \alpha \in \mathrm{Cn}\}$. We also show that only the spaces $C(\alpha)$ for regular cardinals α are needed to generate the subcategory **PsRad**.

From Lemma 1.5.5 and Proposition 1.5.7 we can deduce that

Proposition 1.5.16. *For any class* $\mathbf{A} = \{C(\alpha); \alpha \in A\}$*, where* A *is some class of cardinals, the following holds:* $X \in \mathrm{CH}(\mathbf{A})$ *if and only if closed sets in* X *are precisely the sets closed under limits of* α*-sequences for each* $\alpha \in A$*.*

In particular we get:
$\mathbf{PsRad} = \mathrm{CH}(\{C(\alpha); \alpha \in \mathrm{Cn}\})$,
$\mathbf{Seq} = \mathrm{CH}(C(\omega_0))$ *and*
$\mathrm{CH}(C(\alpha))$ *is the category consisting of spaces in which closed sets are precisely the sets closed under limits of* α*-sequences.*

Although, we have defined the space $C(\alpha)$ for an arbitrary cardinal α, the only important case for us is the case when α is a regular cardinal. The rest of this section is devoted to showing this fact. We start by the definition of regular cardinals.

Whenever we say that a sequence α_ξ of cardinals (or ordinals) converges to a cardinal α, we mean an increasing sequence of smaller ordinals, which converges to α in the topology of $C(\alpha)$.

We say that a cardinal α is a *regular cardinal* if it cannot be obtained as a supremum of set $A = \{\alpha_i; i \in I\}$ of smaller cardinals $\alpha_i < \alpha$ with $\operatorname{card} A < \alpha$. (If this is not the case, it is called *singular*.) It is known that $\aleph_0$ is regular and every successor cardinal is regular.

An equivalent formulation of the definition of regular cardinal is: α is not limit of any increasing transfinite sequence $(\alpha_\xi)_{\xi<\beta}$ with $\beta < \alpha$ and $\alpha_\xi < \alpha$ for each $\xi < \beta$. (Clearly, for an increasing sequence the limit is the same as the supremum.)

The smallest cardinality of a set $A = \{\alpha_i; i \in I\}$ of cardinals smaller than α with $\sup A = \alpha$ is called the *cofinality* of α, it is denoted $\operatorname{cf} \alpha$. (Equivalently, it is the smallest length of a transfinite sequence of cardinals smaller than α converging to α.) Thus we could equivalently define the regular cardinals by $\operatorname{cf} \alpha = \alpha$ and the singular cardinals by $\operatorname{cf} \alpha < \alpha$.

For a more detailed treatment of regular and singular cardinals (and another equivalent reformulation of their definition) see e.g. [HJ, Chapter 9.2].

The reason why we can claim that in the context of coreflective subcategories of **Top** the spaces $C(\alpha)$ for α regular are more important lies in the the following lemma:

Lemma 1.5.17. *Let β be a cardinal and $\alpha = \operatorname{cf}(\beta)$. Then $\operatorname{CH}(C(\alpha)) = \operatorname{CH}(C(\beta))$.*

Proof. Since α is the cofinality of β, there exists an increasing α-sequence $(\beta_\xi)_{\xi<\alpha}$ converging to β (as a sequence of cardinals, which is the same as the convergence in the topology of $C(\beta)$). We claim, that the subspace of $C(\beta)$ on the subset $\{\beta\} \cup \{\beta_\xi; \xi < \alpha\}$ is homeomorphic to $C(\alpha)$. This is clear from the fact that the sequence $(\beta_\xi)_{\xi<\alpha}$ is convergent to β and the intersections of the basic neighborhoods of β with this set are upper sets, thus they have the desired form.

Therefore $C(\alpha)$ is homeomorphic to a subspace of $C(\beta)$ and by Lemma 1.2.3 it is also a quotient of $C(\beta)$, hence it belongs to $\operatorname{CH}(C(\beta))$.

To show the other implication we only need to show that if $\beta \in \overline{V}$ for some subset $V \subseteq \beta$, then there exists an α sequence of points from V converging to β. If we prove this, the claim follows from the characterization of $\operatorname{CH}(C(\alpha))$ in Proposition 1.5.16.

The condition $\beta \in \overline{V}$ means that $\sup V = \beta$. Using the increasing α-sequence $(\beta_\xi)_{\xi<\alpha}$ with $\sup \beta_\xi = \beta$ we can define an α-sequence of points from V by $\gamma_\xi = \min\{\gamma \in V; \gamma \geq \beta_\xi\}$. This sequence clearly converges to β. □

This lemma implies that for every cardinal β we have a regular cardinal α such that the coreflective hulls of $C(\alpha)$ and of $C(\beta)$ are the same. As an easy consequence we get the following

Proposition 1.5.18. $\mathbf{PsRad} = \operatorname{CH}(\{C(\alpha); \alpha$ *is a regular cardinal*$\})$

Note that for a regular cardinal α the open neighborhoods of α in $C(\alpha)$ are precisely the complements of the sets with the cardinality smaller than α which do not contain α. This characterization is more convenient than the original definition and we will use it frequently.

The same is not true if α is singular, since in this case there is a sequence of ordinals converging to α, which has smaller cardinality than α. Complement of this sequence is not closed in $C(\alpha)$.

1.5.4 Other methods of generating coreflective subcategories

Several methods of constructing coreflective subcategories of **Top** are mentioned in [He2, §21-22] and [He3]. Here we include a brief description of results, for the proofs the reader is referred to the above mentioned works or to [Sl1].

Definition 1.5.19. Let $\mathbf{A} \subseteq \mathbf{Top}$. Then we define the following classes of topological spaces.

$\mathbf{A}_{\mathrm{sum}}$ is the class of all topological sums of the spaces from $\mathbf{A}$.

$\mathbf{A}_{\mathrm{loc}}$ is the category of all spaces such that any point has a neighborhood belonging to $\mathbf{A}$.

$\mathbf{A}_{\mathrm{gen}}$ is the class of all $\mathbf{A}$-generated spaces. A topological space X is said to be $\mathbf{A}$-*generated* if $V \subseteq X$ is closed whenever $V \cap A$ is closed for each subspace A of X such that $A \in \mathbf{A}$.

Later we will provide some conditions on $\mathbf{A}$ which imply that coreflective hull of $\mathbf{A}$ in **Top** is equal to some of these classes. But we first include a result about inclusions between these classes.

Proposition 1.5.20 ([He2, Satz 21.2.2]). *For any* $\mathbf{A} \subseteq \mathbf{Top}$ *the following inclusions hold:*

$$\mathbf{A} \subseteq \mathbf{A}_{sum} \subseteq \mathbf{A}_{loc} \subseteq \mathbf{A}_{gen} \subseteq \mathrm{CH}(\mathbf{A}).$$

Definition 1.5.21. A class $\mathbf{A} \subseteq \mathbf{Top}$ is called *map-invariant* if $Y \in \mathbf{A}$ whenever there exists a surjective continuous map $f\colon X \to Y$ with $X \in \mathbf{A}$.

Proposition 1.5.22 ([He2, Satz 21.2.4]). *Let* $\mathbf{A}$ *be a map-invariant class of topological spaces. Then the coreflective hull* $\mathrm{CH}(\mathbf{A})$ *of* $\mathbf{A}$ *consists precisely of all* $\mathbf{A}$*-generated spaces.*

Moreover, the $\mathbf{A}_{gen}$*-coreflection of* X *is the space on the same set in which closed sets are precisely the sets fulfilling the condition, that* $V \cap A$ *is closed in* A *for every subspace* A *of* X *belonging to* $\mathbf{A}$.

Example 1.5.23. Perhaps the simplest example is the coreflective hull of finite spaces. Spaces belonging to this subcategory are precisely the finitely generated spaces (Definition 1.5.14) and it is denoted by **FG**.

Example 1.5.24. An example, which we will often use, is the category $\mathbf{Gen}(\alpha)$ of all α-*generated* spaces. If α is an infinite cardinal, then $\mathbf{Gen}(\alpha)$ is the coreflective hull of the class of all spaces with cardinality at most α, which is clearly map-invariant.

It is known that a topological space is α-generated if and only if $t(X) \leq \alpha$. If $t(X) \leq \alpha$ then $t(Y) \leq \alpha$ holds for each subspace Y of X.

Example 1.5.25. Another class of spaces, which is well-known to be map-invariant, is the class of all compact spaces (we do not require the compact spaces to be Hausdorff). Its coreflective hull is denoted by **CGen** and spaces from **CGen** are called *compactly generated spaces* or *k-spaces*.

The **CGen**-coreflection is obtained from a topological space X by taking as open sets precisely those sets whose intersection with every compact set is open. The space X and its **CGen**-coreflection have the same compact subsets and the topologies of the compact subspaces coincide.

For any topological space we will denote by $o(X)$ the cardinality of topology of this space. (We adopted this notation from [J].)

Example 1.5.26. Let α be a cardinal number and $\mathbf{A}_\alpha = \{X; o(X) < 2^\alpha\}$, i.e., $\mathbf{A}_\alpha$ is the class of all topological spaces having the cardinality of the topology less than 2^α. Since for every surjective continuous map $f\colon X \to Y$ the assignment $U \mapsto f^{-1}(U)$ is an injective map from the topology of Y to the topology of X, the class $\mathbf{A}_\alpha$ is map-invariant.

We will denote by $\mathbf{C}_\alpha$ the class of all $\mathbf{A}_\alpha$-generated spaces, i.e., the coreflective hull of $\mathbf{A}_\alpha$.

If $\mathbf{A}$ has some additional properties apart from being map-invariant, its coreflective hull can be described even simpler.

Definition 1.5.27. We say that $\mathbf{A} \subseteq \mathbf{Top}$ is a *component category* when $\mathbf{A}$ contains all one-point spaces and union of any centered system of subspaces belonging to $\mathbf{A}$ (in any space X) is again in $\mathbf{A}$.

Directly from this definition follows that if $\mathbf{A}$ is a component category, then maximal subspaces of X belonging to $\mathbf{A}$ exist and they form a disjoint cover of X. These subspaces are called $\mathbf{A}$-*components* of X.

Proposition 1.5.28 ([He2, Satz 21.2.6]). *Let $\mathbf{A}$ be a map-invariant component category. Then $\mathbf{A}_{sum} = \mathbf{A}_{loc} = \mathbf{A}_{gen} = \mathrm{CH}(\mathbf{A})$.*

Moreover, the $\mathbf{A}_{sum}$-coreflection of X can be obtained as the topology whose subbase consists of all open sets of the original topology and all $\mathbf{A}$-components.

Example 1.5.29. The class $\mathbf{A}$ of all connected spaces is clearly a map-invariant component category. Therefore the coreflective hull of this class it the subcategory $\mathbf{A}_{\mathrm{sum}}$ of all sums of connected topological spaces.

With a map-invariant component category one can associate also another coreflective subcategory of **Top**.

Proposition 1.5.30 ([He2, Satz 21.2.8]). *Let $\mathbf{A}$ be a map-invariant component category. Let us denote by $\mathbf{A}_L$ the class of all spaces X with the property that all $\mathbf{A}$-components of any open subset of X are open. Then $\mathbf{A}_L$ is a coreflective subcategory of* **Top**.

Example 1.5.31. If $\mathbf{A}$ is the class of all connected spaces, than $\mathbf{A}_L$ consists of all locally connected spaces. A topological space X is said to be *locally connected* if for any $x \in X$ and any open neighborhood U of x there is an open neighborhood $V \subseteq U$, $x \in V$, which is connected (see [E, 6.3.3] or [Wi, Definition 27.7]).

1.5.5 The lattice of coreflective subcategories of Top

The coreflective subcategories of **Top** form a complete large lattice. Infima are the intersections of coreflective subcategories and suprema are the coreflective hulls of unions of coreflective subcategories. This lattice was studied in [He2, §22] and in [He3]. We include here some basic facts which we will need later. Moreover, the overview of basic properties of this lattice provides deeper insight into the problematics of coreflective subcategories of **Top**.

In this section we will denote the lattice of all coreflective subcategories of **Top** by $\mathcal{C}$.

The smallest coreflective subcategory of **Top** containing at least one non-empty space is the subcategory **Disc** consisting of all discrete spaces. This subcategory is the coreflective hull of the one-point space and the one-point space can be obtained as a quotient from any non-empty topological space.

The smallest subcategory which contains a non-discrete space (i.e., the smallest element of $\mathcal{C}$ above **Disc**) is the category $\mathbf{Ind}_{\mathrm{sum}}$ of all sums of indiscrete spaces. The spaces belonging to this subcategory can be characterized as the spaces, in which open and closed subsets are the same. The subcategory $\mathbf{Ind}_{\mathrm{sum}}$ is the coreflective hull of the 2-point indiscrete space I_2.

If a topological space X is not a sum of indiscrete spaces, then it contains a subset which is open and not closed. Using such a subset we can obtain the Sierpiński space S as the quotient of X in the obvious way. Therefore the smallest element of the lattice $\mathcal{C}$ above $\mathbf{Ind}_{\mathrm{sum}}$ is $\mathrm{CH}(S)$. It is known that $\mathrm{CH}(S) = \mathbf{FG}$ is the category of all finitely generated spaces. Recall that a topological space X is *finitely generated* if every intersection of open subsets of X is again open in X. The **FG**-coreflection can be obtained simply by taking all intersections of open sets of the original space for the base of the topology of coreflection. We have already mentioned in Example 1.5.23 that **FG** is the coreflective hull of all finite spaces.

In the context of order theory, the finitely generated spaces are also called Alexandrov discrete spaces or Alexandrov topologies (see e.g. [Hof2]). They are known to correspond to preordered sets.

Another construction of **FG**-coreflection can be obtained by taking all saturated sets as the closed sets of the new topology. A subset V of a topological space x is called *saturated* if $\overline{\{x\}} \cap V \neq \emptyset$ implies $x \in V$.

Note that the coreflective subcategories **Disc**, $\mathbf{Ind}_{\mathrm{sum}}$ and **FG** are hereditary (closed under the formation of subspaces).

At this point the hierarchy of coreflective subcategories of **Top** stops to be linear. For every regular cardinal α there exists an atom $\mathbf{B}_\alpha = \mathrm{CH}(B(\alpha))$ of $\mathcal{C}$ above **FG**.

Definition 1.5.32. For any infinite cardinal α, let $B(\alpha)$ be the topological space on the set $\alpha \cup \{\alpha\}$ with the topology consisting of all sets $B_\beta = \{\gamma \in \alpha \cup \{\alpha\}; \gamma \geq \beta\}$ where $\beta < \alpha$.

The subcategory $\mathbf{B}_\alpha$ is the smallest coreflective subcategory of **Top** with the following two properties: Any intersection of less than α open sets in any

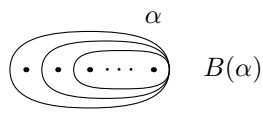


Figure 1.3: The space $B(\alpha)$

space from $\mathbf{B}_\alpha$ is open. There exists a space in $\mathbf{B}_\alpha$ and a system of open sets of this space, which has cardinality α and a non-open intersection. Consequently, for any coreflective subcategory $\mathbf{C}$ of $\mathbf{Top}$ with $\mathbf{C} \supsetneq \mathbf{FG}$ there exists a regular cardinal α such that $\mathbf{B}_\alpha \subseteq \mathbf{C}$.

Clearly, $\mathbf{B}_\alpha \subseteq \mathrm{CH}(C(\alpha))$. Another useful fact is that $\mathbf{B}_\alpha \cap \mathbf{B}_\beta = \mathbf{FG}$ holds for any regular cardinals $\alpha \neq \beta$.

Example 1.5.33. By $\mathbf{Top}(\alpha)$ we denote the subcategory consisting of all spaces in which every intersection of less than α open sets is open. It follows directly from this definition that the subcategory $\mathbf{Top}(\alpha)$ is hereditary. Spaces from the subcategory $\mathbf{Top}(\alpha)$ are called *P_α-spaces* (see e.g. [Gi], [HM] or [CH]). The P_{ω_1}-spaces are known as P-spaces.

Directly from the definition and the basic properties mentioned above it follows that $\mathrm{CH}(C(\alpha)) \subseteq \mathbf{Top}(\alpha) \cap \mathbf{Gen}(\alpha)$ and $\mathbf{Gen}(\alpha) \cap \mathbf{Top}(\beta) = \mathbf{FG}$ holds for any regular cardinals $\alpha < \beta$.

Chapter 2

Hereditary coreflective subcategories of Top

In connection with some questions posed in [HH2, Problem 7], J. Čincura studied in [Č3] hereditary coreflective subcategories of the category **Top**. He proved an elegant characterization of the hereditary coreflective subcategories among all coreflective subcategories of **Top**. Namely he has shown that a coreflective subcategory **C** of **Top** such that $\mathbf{C} \neq \mathbf{Ind}$ is hereditary if and only if it is closed under the formation of prime factors. We will use this characterization frequently, therefore we included it with the proof at the beginning of this chapter.

Using this characterization we can show that every topological space can be embedded into a pseudoradial space. Moreover, every T_1-space can be embedded into a pseudoradial T_1-space. This answers a question posed in the paper [AIT]. We also provide a characterization of coreflective subcategories of **Top** which have the property $\mathrm{S}\mathbf{A} = \mathbf{Top}$, i.e., every topological space can be embedded into a space from this subcategory.

These results were published in the paper [Sl2].

2.1 Heredity and prime factors

In the first part of this section we present the characterization of hereditary coreflective subcategories using prime factors which appeared in [Č3]. This result shows why prime spaces will be one of the basic tools for us.

Definition 2.1.1. A subcategory **C** of **Top** is called *hereditary* if it is closed under the formation of subspaces.

We will study the hereditary coreflective subcategories of **Top**. These subcategories are the classes of topological spaces closed under the topological sums, quotient spaces and subspaces.

Obviously, any intersection of such subcategories is again hereditary and coreflective. This means that for any $\mathbf{B} \subseteq \mathbf{Top}$ there exists the smallest hereditary coreflective subcategory containing $\mathbf{B}$ (it can be obtained simply as the intersection of all such subcategories).

Definition 2.1.2. The smallest hereditary coreflective subcategory of **Top** containing $\mathbf{B}$ is called the *hereditary coreflective hull* of $\mathbf{B}$ in **Top**.

The operation of taking all subspaces of spaces from a given class is often useful in the context of hereditary coreflective subcategories.

Definition 2.1.3. For any $\mathbf{B} \subseteq \mathbf{Top}$ we denote by $\mathrm{S}\mathbf{B}$ the class of all subspaces of spaces from $\mathbf{B}$.

We show that using this operation we can obtain the hereditary coreflective hull in **Top** from the coreflective hull in **Top**. We will need the following simple lemma from general topology.

Lemma 2.1.4. *Let $f\colon X \to Y$ be a quotient map, $A \subseteq Y$ and let f be one-to-one outside $f^{-1}(A)$. Then $f|_{f^{-1}(A)}\colon f^{-1}(A) \to A$ is a quotient map.*

Proof. Let us denote $g := f|_{f^{-1}(A)}\colon f^{-1}(A) \to A$. This map is continuous and surjective. We want to show that U is open in the subspace A of Y, whenever $g^{-1}(U)$ is open in $f^{-1}(A)$ (with the relative topology inherited from X).

If $g^{-1}(U)$ is open in $f^{-1}(A)$, then $g^{-1}(U) = V \cap f^{-1}(A)$ for some open set V in Y. So we can express U as

$$U = g[g^{-1}(U)] = f[g^{-1}(U)] = f[V \cap f^{-1}(A)] = f[V] \cap A. \tag{2.1}$$ {EQU}

We will show that $f[V]$ is open in Y, which implies together with (2.1) the openness of U.

We have $f^{-1}(f[V]) = f^{-1}(f[V] \cap A) \cup f^{-1}(f[V] \setminus A)$. From (2.1) we get $f^{-1}(f[V] \cap A) = g^{-1}(U) = V \cap f^{-1}(A)$. Since f is bijective outside $f^{-1}(A)$, the equality $f^{-1}(f[V] \setminus A) = V \setminus f^{-1}(A)$ holds.

Together we get $f^{-1}(f[V]) = (V \cap f^{-1}(A)) \cup (V \setminus f^{-1}(A)) = V$. Since V is open in X and f is a quotient map, $f[V]$ is open in Y. □

Proposition 2.1.5. *If $\mathbf{C}$ is a coreflective subcategory of* **Top***, then $\mathrm{S}\mathbf{C}$ is a coreflective subcategory of* **Top** *too.*

Proof. Let $X \in \mathrm{S}\mathbf{C}$, i.e., X is a subspace of some $C \in \mathbf{C}$. Let $q\colon X \to Y$ be quotient. We will show that $Y \in \mathrm{S}\mathbf{C}$ by constructing a space D from the coreflective subcategory $\mathbf{C}$ which contains Y as its subspace.

The underlying set of the topological space D will be $Y \cup (C \setminus X)$ (we can w.l.o.g. assume that $Y \cap (C \setminus X) = \emptyset$). We define the topology of D as the quotient topology with respect to the map $f\colon C \to D$ given by $f(c) = c$ for $c \notin X$ and $f(c) = q(c)$ for $c \in X$.

The restriction $f|_{f^{-1}(Y)}$ is precisely the map q. Moreover, the map f is one-to-one outside $f^{-1}(Y)$. This means, that this restriction is a quotient map

by Lemma 2.1.4 and the subspace of D on the subset Y is precisely the space Y (the quotient of X w.r.t. the map q).

We have shown that S**C** is closed under quotients. The closedness under topological sums is clear. So S**C** is coreflective. □

We could note here, that the space D constructed in the above proof is pushout in the category **Top**. Almost identical proof can be used to show that in **Top** the pushout of an embedding along a quotient map is again an embedding.

The above proposition yields a characterization of the hereditary coreflective hull of a coreflective subcategory of **Top**.

Corollary 2.1.6. *For any* $\mathbf{B} \subseteq \mathbf{Top}$ *the hereditary coreflective hull of* $\mathbf{B}$ *is* $\mathrm{S}(\mathrm{CH}(\mathbf{B}))$. (*We will use the shorter notation* $\mathrm{SCH}(\mathbf{B})$.)

Now we can show the main results of this section – the characterization of hereditary coreflective subcategories of **Top** in terms of prime factors.

Definition 2.1.7. We say that a coreflective subcategory **C** of **Top** is *nontrivial* if $\mathbf{FG} \subseteq \mathbf{C}$.

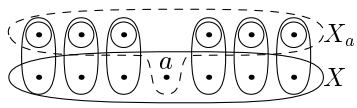
Proposition 2.1.8 ([Č3, Proposition 3.5]). *Let* **C** *be a nontrivial coreflective subcategory of* **Top**. *If* **C** *is hereditary then for every* $a \in X \in \mathbf{C}$ *the prime factor* X_a *belongs to* **C**.

Proof. Let S be the Sierpiński space on the set $\{0, 1\}$ with 1 being the isolated point. Since **C** is nontrivial, $S \in \mathbf{C}$.

If a is an isolated point of X, then X_a is discrete and $X_a \in \mathbf{C}$. So we can assume that a is not-isolated.

Let us denote by Y the space on the set $((X \setminus \{a\}) \times \{0, 1\}) \cup \{a\}$ with the quotient topology w.r.t. the map $p\colon X \sqcup \coprod_{x \in X \setminus \{a\}} (\{x\} \times S) \to Y$ given by $p(a) = a$, $p(x) = (x, 0)$ pre $x \in X \setminus \{a\}$ and $p(x, y) = (x, y)$ for $(x, y) \in \coprod_{x \in X \setminus \{a\}} (\{x\} \times S)$.

Since **C** is coreflective, we have $Y \in \mathbf{C}$. The subspace of Y on the subset $((X \setminus \{a\}) \times \{1\}) \cup \{a\}$ is homeomorphic to X_a. **C** is hereditary, thus $X_a \in \mathbf{C}$ as well. □



Theorem 2.1.9. *Let* **C** *be a nontrivial coreflective subcategory of* **Top**. *The subcategory* **C** *is hereditary if and only if it is closed under the formation of prime factors, i.e, for each* $X \in \mathbf{C}$ *and each* $a \in X$ *the prime factor* X_a *of* X *at* a *belongs to* **C**.

Proof. One implication is proved in Proposition 2.1.8, the opposite one follows from Lemma 1.2.5. □

We close this section by including a few examples of hereditary coreflective subcategories. The simplest examples are the subcategories **Disc**, $\mathbf{Ind}_{\mathrm{sum}}$ and **FG**. The subcategory $\mathbf{Top}(\alpha)$ of P_α-spaces was already mentioned in Example 1.5.33. This subcategory is easily shown to be hereditary. Another example is the subcategory $\mathbf{Gen}(\alpha)$ of α-generated spaces from Example 1.5.24.

The subcategory $\mathbf{Gen}(\alpha)$ is the coreflective hull of the map-invariant class of all spaces with the cardinality at most α. This map-invariant class is hereditary. It is natural to ask whether the coreflective hull of each hereditary map-invariant class is a hereditary coreflective subcategory of **Top**. We will show in Examples 5.4.10 and 5.4.11 that the subcategory $\mathbf{C}_\alpha$ defined in Example 1.5.26 is a counterexample to this conjecture.

Example 2.1.10. The subcategory **SSeq** of subspaces of sequential spaces is a hereditary coreflective subcategory by Proposition 2.1.5. Such spaces are called *subsequential.* They were studied by S. P. Franklin and M. Rajagopalan in [FR].

We could do the same construction with the pseudoradial spaces instead of the sequential spaces. The subcategory of all subspaces of pseudoradial spaces, which we obtain in this way, is the whole **Top**. The following section is devoted to the proof of this fact.

2.2 Subspaces of pseudoradial spaces

The main goal of this section is to answer the question posed in [AIT]: Can every T_1-space be embedded into a pseudoradial T_1-space? We start with a simpler question – can every topological space be embedded into a pseudoradial space? In the language of coreflective subcategories of **Top** we can express this question as **Top** = **SPsRad**. (By **SPsRad** we denote the category consisting of all subspaces of pseudoradial spaces.) This question is answered affirmatively in Theorem 2.2.3. The same holds for T_0- and T_1-spaces, but not for T_2-spaces, as Example 2.2.7 shows.

The methods used in the proof that **SPsRad** = **Top** enable us to find the smallest coreflective subcategory **A** of **Top** such that S**A** = **Top**, this can be considered as a kind of characterizations of such subcategories.

2.2.1 Embeddings into pseudoradial spaces

In this section we answer the question posed by A. V. Arhangel'skiĭ, R. Isler and G. Tironi in their paper [AIT] by showing that every (T_1-)space can be embedded into a pseudoradial (T_1)-space.

This question was asked again in the survey [Ny]. Partial answer, under the assumption $\mathfrak{p} = \mathfrak{c}$ (which is consequence of Martin's Axiom), was provided by J. Zhou in [Z]. (He obtained a stronger result for prime spaces that any countable prime T_2-space is a subspace of a Hausdorff sequential space.)

The main result of this section was also proved in E. Murtinová's master thesis [Mu], the proof used there is similar to ours. I was not aware of her result at the time of publishing the paper [Sl2] (it was not published elsewhere).

We want to show that **SPsRad** = **Top**. By Lemma 1.2.5 any topological space is in the coreflective hull of its prime factors, so we only need to show that any prime space is in **SPsRad**.

To make our result a little bit sharper, we introduce the notion of β-sequential space, where β is an infinite cardinal. A topological space X is *β-sequential* if closed subsets in X are precisely the sets closed under limits of α-sequences for all $\alpha \leq \beta$.

In other words, a subset V in a β-sequential space is closed if and only if it is equal to its *β-sequential closure,* i.e., the smallest subset containing V and closed under limits of α-sequences for all $\alpha \leq \beta$. (There is no standard terminology for these notions, this one seems reasonable to us.) In this section we will use the notation $\widetilde{V}$ for the β-sequential closure of V.

By Lemma 1.5.6 and Proposition 1.5.7 the class of all β-sequential spaces is the coreflective hull of the set $\{C(\alpha); \alpha \leq \beta\}$.

This subcategory will be denoted by $\mathbf{Psrad}(\beta)$.

We have already mentioned that our strategy is to start by embedding each prime space into a pseudoradial space. Since every prime space is a T_0-space, it is a subspace of some power of the Sierpiński space S by Theorem 1.4.2.

Recall that the Sierpiński space is the two-point space in which only one point is isolated. For the sake of notation we will assume that it is the topological space on the set $\{0, 1\}$ with open sets $\emptyset$, $\{0\}$ and $\{0, 1\}$, i.e., the isolated point is 0.

The core of the proof of the main result is the following proposition.

Proposition 2.2.1. *If β is an infinite cardinal, then the topological power S^β of the space S is a β-sequential space.*

Proof. Let γ be the smallest cardinal such that S^γ is not β-sequential. We want to show that $\gamma > \beta$. Since for any cardinal $\alpha \leq \omega_0$ the space S^α is sequential (it is first-countable), $\gamma > \omega_0$. Assume that $\gamma \leq \beta$. Since a space is β-sequential if and only if $\widetilde{U} = \overline{U}$ for any subset of X, there exists a subset U of S^γ with $\overline{U} \setminus \widetilde{U} \neq \emptyset$.

Let $t \in \overline{U} \setminus \widetilde{U}$, $A = \{\eta \in \gamma; t(\eta) = 0\}$ and $\varkappa = \operatorname{card} A$. Clearly, $A \neq \emptyset$ and $\varkappa \leq \gamma$. Consider the subspace $K = \{s \in S^\gamma;\ s(\eta) = 1 \text{ for each } \eta \in \gamma \setminus A\}$. The space K is a closed subspace of S^γ, $t \in K$ and, obviously, K is homeomorphic to the space $S^\varkappa$.

Let us define a map $g\colon S^\gamma \to K$ by

$$g(f)(x) = \begin{cases} f(x), & \text{if } x \in A \\ 1, & \text{if } x \notin A \end{cases}$$

The map g is continuous, hence $t = g(t) \in \overline{g[U]}$. Clearly, for every $s \in S^\gamma$ $g(s) \in \widetilde{\{s\}}$ (the constant sequence $(s)_{n<\omega_0}$ converges to $g(s)$) and therefore $\widetilde{g[U]} \subseteq \widetilde{U}$.

If $\varkappa < \gamma$, then K is β-sequential and therefore $\overline{g[U]} = \widetilde{g[U]}$. This implies that $t \in \widetilde{U}$. Thus, we obtain that $\varkappa = \gamma$.

In this case there exists a homeomorphism $f\colon K \to S^\gamma$ such that $f(t) = t_0$ where $t_0(\eta) = 0$ for each $\eta \in \gamma$. Without loss of generality we can suppose that $K = S^\gamma$ and $t = t_0$ (and, obviously, g is the identity map). For each $\xi \in \gamma$ let f_ξ denote the element of S^γ given by

$$f_\xi(x) = \begin{cases} 0, & \text{for } x < \xi, \\ 1, & \text{for } x \geq \xi. \end{cases}$$

It is easy to see that the γ-sequence $(f_\xi)_{\xi\in\gamma}$ converges to t_0 in S^γ. Since $t_0 \in \overline{U}$ and $\overline{\{t_0\}} = S^\gamma$, we obtain that $\overline{U} = S^\gamma$ and therefore $f_\xi \in \overline{U}$ for each $\xi \in \gamma$. Put $A_\xi = \{\eta \in \gamma; f_\xi(\eta) = 0\} = \{\eta \in \gamma; \eta < \xi\}$. Then $\operatorname{card} A_\xi < \gamma$ for each $\xi \in \gamma$ and according to the preceding part of proof (the case $\varkappa < \gamma$) $f_\xi \in \widetilde{U}$. Hence, $t_0 \in \widetilde{U}$ contradicting our assumption. Thus, $\gamma > \beta$ and S^β is β-sequential. □

Using the above proposition we can show easily

Theorem 2.2.2. $\mathbf{Gen}(\alpha) \subseteq \mathbf{SPsrad}(2^\alpha)$ *for every infinite cardinal* α.

Proof. The subcategory $\mathbf{Gen}(\alpha)$ is a hereditary subcategory of $\mathbf{Top}$, therefore it is closed under prime factors. It is the coreflective hull of spaces of cardinality at most α and since each topological space can be obtained from its prime factors (Lemma 1.2.5), we see that $\mathbf{Gen}(\alpha)$ is the coreflective hull of prime spaces P with $\operatorname{card} P \leq \alpha$. Therefore it suffices to show that these prime spaces are subspaces of 2^α-sequential spaces.

Every prime space P is a T_0-space. Moreover, a trivial upper bound on $w(P)$ is the cardinality of the powerset of P, which is at most 2^α. Thus it is a subspace of S^{2^α} (Theorem 1.4.2) and this space is 2^α-sequential by Proposition 2.2.1. □

Since each topological space belongs to some $\mathbf{Gen}(\alpha)$, from the above result we obtain easily:

Theorem 2.2.3. *Every topological space is a subspace of a pseudoradial space. Moreover, every* T_0*-space is a subspace of a pseudoradial* T_0*-space.*

Proof. The first part follows directly from Theorem 2.2.2. To see the second claim it is enough to combine Theorem 1.4.2 and Proposition 2.2.1. □

Recall that a topological space X is *sober* if for every irreducible closed subset $V \subseteq X$ there exist precisely one point $x \in X$ such that $V = \overline{\{x\}}$. (A closed set V in X is irreducible, if it is not a union of two closed proper subsets.) Sober spaces are precisely the epireflective hull of S in $\mathbf{Top}_0$. For an overview of basic facts about the sober spaces see e.g. [Hof2].

Since S is sober and the sober spaces are productive, all powers S^α are sober as well. This leads to

Corollary 2.2.4. *Every T_0-space is a subspace of a pseudoradial sober space.*

Our next goal is to extend the results of Theorem 2.2.3 to T_1-spaces as well. The space S^α is not T_1, but we can modify its topology to a T_1-topology without violating the pseudoradiality.

Recall that the *cofinite topology* on a given set X is the topology in which open sets are precisely $\emptyset$ and the complements of finite sets. This topology is the coarsest T_1-topology on X.

For any cardinal number β, let $(S^\beta)_1$ be the topological space on the set $\{0,1\}^\beta$ with the topology which is the join of the product topology of S^β and the cofinite topology on the set $\{0,1\}^\beta$. If β is finite, then $(S^\beta)_1$ is discrete.

The basic idea of the proof that $(S^\beta)_1$ is β-sequential is the same as in the proof of Proposition 2.2.1.

Proposition 2.2.5. *If β is an infinite cardinal, then $(S^\beta)_1$ is β-sequential.*

Proof. The collection $\mathcal{B}_1 = \{U_M; M \subseteq \alpha, M \text{ is finite}\}$, where

$$U_M = \{f \in \{0,1\}^\alpha; f(m) = 0 \text{ for each } m \in M\}$$

is the canonical base for the product topology S^α. Clearly

$$\mathcal{B} = \{U_M \setminus F; M \subseteq \alpha, M \text{ is finite}, F \subseteq \{0,1\}^\alpha, F \text{ is finite}\}$$

is a base for the topology of the space $(S^\beta)_1$.

We have to show that if $t \in \overline{U} \setminus U$ then $t \in \widetilde{U}$. (By $\widetilde{U}$ we denote the α-sequential closure of U in $(S^\beta)_1$.) Let us put

$$A_t = \{\eta \in \alpha; t(\eta) = 0\}.$$

Assume that, on the contrary, there exist some $t \in \{0,1\}^\alpha$ and $U \subseteq \{0,1\}^\alpha$ such that $t \in \overline{U} \setminus U$ and $t \notin \widetilde{U}$. Let β be the smallest cardinal number such that $\beta = \operatorname{card} A_t$ for some t and U satisfying $t \in \overline{U} \setminus U$ and $t \notin \widetilde{U}$.

First let β be finite, i.e. let A_t be a finite subset of α. Then U_{A_t} is a neighborhood of t, thus there exists $f_1 \in U \cap U_{A_t}$. Since $U_{A_t} \setminus \{f_1\}$ is a neighborhood of t, there is $f_2 \in U \cap (U_{A_t} \setminus \{f_1\})$. In a similar way we can find for every $n < \omega_0$, $n \geq 2$, an $f_n \in U \cap (U_{A_t} \setminus \{f_1, \dots, f_{n-1}\})$. We claim that f_n converges to t.

Every basic neighborhood of t has the form $U_B \setminus F$, where $F \subseteq S^\alpha$ and $B \subseteq A_t$ are finite subsets. U_B contains all terms of the sequence $(f_n)_{n<\omega_0}$ and by omitting the finite subset F we omit only finitely many of them, since this sequence is one-to-one.

Thus β is not finite and $\omega_0 \leq \beta = \operatorname{card} A_t \leq \alpha$. Let us arrange all members of A_t into a one-to-one β-sequence. Hence, $A_t = \{a_\xi; \xi < \beta\}$. Let us define a function $f_\gamma \colon \alpha \to S$ by

$$f_\gamma(x) = \begin{cases} 0, & \text{if } x = a_\xi \text{ for some } \xi < \gamma \\ 1, & \text{otherwise,} \end{cases}$$

for every $\gamma < \beta$.

If $U_B \setminus F$ is a basic neighborhood of f_γ, then $(U_B \setminus F) \cup \{t\}$ is a neighborhood of t. Hence $f_\gamma \in \overline{U}$. Since the cardinality of the set $A_\gamma = \{a_\xi, \xi < \gamma\} = \{\eta \in \beta; f_\gamma(\beta) = 0\}$ is less then β and $f_\gamma \in \overline{U}$, we get $f_\gamma \in \widetilde{U}$.

It only remains to show that the sequence f_γ converges to t. Any basic neighborhood of t has the form $U_B \setminus F$, where $B \subseteq A_t$, B and F are finite. Let $\delta_1 = \sup\{\xi : a_\xi \in B\}$ and $\delta_2 = \sup\{\xi : f_\xi \in F\}$. Since F and B are finite, $\delta_1, \delta_2 < \alpha$. Let $\delta = \max\{\delta_1, \delta_2\}$. Then $f_\gamma \in U_B \setminus F$ for each $\gamma > \delta$.

Thus $t \in \widetilde{U}$, a contradiction. □

Theorem 2.2.6. *Every T_1-space is a subspace of a pseudoradial T_1-space.*

Proof. Let X be a T_1-space. Then there exists an embedding $e\colon X \hookrightarrow S^\alpha$ of X into some power S^α of S. Since X is T_1, $e\colon X \hookrightarrow (S^\beta)_1$ is an embedding as well. $(S^\beta)_1$ is a T_1-space and it is pseudoradial by Proposition 2.2.5. □

As the following example shows, the same is not true for Hausdorff spaces.

Example 2.2.7. Let X be any compact T_2-space which is not sequential. Suppose that X can be embedded into a sequential T_2-space Y. Then X is closed subspace of Y (every compact subspace of Hausdorff space is closed). By Lemma 1.5.13 every closed subspace of a sequential space is again sequential. Therefore X is itself sequential, a contradiction.

The Stone-Čech compactification $\beta\mathbb{N}$ of the discrete countable space is an example of such a space, since there are no non-trivial convergent sequences in $\beta\mathbb{N}$ ([E, Corollary 3.6.15]).

2.2.2 Coreflective subcategories with $\mathrm{S}\mathbf{A} = \mathbf{Top}$

We have already seen that for the coreflective subcategory **PsRad** of all pseudoradial spaces **SPsRad** = **Top** holds. It is natural to ask whether we could employ similar methods in order to characterize other coreflective subcategories with the property $\mathrm{S}\mathbf{A} = \mathbf{Top}$, i.e., every topological space can be embedded into a space from **A**.

This question is connected with [HH2, Problem 7], where H. Herrlich and M. Hušek are interested in the characterization of coreflective subcategories of **Top** such that their hereditary coreflective hull is **Top** and their hereditary coreflective kernel is **FG**. We will deal with the categories having the property $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$ later in Section 4.2. (Hereditary coreflective kernel of **A**, $\mathrm{HCK}(\mathbf{A})$, is the largest hereditary coreflective subcategory of **Top** contained in the subcategory **A**.)

We provide several classes of spaces that generate the smallest coreflective category of **Top** with the property $\mathrm{S}\mathbf{A} = \mathbf{Top}$. This can be considered as a kind of characterizations of such coreflective subcategories. We next show that the powers of the Sierpiński space form such class.

The key property of the spaces S^α used in the proof of Theorem 2.2.9 is formulated in the following proposition. We could note that from the categorical

viewpoint this property says that each S^α is an absolute retract in the category **Top**. An object C is an *absolute retract* in a concrete category **A** over **Set** provided that any embedding e with domain C there exists a retraction r with $r \circ e = id_C$.

Proposition 2.2.8. *Let α be an infinite cardinal and S^α be a subspace of X. Then there exists a retraction $f\colon X \to S^\alpha$.*

Proof. Let S^α be a subspace of a topological space X. For each $a \in \alpha$ let $p_a\colon S^\alpha \to S$ denote the a-th projection of topological power S^α onto S. The set $(p_a)^{-1}(0)$ is open in S^α so that there exists an open subset U_a in X such that $U_a \cap S^\alpha = (p_a)^{-1}(0)$. The map $f_a\colon X \to S$ given by $f_a(x) = 0$ for each $x \in U_a$ and $f_a(x) = 1$ otherwise is a continuous extension of $p_a\colon S^\alpha \to S$. The map $f = \langle f_a \rangle\colon X \to S^\alpha$ given by $f_a = p_a \circ f$ for each $a \in \alpha$ is continuous and the restriction $f|_{S^\alpha}$ is the identity map on S^α. Hence f is a retraction. □

Theorem 2.2.9. *Let* **A** *be a coreflective subcategory of* **Top**. *Then* $\mathrm{S}\mathbf{A} = \mathbf{Top}$ *if and only if $S^\alpha \in \mathbf{A}$ for every infinite cardinal α.*

Proof. $\boxed{\Rightarrow}$ Let $\mathrm{S}\mathbf{A} = \mathbf{Top}$. Then S^α is a subspace of some $X \in \mathbf{Top}$. By Proposition 2.2.8 there exists a retraction from X to S^α. Every retraction is a quotient map, hence $S^\alpha \in \mathbf{A}$.

$\boxed{\Leftarrow}$ If $S^\alpha \in \mathbf{A}$ for all cardinals α, then all prime spaces belong to $\mathrm{S}\mathbf{A}$. Since $\mathrm{S}\mathbf{A}$ is coreflective in **Top**, this implies that $\mathrm{S}\mathbf{A}$ contains all topological spaces. □

Corollary 2.2.10. *If $\{\mathbf{A}_i, i \in I\}$ is a nonempty collection of coreflective subcategories of* **Top** *such that* $\mathrm{S}\mathbf{A}_i = \mathbf{Top}$ *for each $i \in I$ and $\mathbf{A} = \bigcap\{\mathbf{A}_i, i \in I\}$, then* $\mathrm{S}\mathbf{A} = \mathbf{Top}$.

If, moreover, for each $i \in I$ the coreflective hereditary kernel of $\mathbf{A}_i$ is **FG**, *then, obviously, the coreflective hereditary kernel of* **A** *is again* **FG**.

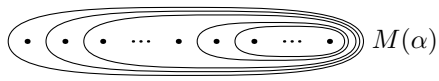
Corollary 2.2.11. *The class* $\mathbf{A}_0 = \mathrm{CH}(\{S^\alpha; \alpha \in \mathrm{Cn}\})$ *is the smallest coreflective subcategory of* **Top** *such that* $\mathrm{S}\mathbf{A} = \mathbf{Top}$.

The coreflective hereditary kernel of $\mathbf{A}_0$ is **FG** (since $\mathbf{FG} \subseteq \mathbf{A} \subseteq \mathbf{PsRad}$ and the hereditary coreflective kernel of **PsRad** is **FG** – see the note after Proposition 4.1.2).

Since S^α is an absolute retract in the category $\mathbf{Top}_0$ too, Theorem 2.2.9 and Corollaries 2.2.10 and 2.2.11 remain valid after replacing **Top** by $\mathbf{Top}_0$ (the category of T_0-spaces).

We next construct another class $\{M(\alpha); \alpha \text{ is a regular cardinal}\}$ of spaces, which generates the same coreflective subcategory of **Top**.

Let α be an infinite cardinal. We will use the sets $B_\beta = \{\gamma \in \alpha \cup \{\alpha\}; \gamma \geq \beta\}$ for each $\beta \in \alpha$. (The same sets were used in the definition of spaces $C(\alpha)$ and $B(\alpha)$.) We denote by $M(\alpha)$ the topological space on the set $\alpha \cup \{\alpha\}$ with the topology consisting of all B_β's, β being a non-limit ordinal less then α or $\beta = 0$.



We next show that the spaces $M(\alpha)$ are absolute retracts in **Top**.

Proposition 2.2.12. *Let α be an infinite cardinal and $M(\alpha)$ be a subspace of X. Then there exists a retraction $f\colon X \to M(\alpha)$.*

Proof. For every non-limit ordinal $\beta < \alpha$ denote by U_β the union of all open subsets of X with $U \cap M(\alpha) = B_\beta$ and put $U_0 = X$. Clearly, if $0 \leq \beta < \beta' < \alpha$ then $U_\beta \supsetneqq U_{\beta'}$ and $U_\beta \cap M(\alpha) = B_\beta$ for each $\beta < \alpha$. Define $f\colon X \to M(\alpha)$ by

$$f(x) = \sup\{\beta \in \alpha : x \in U_\beta\}.$$

Obviously, $f^{-1}(B_\beta) = U_\beta$ for every non-limit ordinal $\beta < \alpha$. Thus f is continuous. Moreover we have $f(\beta) = \beta$ for $\beta \in M(\alpha)$ and f is a retraction. □

Theorem 2.2.13. *Let* **A** *be a coreflective subcategory of* **Top** *and α be an infinite cardinal. The following statements are equivalent:*

(i) $\mathbf{Psrad}(\alpha) \subseteq \mathrm{S}\mathbf{A}$

(ii) $S^\alpha \in \mathbf{A}$

(iii) $M(\alpha) \in \mathbf{A}$

Proof. (i) ⇒ (ii) By Proposition 2.2.1 $S^\alpha \in \mathbf{Psrad}(\alpha)$. Hence $S^\alpha \in \mathrm{S}\mathbf{A}$, i.e. S^α is a subspace of a space $X \in \mathbf{A}$. Since S^α is an absolute retract in **Top**, there exists a retraction $f\colon X \to S^\alpha$, thus $S^\alpha \in \mathbf{A}$.

(ii) ⇒ (iii) Let $S^\alpha \in \mathbf{A}$. The weight of the space $M(\alpha)$ is $w(M(\alpha)) = \alpha$, therefore $M(\alpha)$ is a subspace of S^α by Theorem 1.4.2. Then by Proposition 2.2.12 there exists a retraction $g\colon S^\alpha \to M(\alpha)$ and $M(\alpha) \in \mathbf{A}$.

(iii) ⇒ (i) Let $M(\alpha) \in \mathbf{A}$. Clearly, $M(\beta)$ is a subspace of $M(\alpha)$ for every $\beta < \alpha$. ($M(\beta)$ is the subspace on the set $\beta \cup \{\beta\}$.) Thus for every $\beta \leq \alpha$ we have $M(\beta) \in \mathrm{S}\mathbf{A}$ and $C(\beta) = (M(\beta))_\beta \in \mathrm{S}\mathbf{A}$ (using Theorem 2.1.9). Therefore $\mathbf{Psrad}(\alpha) = \mathrm{CH}(\{C(\beta); \beta \leq \alpha\}) \subseteq \mathrm{S}\mathbf{A}$. □

Corollary 2.2.14. *Let α be an infinite cardinal. Then* $\mathrm{CH}(M(\alpha)) = \mathrm{CH}(S^\alpha)$ *and this is the smallest coreflective subcategory of* **Top** *with* $\mathbf{Psrad}(\alpha) \subseteq \mathrm{S}\mathbf{A}$.

Corollary 2.2.15. $\mathrm{CH}(\{M(\alpha); \alpha \textit{ is regular cardinal}\}) = \mathrm{CH}(\{S^\alpha; \alpha \in \mathrm{Cn}\})$

Corollary 2.2.16. *Let* **A** *be a coreflective subcategory of* **Top**. *Then* $\mathrm{S}\mathbf{A} = \mathbf{Top}$ *if and only if $M(\alpha) \in \mathbf{A}$ for every regular cardinal α.*

We have found two classes of spaces such that if a coreflective subcategory **A** of **Top** contains all these spaces, then $\mathrm{S}\mathbf{A} = \mathbf{Top}$. The following proposition suggests, how other classes of spaces with this property could be found.

Proposition 2.2.17. *Let* $\mathbf{A}$ *be a coreflective subcategory of* **Top**. *Then* $\mathrm{S}\mathbf{A} = \mathbf{Top}$ *if and only if for every regular cardinal* α *the subcategory* $\mathbf{A}$ *contains a space* X *such that there exists* $x \in X$ *with* $t(X, x) = \alpha$ *and for* $\alpha = \omega_0$ *the prime factor* X_x *of* X *at* x *is, moreover, not finitely generated.*

Proof. One direction follows from $t(M(\alpha), \alpha) = \alpha$ and Theorem 2.2.13.

Now let $t(X, x) = \alpha$ and $X \in \mathbf{A}$. Then there exist $V \subseteq X$ and $x \in X$ with $\operatorname{card} V = \alpha$, $x \in \overline{V}$ and $x \notin \overline{U}$ for any $U \subseteq V$, $\operatorname{card} U < \alpha$. Let Y be the subspace of X on the set $V \cup \{x\}$. Y belongs to $\mathrm{S}\mathbf{A}$ and by Theorem 2.1.9 Y_x also belongs to $\mathrm{S}\mathbf{A}$.

Next we want to prove that $B(\alpha) \in \mathrm{S}\mathbf{A}$.

We claim that the topological space Y_x is finer than $B(\alpha)$. Indeed, if $x \in U \subseteq Y$ and $\operatorname{card}(V \setminus U) < \alpha$ then $x \notin \overline{V \setminus U}$ and U is neighborhood of x in Y, hence U is open in Y_x. Clearly, the set $\{x\}$ is not open in Y.

Since $\operatorname{card} Y = \alpha$, we can assume that Y is a topological space on the set $\alpha \cup \{\alpha\}$ and $x = \alpha$. For every $\gamma < \delta \leq \alpha$ let $S_{\gamma\delta}$ be a Sierpiński topological space on the set $\{\gamma, \delta\}$ with the set $\{\delta\}$ open. A subset $U \subseteq \alpha \cup \{\alpha\}$ is open in $B(\alpha)$ if and only if it is open in Y_x and $U \cap \{\gamma, \delta\}$ is open in $S_{\gamma\delta}$ for every $\gamma < \delta \leq \alpha$ (i.e. U contains with $\gamma \in U$ every $\delta > \gamma$). Thus $B(\alpha)$ is a quotient space of $Y_x \sqcup (\coprod S_{\gamma\delta})$ and $B(\alpha) \in \mathrm{S}\mathbf{A}$.

Then the prime factor $(B(\alpha))_\alpha = C(\alpha)$ belongs to $\mathrm{S}\mathbf{A}$ for every regular cardinal α, hence $\mathbf{PsRad} \subseteq \mathrm{S}\mathbf{A}$ and by Theorem 2.2.3 $\mathrm{S}\mathbf{A} = \mathbf{Top}$. □

As an application of results from this section we reprove the well-known result of V. Kannan [K2] on the classes of spaces closed under all four basic topological operations – subspaces, products, sums and quotients. (In fact he proved a stronger result than the following proposition - he has shown that there is no subcategory of **Top** which is at the same time reflective and coreflective.)

Proposition 2.2.18. *If* $\mathbf{A}$ *is a subcategory of* **Top** *which is simultaneously epireflective and coreflective, then* $\mathbf{A} = \mathbf{Top}$.

Proof. Let $\mathbf{A}$ be such a category. Then it contains one-point space (the empty product) and the 2-point discrete space D_2 (the sum of two one-point spaces). By productivity $D_2^{\omega_0} \in \mathbf{A}$. The space $D_2^{\omega_0}$ is not a sum of indiscrete spaces, therefore $S \in \mathbf{A}$ and $S^\alpha \in \mathbf{A}$ for each infinite cardinal α. Thus we get by Theorem 2.2.9 that $\mathbf{A} = \mathrm{S}\mathbf{A} = \mathbf{Top}$. □

Chapter 3

Generators of hereditary coreflective subcategories

The purpose of this chapter is to generalize the results from the paper [FR] of S. P. Franklin and M. Rajagopalan dealing with subsequential spaces to the category SCH(A) which is the hereditary coreflective hull of a single prime space A. We have already seen how we can generalize results obtained for sequential spaces to similar situation. Therefore it is not very hard to generalize the methods used in [FR] like the sequential sum and the canonical sets of generators constructed using the sequential sum to the analogous constructions appropriate for our situation.

A countable generator of the subcategory $\mathbf{SSeq} = \mathrm{SCH}(C(\omega_0))$ of subsequential spaces is constructed in [FR] using maximal almost disjoint families on a countable set. We succeeded to construct a generator $(A_\omega)_a$ of the subcategory SCH(A) such that $\mathrm{card}(A_\omega)_a = \mathrm{card}\, A$. Our construction is different from that one used in [FR] for the special case $A = C(\omega_0)$. In this special case $A_\omega = S_\omega$ holds, where S_ω is the space defined in [AF].

In the paper [TU] the space S_ω and some related spaces were studied. The authors defined here a space $\tau_{\vec{\mathcal{F}}}$ such that A_ω is a special case of this space.

The results of Sections 3.1, 3.2 and Subsection 3.3.1 have been published in [Sl3]. Proposition 3.3.11 is included in the (not yet published) paper [Sl4].

3.1 A-sum

The basic tool in the construction of a countable generator of **SSeq** in [FR] is the sequential sum. Now we introduce the A-sum, whose construction is similar to the construction of the sequential sum. The difference is that in the definition of the A-sum we use the prime space A in place of $C(\omega_0)$. Both, the A-sum and the sequential sum, are special cases of the brush from [K4]. The product $*$ defined in [Č3] is a special case of the A-sum (in this case, every summand is the same prime space).

Definition 3.1.1. Let A be a prime space with the accumulation point $a \in A$. Let us denote $B := A \setminus \{a\}$. For each $b \in B$ let X_b be a topological space and $x_b \in X_b$. Then the *A-sum* $\sum_A \langle X_b, x_b \rangle$ is the topological space on the set $F = A \cup (\bigcup_{b \in B} \{b\} \times (X_b \setminus \{x_b\}))$ such that the map $\varphi\colon A \sqcup (\coprod_{b \in B} X_b) \to F$ given by $\varphi(x) = x$ for $x \in A$, $\varphi(x) = (b, x)$ for $x \in X_b \setminus \{x_b\}$ and $\varphi(x_b) = b$ for every $b \in B$ is a quotient map. (We assume A and all $\{b\} \times X_b$ to be disjoint.) The map φ will be called the *defining map* of the A-sum.

Often it will be clear from the context what we mean under A and we will abbreviate the notation of the A-sum to $\sum \langle X_b, x_b \rangle$ or $\sum X_b$.

The A-sum is obtained simply by identifying every $x_b \in X_b$ with the point $b \in A$. It is easy to see that A is a subspace of the A-sum $\sum \langle X_b, x_b \rangle$ and the subspace $\varphi[X_b]$ is homeomorphic to X_b. The subspaces $\varphi[X_b]$ will be called the *bristles* of the A-sum. The point $b \in \sum X_b$ is said to be the *root* of this bristle. (This terminology follows [K4, Construction 4.2.1] where V. Kannan defines a notion of a brush.)

The A-sum is defined using topological sum and quotient map, thus if $\mathbf{A}$ is a coreflective subcategory of **Top** and $\mathbf{A}$ contains A and all X_b's, then the A-sum $\sum X_b$ belongs to $\mathbf{A}$.

If we take the space $C(\omega_0)$ for the space A, we obtain precisely the construction of the *sequential sum* as it was defined in [AF] and [FR].

The following lemma follows easily from the definition of the A-sum.

Lemma 3.1.2. *Let A be a prime space with the accumulation point a and $B = A \setminus \{a\}$. A subset $U \subseteq \sum_A \langle X_b, x_b \rangle$ is open (closed) in $\sum X_b$ if and only if $U \cap A$ is open (closed) in A and $U \cap \varphi[X_b]$ is open (closed) in $\varphi[X_b]$ for every $b \in B$.*

For every $b \in B$ let X_b and Y_b be topological spaces, $x_b \in X_b$, $y_b \in Y_b$ and let $f_b\colon X_b \to Y_b$ be a function with $y_b = f_b(x_b)$. Then we can define a map

$$f =: \sum f_b\colon \sum_A \langle X_b, x_b \rangle \to \sum_A \langle Y_b, y_b \rangle$$

by $f(b, x) = (b, f_b(x))$ for $x \in X_b \setminus \{x_b\}$ and $f(x) = x$ for $x \in A$. Let us note that $f \circ \varphi_1|_{X_b} = \varphi_2|_{Y_b} \circ f_b$ where φ_1 and φ_2 are the defining maps of the A-sums $\sum X_b$ and $\sum Y_b$ respectively.

We next show that continuity, embeddings and quotient maps are preserved by this construction.

Lemma 3.1.3. *Let A be a prime space with the accumulation point a and $B = A \setminus \{a\}$. For every $b \in B$ let $f_b\colon X_b \to Y_b$ be a map between topological spaces $x_b \in X_b$, $y_b \in Y_b$ and $f_b(x_b) = y_b$.*

(i) *If all f_b's are continuous, then $\sum f_b$ is continuous.*

(ii) *If all f_b's are quotient maps, then $\sum f_b$ is a quotient map.*

(iii) *If all f_b's are embeddings, then $\sum f_b$ is an embedding.*

(iv) *If all f_b's are homeomorphisms, then $\sum f_b$ is a homeomorphism.*

(v) *Let C be a prime subspace of A. Then $\sum_C \langle X_b, x_b \rangle$ is a subspace of the space $\sum_A \langle X_b, x_b \rangle$.*

Proof. Put $f = \sum f_b$ and let φ_1, φ_2 be the defining maps of the A-sums $\sum \langle X_b, x_b \rangle$, $\sum \langle Y_b, y_b \rangle$ respectively. Let us denote the map $id_A \sqcup (\coprod_{b \in B} f_b)$ by h. Then the following diagram commutes.

$$\begin{tikzcd} A \sqcup (\coprod X_b) \arrow[r, "h"] \arrow[d, "\varphi_1"'] & A \sqcup (\coprod Y_b) \arrow[d, "\varphi_2"'] \\ \sum \langle X_b, x_b \rangle \arrow[r, "f"'] & \sum \langle Y_b, y_b \rangle \end{tikzcd}$$

The validity of (i) and (ii) follows easily from the commutativity of this diagram and from the fact that φ_1, φ_2 are quotient.

(iii) Now, suppose that all f_b's are embeddings. W.l.o.g. we can assume that $X_b \subseteq Y_b$ and f_b is the inclusion of X_b into Y_b for every $b \in B$. Let X' be the subspace of the space $\sum Y_b$ on the set $\sum X_b$. We have the following situation:

$$\begin{tikzcd} A \sqcup (\coprod X_b) \arrow[r, hook, "h"] \arrow[d, "\varphi_1"'] & A \sqcup (\coprod Y_b) \arrow[d, "\varphi_2"'] \\ X' \arrow[r, hook, "f"'] & \sum Y_b \end{tikzcd}$$

We only need to prove that X' has the quotient topology with respect to φ_1, because this implies that $X' = \sum X_b$ and f is embedding of $X' = \sum X_b$ to $\sum Y_b$. But φ_2 is one-to-one outside the set $A \sqcup (\coprod X_b)$ and Lemma 2.1.4 implies that φ_1 is a quotient map.

(iv) is an easy consequence of (ii) and (iii). (v) follows easily from the definition of the A-sum. □

Let us note, that if for every $b \in B$ the map f_b is an embedding which maps isolated points of X_b to isolated points of Y_b, the embedding $\sum f_b$ has the same property.

Corollary 3.1.4. *Let A be a prime space with the accumulation point a and let C be a prime subspace of A. For every $b \in A \setminus \{a\}$ let X_b be a topological space and $x_b \in X_b$. For every $c \in C$ let Y_c be a subspace of X_c such that $x_c \in Y_c$. Then $\sum_C \langle Y_c, x_c \rangle$ is a subspace of the space $\sum_A \langle X_b, x_b \rangle$.*

3.2 The sets TS_γ, TSS_γ

In the rest of this chapter we will assume that we are given a prime space A with the accumulation point a. In this section α will denote the tightness $t(A)$ of the prime space A.

We define for every $\gamma < \alpha^+$ two families TS_γ and TSS_γ of topological spaces in a similar manner as in [FR] for $A = C(\omega_0)$. Our goal is to show that the sets TSS_γ together generate SCH(A) as a coreflective subcategory of **Top**. This will help us to construct a single space A_ω whose coreflective hull is SCH(A) in the next section.

Let $TS_0 = \emptyset$ and TS_1 be the set of all prime subspaces of A.

If $\beta \geq 1$ is an ordinal, then $TS_{\beta+1}$ consists of all B-sums $\sum_B \langle X_b, x_b \rangle$ where B is a prime subspace of A, each $X_b \in TS_\beta$ and $x_b = a$.

If $\gamma > 0$ is a limit ordinal, then $TS_\gamma = \bigcup_{\beta<\gamma} TS_\beta$.

Sometimes, if we want to emphasize which prime space A is used to construct this set, we use the notation $TS_\gamma(A)$.

Every space belonging to TS_γ contains some prime subspace B of A and therefore it contains a. All sets from TS_γ are constructed from A using B-sums, where $B \in \mathrm{CH}(A)$, thus $TS_\gamma(A) \subseteq \mathrm{CH}(A)$ for each γ.

The following lemma is a generalization of [FR, Lemma 6.2].

Lemma 3.2.1. *Let X be a topological space and $M \subseteq X$. If $p \in M_\beta \setminus M_\gamma$ for any $\gamma < \beta$, then there exists a space $S \in TS_\beta$ and a continuous map $f\colon S \to X$, which maps all isolated points of S into M and maps only the point a to p.*

Proof. For $\beta = 1$ the claim follows from the definition of M_1.

From the definition of M_β it follows that β is a non-limit ordinal. According to Proposition 1.5.12 $\beta < \alpha^+$. Suppose the assertion is true for any subset M of X and for any $\beta' < \beta$.

For a non-limit $\beta > 1$ there exists a prime subspace B of A and a continuous map $f\colon B \to X$ such that $f(a) = p$ and $f[B \setminus \{a\}] \subseteq M_{\beta-1}$.

If $\beta-1$ is non-limit, we can moreover assume that $f[B\setminus\{a\}] \subseteq M_{\beta-1}\setminus M_{\beta-2}$. (If necessary, we choose $B' = \{a\} \cup \{b \in B : f(b) \in M_{\beta-1}\setminus M_{\beta-2}\}$ and $f' = f|_{B'}$. B' is a prime subspace of A, otherwise we would get $x \in M_{\beta-1}$.)

If $\beta - 1$ is a limit ordinal, then for each point $x \in M_{\beta-1}$ there exists the smallest ordinal $\gamma < \beta-1$ such that $x \in M_\gamma$. Obviously, γ is a non-limit ordinal.

Thus for each $x \in f[B \setminus \{a\}]$ there exists a continuous map $f_x\colon S_x \to X$, where $S_x \in TS_{\beta-1}$, which sends all isolated points of S_x into M and a to x.

Then $\sum_B \langle S_{f(b)}, a \rangle \in TS_\beta$ and we can define a map $g\colon \sum_B \langle S_{f(b)}, a \rangle \to X$ such that $g|_B = f$ and $g|_{\{x\}\times(S_x\setminus\{a\})}(x, y) = f_x(y)$ for $y \in S_x \setminus \{a\}$. Clearly, g maps isolated points into M. It remains only to show that g is continuous.

The defining map $\varphi\colon B \sqcup (\coprod_{b\in B\setminus\{a\}} S_{f(b)}) \to \sum \langle S_{f(b)}, a \rangle$ is a quotient map. Hence $g\colon \sum \langle S_{f(b)}, a \rangle \to X$ is continuous if and only if $g \circ \varphi$ is continuous. But $g \circ \varphi|_B = f$ and $g \circ \varphi|_{S_x} = f_x$ are continuous, thus g is continuous. □

For any $S \in TS_\gamma$ we denote by $P(S)$ the subspace of the space S which consists of all isolated points of S and of the point a. Clearly, $P(S)$ is a prime space. We denote by TSS_γ the set of all spaces $P(S)$ where $S \in TS_\gamma$. The above lemma implies:

Lemma 3.2.2. *If $p \in M_\beta$ and $p \notin M_\gamma$ for any $\gamma < \beta$, then there exists a space $T \in TSS_\beta$ and a continuous map $f\colon T \to X$, which maps all isolated points of the space T into M and such that $f(a) = p$.*

Proposition 3.2.3. SCH(A) *is generated by the set* $\bigcup_{\gamma<\alpha^+} TSS_\gamma$.

Proof. Let $X \in$ SCH(A). According to Lemma 1.5.6 it suffices to prove that for any subset $M \subseteq X$ and any $x \in \overline{M} \setminus M$ there exists $T \in \bigcup_{\gamma<\alpha^+} TSS_\gamma$ and a continuous map $f\colon M \to X$ such that $f(a) = x$ and $f[T \setminus \{a\}] \subseteq M$.

Since $X \in$ SCH(A) there exists $Y \in$ CH(A) such that X is a subspace of Y. Denote by $\overline{M}^Y$ the closure of M in Y. Then $\overline{M} = \overline{M}^Y \cap X$ and $x \in \overline{M}^Y \setminus M$ in Y. By Proposition 1.5.7 $\overline{M}^Y = M_{\alpha^+} = \bigcup_{\beta<\alpha^+} M_\beta$. Let β be the smallest ordinal with $x \in M_\beta$. Then $\beta > 0$ and $x \notin M_\gamma$ for any $\gamma < \beta$. By Lemma 3.2.1 there exists $S \in TS_\beta$ and a continuous map $f\colon S \to Y$ with $f(a) = x$ and $f(c) \in M$ for any isolated point of S. Then $P(S) \in TSS_\gamma$ and $f[P(S)] \subseteq X$. Hence $f|_{P(S)}\colon P(S) \to X$ is a continuous map satisfying the required conditions. Consequently, $X \in$ CH$(\bigcup_{\gamma<\alpha^+} TSS_\gamma)$. □

It can be easily seen that if we define the sets $T'S_\gamma$, $\gamma < \alpha^+$, similarly as the sets TS_γ but we use only the A-sums (and not all B-sums for all prime subspaces B of A) and then we put $T'SS_\gamma = \{P(S) : S \in T'S_\gamma\}$ we obtain the set $\bigcup_{\gamma<\alpha^+} T'SS_\gamma$ which also generates SCH(A). This follows from the fact that any space from $\bigcup_{\gamma<\alpha^+} TSS_\gamma$ is a prime subspace of some space from $\bigcup_{\gamma<\alpha^+} T'SS_\gamma$.

One can also note, that the same proofs work for analogous claims on a class $\mathbf{A}$ of prime spaces, since Lemma 1.5.6 holds for a family of prime spaces as well. In this case we would have to put $\alpha = \sup\{t(A); A \in \mathbf{A}\}$, see Proposition 1.5.12.

3.3 The spaces A_ω and $(A_\omega)_a$

In this section we define, for any prime space A, the space A_ω and show that its prime factor is a generator of SCH($\mathbf{A}$) – this is the main result of this section.

Apart from this we show some properties of these two spaces. Namely, we show that A_ω is homogeneous and zero-dimensional whenever A is T_2. We also show that $(A_\omega)_a$ can be embedded into a zero-dimensional T_2-space belonging to CH(A) for a prime T_2-space A.

3.3.1 $(A_\omega)_a$ generates SCH(A)

The space A_ω is defined similarly as S_ω in [AF] using the A-sum and the prime space A instead of the sequential sum and the space $C(\omega_0)$. We start by defining the space A_n for each $n \in \mathbb{N}$ putting $A_1 = A$ and $A_{n+1} = \sum_A \langle A_n, a\rangle$. Clearly, A_1 is a subspace of A_2 and if A_{n-1} is a subspace of A_n, then, according to Lemma 3.1.3, $A_n = \sum_A \langle A_{n-1}, a\rangle$ is a subspace of $A_{n+1} = \sum_A \langle A_n, a\rangle$. Hence, A_n is a subspace of A_{n+1} for each $n \in \mathbb{N}$.

The Figure 3.1 represents the space A_3 for $A = C(\omega_0)$.

Let us mention another possibility, how the spaces A_n can be constructed.

Lemma 3.3.1. *Let $n \geq 2$. The space A_n can be obtained from A_{n-1} by attaching the prime space A to every isolated point of A_{n-1}.*

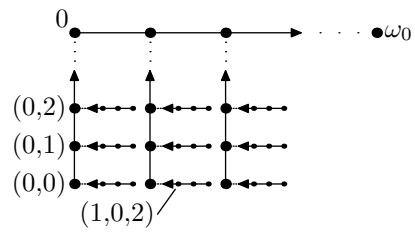


Figure 3.1: The space A_3 for $A = C(\omega_0)$

Proof. To simplify the notation we denote $B = A \setminus \{a\}$ in this proof.

For $n = 2$ the claim follows directly from the definition of A_2.

Suppose the claim of the lemma holds for n. I.e., we have a quotient map $f\colon A_{n-1} \sqcup (\coprod_{x\in B^{n-1}} A) \to A_n$ which corresponds to the attaching the space A to each $x \in B^{n-1}$.

Let $f_b = f$ for each $b \in B$. Then $\sum f_b\colon \sum(A_{n-1} \sqcup (\coprod_{x\in B^{n-1}} A)) \to \sum A_n$ is quotient by Lemma 3.1.3. Clearly, $\sum A_n = A_{n+1}$.

One can see that $\sum(A_{n-1} \sqcup (\coprod_{x\in B^{n-1}} A)) \cong (\sum A_{n-1}) \sqcup (\coprod_{x\in B^n} A)$ and the map $\sum f_b$ is (up to a homeomorphism) precisely the map from the claim for $n + 1$. □

Definition 3.3.2. The space A_ω is the space on the set $\bigcup_{n\in\mathbb{N}} A_n$ such that a subset U of $\bigcup_{n\in\mathbb{N}} A_n$ is open in A_ω if and only if $U \cap A_n$ is open in A_n for every $n \in \mathbb{N}$.

It is obvious that for every $n \in \mathbb{N}$ the space A_n is a subspace of A_ω and A_ω is a quotient space of the topological sum $\coprod_{n\in\mathbb{N}} A_n$. Consequently, A_ω belongs to $\mathrm{CH}(A)$. Observe that A_ω can be understood as the inductive limit of its subspaces A_n, $n \in \mathbb{N}$.

Similarly as the space S_ω in [AF] the space A_ω has the following important property.

Proposition 3.3.3. $A_\omega = \sum_A \langle A_\omega, a\rangle$

Proof. Put $X = \sum_A \langle A_\omega, a\rangle$. For each $n \in \mathbb{N}$ the space A_n is a subspace of A_ω and it follows that $A_{n+1} = \sum_A \langle A_n, a\rangle$ is a subspace of X (Lemma 3.1.3). Obviously, $A = A_1$ is also a subspace of X. Therefore A_n is a subspace of X for each $n \in \mathbb{N}$. Clearly, $X = \bigcup_{n\in\mathbb{N}} A_n$.

To finish the proof it suffices to check that X has the final topology with respect to embeddings of A_n's. More precisely, we will show that if U is a subset of X and $U \cap A_n$ is open in A_n for each $n \in \mathbb{N}$, then U is open in X.

Let us denote by A_n^b the subspace of X on the set $\{b\} \cup (\{b\} \times (A_n \setminus \{a\}))$ and by A_ω^b the subspace of X on the set $\{b\} \cup (\{b\} \times (A_\omega \setminus \{a\}))$. (This means that A_ω^b is the bristle of the A-sum with the root b and A_n^b is the subset of this bristle corresponding to the subspace A_n of A_ω.) Clearly A_n^b is homeomorphic to A_n and A_ω^b is homeomorphic to A_ω, A_n^b is a subspace of A_ω^b and subset V of A_ω^b is open in A_ω^b if and only if $V \cap A_n^b$ is open in A_n^b for each $n \in \mathbb{N}$.

If $U \subseteq X$ and for all $n \in \mathbb{N}$ the intersection $U \cap A_n$ is open in A_n, then $U \cap A$ is open in A and $U \cap A_{n+1}$ is open in $A_{n+1} = \sum_A \langle A_n, a\rangle$ for all $n \in \mathbb{N}$. Then $U \cap A_n^b$ is open in A_n^b for each $n \in \mathbb{N}$ and $b \in A \setminus \{a\}$ and it follows that $U \cap A_\omega^b$ is open in A_ω^b for each $b \in B \setminus \{a\}$. Hence, U is open in X. □

Lemma 3.3.4. *If A is infinite, then* $\operatorname{card} A_\omega = \operatorname{card} A$.

Proof. $\operatorname{card} A_\omega = \operatorname{card} \bigcup_{n\in\mathbb{N}} A_n = \operatorname{card} A$ □

Lemma 3.3.5. *For every ordinal γ, $1 \le \gamma < \alpha^+$ and every space $S \in TS_\gamma$ the space S is a subspace of A_ω. (Clearly, the point a of S coincides with the point a of A_ω.)*

Proof. If $\gamma = 1$ then $S = B$ is a prime subspace of $A = A_1$. Let γ be an ordinal, $1 < \gamma < \alpha^+$ and suppose that the assertion holds for every ordinal β, $1 \le \beta < \gamma$. If $S = \sum_B X_b \in TS_\gamma$ then, for each $b \in B \setminus \{a\}$, $X_b \in TS_\beta$ with $1 \le \beta < \gamma$. Hence, for each $b \in B \setminus \{a\}$, X_b is a subspace of A_ω and, according to Corollary 3.1.4 S is a subspace of $A_\omega = \sum_A \langle A_\omega, a\rangle$. □

The following two theorems are the principal results of this chapter.

Theorem 3.3.6. *Let $(A_\omega)_a$ be the prime factor of the space A_ω at a. Then $(A_\omega)_a$ is a prime space,* $\mathrm{CH}((A_\omega)_a) = \mathrm{SCH(A)}$ *and* $\operatorname{card}(A_\omega)_a = \operatorname{card} A$.

Proof. Evidently, $(A_\omega)_a$ is a prime space and $\operatorname{card}(A_\omega)_a = \operatorname{card} A$. Since A_ω belongs to $\mathrm{CH}(A) \subseteq \mathrm{SCH(A)}$, according to Theorem 2.1.9 $(A_\omega)_a$ belongs to SCH(A). Hence, it suffices to check that $\bigcup_{\gamma<\alpha^+} TSS_\gamma \subseteq \mathrm{CH}((A_\omega)_a)$.

Let $T \in \bigcup_{\gamma<\alpha^+} TSS_\gamma$. Then there exists an ordinal γ, $1 \le \gamma < \alpha^+$, and $S \in TS_\gamma$ such that $T = P(S)$. By Lemma 3.3.5 S is a subspace of A_ω and, clearly, it follows that $T = P(S)$ is a subspace of $(A_\omega)_a$. Consequently, by Lemma 1.2.3 there exists a quotient map $(A_\omega)_a \to T$ and we obtain that T belongs to $\mathrm{CH}((A_\omega)_a)$. □

Theorem 3.3.7. *Let X be a topological space. Then there is a prime space $((A_X)_\omega)_a$ such that* $\mathrm{SCH}(X) = \mathrm{CH}(((A_X)_\omega)_a)$ *and* $\operatorname{card}((A_X)_\omega)_a = \operatorname{card} X$.

Proof. Theorem 2.1.9 implies that $\mathrm{SCH}(X)$ is the coreflective hull of the prime factors of X. By the operation $\bigvee$ (see Proposition 1.5.10) we can construct from the (non-finitely generated) prime factors of X a single prime space A_X, which generates the same coreflective subcategory. Moreover, $\operatorname{card} A_X = \operatorname{card} X$. The rest follows easily from Theorem 3.3.6. □

As a special case of the construction provided in Theorem 3.3.6 we can obtain a countable generator of the subcategory **SSeq** of all subsequential spaces. Since $\mathbf{Seq} = \mathrm{CH}(C(\omega_0))$, it suffices to put $A = C(\omega_0)$. The obtained space A_ω is the same as the space S_ω defined in [AF], which in fact inspired our construction. We have already mentioned that a different countable generator for this subcategory was constructed in [FR].

Another example, which turns out to be useful, is the countable generator $(C(\alpha)_\omega)_\alpha$ for the subcategory $\mathrm{SCH}(C(\alpha))$ with cardinality α. Using the wedge sum $\bigvee$ (see Proposition 1.5.10) we can construct a generator of the same cardinality for the subcategory $\mathbf{SPsrad}(\alpha)$.

We include here also some other facts concerning the space A_ω and its prime factors, which we will need later.

Recall (see Example 1.5.33) that a topological space X belongs to $\mathbf{Top}(\omega_1)$ if and only if every countable intersection of open subsets of X is open in X and $\mathbf{Top}(\omega_1)$ is a hereditary coreflective subcategory of $\mathbf{Top}$. If the space A belongs to $\mathbf{Top}(\omega_1)$ we can find smaller (and simpler) set of generators of SCH(A) than the set $\bigcup_{\gamma<\alpha^+} TSS_\gamma(A)$ constructed in Proposition 3.2.3. Observe that if $A \in \mathbf{Top}(\omega_1)$ then $\mathrm{SCH(A)} \subseteq \mathbf{Top}(\omega_1)$.

Proposition 3.3.8. *If* $A \in \mathbf{Top}(\omega_1)$ *then* $\mathrm{SCH(A)} = \mathrm{CH}(\{P(A_n); 0 < n < \omega_0\})$.

Proof. It suffices to show that $(A_\omega)_a \in \mathrm{CH}(\{P(A_n); 0 < n < \omega_0\})$. Clearly, each $P(A_n)$ is a subspace of $(A_\omega)_a$. Denote by $i_n \colon P(A_n) \hookrightarrow (A_\omega)_a$ the corresponding embedding and let $f := [i_n] \colon \coprod_{n\in\mathbb{N}} P(A_n) \to (A_\omega)_a$. It is easy to see that this map is surjective. We claim that f is also a quotient map.

It suffices to show that if $a \in U \subseteq (A_\omega)_a$ and $U \cap P(A_n)$ is open in $P(A_n)$ for each $0 < n < \omega_0$, then U is open in $(A_\omega)_a$. Since $P(A_n)$ is a subspace of A_ω, there exists an open subset W_n of A_ω such that $W_n \cap P(A_n) = U \cap P(A_n)$. Put $W = \bigcap_{0<n<\omega_0} W_n$. The set W is open in A_ω since A_ω belongs to $\mathbf{Top}(\omega_1)$. We have $W \cap P(A_n) \subseteq U \cap P(A_n)$ and $\bigcup_{0<n<\omega_0} P(A_n) = A_\omega$, therefore $W \subseteq U$. Obviously, $a \in W$. Hence, the set U is open in $(A_\omega)_a$. □

3.3.2 A_ω is zero-dimensional

It was shown in [AF] that the space S_ω is zero-dimensional and homogenous. Since the construction of A_ω is a generalization of S_ω, one can ask whether the same holds for the space A_ω.

We show that if A is a prime T_2-space, then the space A_ω is zero-dimensional and homogeneous. As we can see from Lemma 1.5.15, the condition that A is T_2 is not too restrictive.

To show that the T_2 separation axiom cannot be omitted it suffices to take any non-Hausdorff prime space. Such a space contains the Sierpiński space S as a subspace. Clearly, S is not zero-dimensional and the one-point set $\{a\}$ is closed in A_ω, so there exists no homeomorphism mapping the point a to a point b such that $\{b\}$ is not closed.

In order to show that A_ω is zero-dimensional, we will construct a clopen base for this space.

We first show that $\mathcal{B} = \{U \subseteq A_\omega; a \in U, U \text{ is open in } A_\omega \text{ and } U \text{ fulfills (3.1)}\}$ is a base for A_ω at the point a.

{j4:EQ1}
$$(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) \in U \Rightarrow (x_1, x_2, \ldots, x_n) \in U \tag{3.1}$$

Lemma 3.3.9. *The family $\mathcal{B}$ defined above is a base for A_ω at the point a.*

Proof. Let V be an open neighborhood of a in A_ω. We want to find $U \in \mathcal{B}$ such that $U \subseteq V$. Let us put

$$\begin{aligned} U_1 &:= V \cap A_1 \\ U_2 &:= V \cap A_2 \cap [U_1 \cup (U_1 \times (A \setminus \{a\}))] \\ U_{n+1} &:= V \cap A_{n+1} \cap [U_n \cup (U_n \times (A \setminus \{a\}))] \end{aligned}$$

and $U := \bigcup_{n\in\mathbb{N}} U_n$.

Observe that $U_n \subseteq U_{n+1}$ for each n, $a \in U \subseteq V$, each U_n is open in A_n and $U \cap A_n = U_n$. (The sets $U_n \cup (U_n \times (A \setminus \{a\}))$ are open by Lemma 3.3.1.) Hence U is open in A_ω.

If $(x_1, \dots, x_{n+1}) \in U_{n+1}$ then $(x_1, \dots, x_{n+1}) \in U_n \times (A \setminus \{a\})$ and $(x_1, \dots, x_n) \in U_n$. By induction we get that (3.1) holds for U. □

Lemma 3.3.10. *All sets in $\mathcal{B}$ are clopen.*

Proof. Let $U \in \mathcal{B}$. If $x = (x_1, \dots, x_k) \notin U$, then no point of the form $(x_1, \dots, x_k, y_{k+1}, \dots, y_{k+l})$ belongs to U, i.e. for the open neighborhood $U_x = \{(x_1, \dots, x_k)\} \cup \{(x_1, \dots, x_k)\} \times (A_\omega \setminus \{a\})$ the equality $U_x \cap U = \emptyset$ holds.

Hence $x \in \operatorname{int}(A_\omega \setminus U)$, $A_\omega \setminus U$ is open, U is closed. □

We have already noticed in that the bristles in an A-sum are homeomorphic to the original spaces. By Proposition 3.3.3 we get in this case that A_ω is homeomorphic to its proper subspace on the subset $\{b\} \cup (\{b\} \times (A_\omega \setminus \{b\}))$. In fact this homeomorphism is given by $h_b(a) = b$ and $h_b(x_1, \dots, x_n) = (b, x_1, \dots, x_n)$.

If A is T_2 then the bristles are clopen subspaces of A_ω. Using the homeomorphisms of the form h_b repeatedly we obtain from the clopen base $\mathcal{B}$ at a a clopen base at each point of A_ω. Thus we get finally

Proposition 3.3.11. *Let A be a prime T_2-space. Then the space A_ω is zero-dimensional for any prime space A.*

3.3.3 A_ω is homogeneous

As we have already mentioned, the space A_ω is homogeneous for any prime T_2-space. For the proof of this result we will need the following lemma.

Lemma 3.3.12. *Let A be a prime T_2-space. Any subspace of A obtained by omitting one isolated point of A is homeomorphic to A.*

Proof. Let $b \in A \setminus \{a\}$, where a denotes the accumulation point of A. We will show that A is homeomorphic to its subspace $A \setminus \{b\}$. We consider the following two possibilities: Either all closed subsets of $A \setminus \{a\}$ are finite or there is at least one infinite closed subset in A, which does not contain a.

If all closed subsets of isolated points in some prime space are finite, the topology of this space is fully determined by the cardinality of the underlying

set. Therefore A and $A \setminus \{b\}$ are homeomorphic in this case. (Note that every prime T_2-space is infinite.)

Now let D be an infinite closed subset of A not containing the accumulation point. We can assume $b \in D$, since $\{b\}$ is closed as well. Clearly, D is a clopen subset and A can be obtained as the topological sum of the discrete subspace D and the subspace on the set $A \setminus D$, $A = D \sqcup (A \setminus D)$. Similarly, $A \setminus \{b\} = (D \setminus \{b\}) \sqcup (A \setminus D)$ and any bijection $D \to D \setminus \{b\}$ gives rise to a homeomorphism of these two spaces. □

Proposition 3.3.13. *Let A be a prime T_2-space with the accumulation point a. Then the space A_ω is homogenous.*

Proof. It suffices to find for any point $b \in A \subseteq A_\omega$, $b \neq a$, a homeomorphism $h\colon A_\omega \to A_\omega$ such that $h(b) = a$. If we have such a homeomorphism, then by induction we can easily show that there exists a homeomorphism h' with $h'(x) = y$ for any pair $x, y \in A_\omega$.

Let $b \in A \subseteq A_\omega$. Let us denote by B the subspace of A_ω which is the bristle with the root b. (I.e., B is the subspace on the set $\{b\} \cup \{b\} \times (A_\omega \setminus \{b\})$.)

We first notice that the space A_ω is homeomorphic to its subspace $A_\omega \setminus B$. This follows from the fact that $A_\omega \setminus B$ is in fact the $(A \setminus \{b\})$-sum of copies of A_ω and from the existence of a homeomorphism between A and its subspace $A \setminus \{b\}$ (Lemma 3.3.12).

Moreover, the subset B is clopen in A_ω, since $\{b\}$ is clopen in A. This means that A_ω is the topological sum of the subspaces B and $A_\omega \setminus B$. Thus the homeomorphism $g\colon B \to A_\omega \setminus B$ such that $g(b) = a$ together with $g^{-1}\colon A_\omega \setminus B \to B$ give the autohomeomorphism $h = [g, g^{-1}]$ of $A_\omega = B \sqcup (A_\omega \setminus B)$ fulfilling $h(b) = a$. □

3.3.4 $(A_\omega)_a \in \mathrm{S}(\mathrm{CH}(A) \cap \mathbf{ZD_0})$

In the connection with the subsequential spaces also the question whether a T_2-space can be embedded into a sequential T_2-space was studied. Such spaces are called *T_2-subsequential*. Some observations and questions about them can be found already in [FR]. In [GH] it was shown that not every subsequential space which is T_2 must be T_2-subsequential.

Therefore it is reasonable to ask whether the generator $((C(\omega_0))_\omega)_{\omega_0}$ of the subcategory **SSeq** is T_2-subsequential. We can prove a more general result - for any prime T_2-space A the space $(A_\omega)_a$ can be embedded into a zero-dimensional T_2 space which belongs to the coreflective hull $\mathrm{CH}(A)$ of A.

Before constructing this space X we provide another useful description of the topology of A_ω.

To make our notation simpler, we will use B instead of $A \setminus \{a\}$ in the rest of this section.

The underlying set of A_ω is $\bigcup_{n=1}^{\infty} A_n$. Since $A_n = \{a\} \cup (\bigcup_{k=1}^{n} B^k)$, this can be rewritten as $A_\omega = \{a\} \cup (\bigcup_{n=1}^{\infty} B^n)$, so the points of the space A_ω are sequences of elements of B and the point a. For every such sequence $x =$

$(x_1, \dots, x_n) \in A_\omega$ we define a map $f_x \colon A \to A_\omega$ by $f(a) = x$ and $f(b) = (x_1, \dots, x_n, b)$ for $b \in A \setminus \{a\}$. We define f_a in a similar way: $f_a(b) = b$ for each $b \in A$. So f_a is simply the inclusion of A into A_ω.

Lemma 3.3.14. *The space A_ω has the quotient topology with respect to the map $[f_x] \colon \coprod_{x \in A_\omega} A \to A_\omega$.*

Proof. It suffices to observe that each A_n has the final topology with respect to the family f_x, $x \in A_{n-1} = \{a\} \cup (\bigcup_{k<n} B^k)$. (This can be shown directly from the definition of A_n by induction.) The space A_ω has the final topology w.r.t. the embeddings $i_n \colon A_n \hookrightarrow A_\omega$. By combining these final sinks we obtain precisely the family f_x, $x \in A_\omega$. (Some f_x's occur repeatedly, but this does not change the final topology.) $\square$

We next construct a new space X from A_ω by attaching a copy of the prime space A to each point of A_ω. I.e. $X \cong A_\omega \triangle_a A$, but we describe the underlying set and the topology of X in another way than in the definition of $X \triangle_b Y$ in order to make the notation in the following proofs simpler. The space X is illustrated by the Figure 3.2.

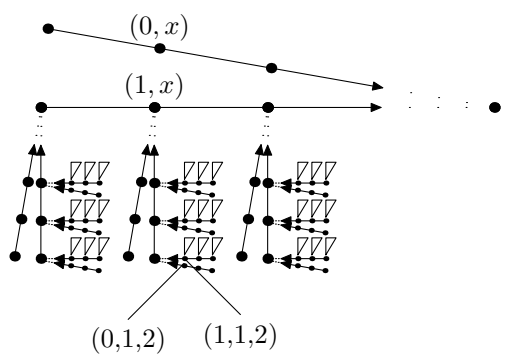


Figure 3.2: The space X for $A = C(\omega_0)$

Formally, let $X := \{a\} \cup \{0,1\} \times (\bigcup_{n \in \mathbb{N}} B^n)$. Let the topology of X be final with respect to the family of maps $g_x, h_x \colon A \to X$, $x \in A_\omega$, defined as follows: $g_a(a) = h_a(a) = a$, $g_a(b) = (0, b)$ for any $b \in B$, $h_a(b) = (1, b)$ for any $b \in B$; for $x = (x_1, \dots, x_n)$ we put $g_x(a) = h_x(a) = (1, x_1, \dots, x_n)$ and $g_x(b) = (1, x_1, \dots, x_n, b)$, $h_x(b) = (0, x_1, \dots, x_n, b)$ for $b \in B$.

Let us denote by Y the subspace of X on the set $\{a\} \cup (\{0\} \times A_\omega)$ and by Z the subspace on the set $\{a\} \cup (\{1\} \times A_\omega)$. Let us define $\varphi_i \colon A_\omega \to X$ for $i = 0, 1$ by $\varphi_i(a) = a$ and $\varphi_i(x_1, \dots, x_n) = (i, x_1, \dots, x_n)$. Clearly, $Y = \varphi_0[A_\omega]$ and $Z = \varphi_1[A_\omega]$.

We would like to show that Y is homeomorphic to $(A_\omega)_a$. The desired homeomorphism will be φ_0.

Let us denote by $q \colon \coprod A \to X$ the quotient map which is obtained as the combination of the maps g_x, h_x defined above. Observe that the restriction

$q|_{q^{-1}(Z)} = \varphi_1$ is essentially the same as the map $[f_x]$ from Lemma 3.3.14. The map q is one-to-one outside Z, thus by Lemma 2.1.4 this restriction is a quotient map and Z is homeomorphic to A_ω, the homeomorphism is the map φ_1.

Hence X is indeed homeomorphic to $A_\omega \triangle_a A$.

Let us also define a map $\varphi\colon X \to X$ by $\varphi(a) = a$ and $\varphi(0,x) = \varphi(1,x) = (1,x)$. One can readily verify that φ is continuous.

Lemma 3.3.15. *The map $\varphi_0\colon (A_\omega)_a \to X$ defined above is a homeomorphic embedding.*

Proof. The space $Y = \varphi_0[(A_\omega)_a]$ is a prime space, so only the neighborhoods of the accumulation point a are important in the proof.

The map φ_0 is clearly injective. We first show that it is also continuous. If V is any open neighborhood of a in X then $V \supseteq U := V \cap \varphi^{-1}(V)$. The subset U is an open neighborhood of a, which moreover has the property $\varphi^{-1}(U) \subseteq U$, in other words, $(1,x) \in U \Rightarrow (0,x) \in U$. Note that $\varphi_1 = \varphi \circ \varphi_0$. Hence $\varphi_1^{-1}(U) = \varphi_0^{-1}(\varphi^{-1}(U)) \subseteq \varphi_0^{-1}(U) \subseteq \varphi_0^{-1}(V)$. The set $\varphi_1^{-1}(U)$ is a neighborhood of a in A_ω (φ_1 is continuous), so $\varphi_0^{-1}(V)$ is open in the prime space $(A_\omega)_a$.

On the other hand, if V is an open neighborhood of a in $(A_\omega)_a$, then $V \supseteq U \ni a$ for some open set in A_ω. For the set $W := \varphi_0[V] \cup \varphi_1[U]$ the equality $W \cap Y = \varphi_0[V]$ holds and the set W is open in X. (To see that W is open just note that $W \cap Z = \varphi_1[U]$ is open in Z and if $(1,x) \in W$ then the set $h_x^{-1}(W) \supseteq g_x^{-1}(W) = g_x^{-1}(\varphi_1[U])$ is open in A. The same argument works for f_a and g_a.) $\square$

Since the space A_ω is zero-dimensional for a prime T_2-space A (Lemma 1.4.1 and Proposition 3.3.11) and $X \cong A_\omega \triangle_a A$, we see that X is zero-dimensional T_2-space as well.

Proposition 3.3.16. *If A is a prime T_2-space with the accumulation point a, then $(A_\omega)_a$ is a subspace of the zero-dimensional T_2-space $A_\omega \triangle_a A \in \mathrm{CH}(\mathbf{A})$.*

Corollary 3.3.17. *The space $(A_\omega)_a$ for $A = C(\omega_0)$ is T_2-subsequential.*

Chapter 4

Hereditary coreflective kernel

In [HH2, Problem 7] H. Herrlich and M. Hušek suggested to study the coreflective subcategories $\mathbf{A}$ of $\mathbf{Top}$ fulfilling the following two conditions. Firstly, $\mathbf{A}$ is large enough to contain each topological space as a subspace of some space from $\mathbf{A}$. This is precisely the condition $\mathrm{S}\mathbf{A} = \mathbf{Top}$ which we have investigated in Section 2.2.2. Secondly, $\mathbf{A}$ is small enough not to contain many hereditary coreflective subcategories. More precisely, the hereditary coreflective kernel of $\mathbf{A}$ is $\mathbf{FG}$. In this chapter we provide some results which can lead to better understanding what "small enough" means.

The results of Section 4.2 were published in [Sl3].

4.1 Definition and basic properties

We have already mentioned the hereditary coreflective kernel in section 2.2.2. Here we recall the definition and introduce the notation for the hereditary coreflective kernel.

Definition 4.1.1. Let $\mathbf{C}$ be a subcategory of $\mathbf{Top}$. The subcategory $\mathbf{B} \subseteq \mathbf{C}$ is said to be the *hereditary coreflective kernel* of $\mathbf{C}$ if $\mathbf{B}$ is the largest subclass of $\mathbf{C}$ which is at the same time hereditary and coreflective in $\mathbf{Top}$.

The hereditary coreflective kernel of $\mathbf{C}$ will be denoted by $\mathrm{HCK}(\mathbf{C})$.

Since we would like to study the coreflective subcategories of $\mathbf{Top}$ with the property $\mathrm{HCK}(\mathbf{C}) = \mathbf{FG}$, the following criterion will be useful.

Proposition 4.1.2 ([Č3, Theorem 4.8]). *Let* $\mathbf{C}$ *be a coreflective subcategory of* $\mathbf{Top}$ *with* $\mathbf{FG} \subseteq \mathbf{C}$. *Then* $\mathrm{HCK}(\mathbf{C}) = \mathbf{FG}$ *if and only if* $\mathrm{SCH}(C(\alpha)) \not\subseteq \mathbf{C}$ *for each regular cardinal* α.

Proof. It suffices to show that if $\mathbf{B}$ is a hereditary coreflective subcategory of $\mathbf{Top}$ and $\mathbf{B} \supsetneq \mathbf{FG}$, then $\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{B}$ for some regular cardinal α. Any such

subcategory contains some space $B(\alpha)$ (see the note after Definition 1.5.32) and by Proposition 2.1.8 it contains $(B(\alpha))_\alpha = C(\alpha)$ as well. $\square$

Using this result one can show that $\mathrm{HCK}(\mathbf{PsRad}) = \mathrm{HCK}(\mathbf{CGen}) = \mathbf{FG}$. This was shown in [Č3]. (It suffices to show that $A_2^- \notin \mathbf{PsRad}$, resp. $A_2^- \notin \mathbf{CGen}$, for $A = C(\alpha)$.)

Proposition 4.1.2 shows the connection between the problem of determining the hereditary coreflective kernel $\mathrm{HCK}(\mathbf{C})$ and the prime spaces $C(\alpha)$. Therefore we will apply throughout this chapter several results, which we have proved for an arbitrary prime space, to the space $C(\alpha)$.

The hereditary coreflective kernel in general need not exist. Several examples, with a more detailed analysis than we provide here, can be found in [Č3, Section 4]. We present here only one of them to show how the apparatus built in the preceding chapters can be used for this purpose.

In the proof of the following lemma we use precisely the construction from [Č3], but we describe it using the A-sum and our proof relies on some results about (hereditary) coreflective hulls of prime spaces, which can be found in the preceding chapters.

Lemma 4.1.3. *Let $\beta < \alpha$ be regular cardinals. Then the coreflective subcategory $\mathbf{C} := \mathrm{CH}(\mathrm{SCH}(C(\alpha)) \cup \mathrm{SCH}(C(\beta)))$ is not hereditary.*

Proof. Let us denote $X := \sum_{C(\alpha)} C(\beta)$ and $Y := P(X)$. I.e., the underlying set of X is $\{\alpha\} \cup \alpha \cup \alpha \times \beta$ and Y is the subspace of X on the subset $\{\alpha\} \cup \alpha \times \beta$. Clearly, $X \in \mathbf{C}$, so if we show that $Y \notin \mathbf{C}$ then $\mathbf{C}$ is not hereditary.

We define maps $q_\alpha \colon Y \to C(\alpha)$ and $q_\beta \colon Y \to C(\beta)$ in the following way:
$q_\alpha(\alpha) = \alpha$ and $q_\alpha(\eta, \xi) = \eta$ for $\eta \in \alpha$, $\xi \in \beta$;
$q_\beta(\alpha) = \beta$ and $q_\beta(\eta, \xi) = \xi$ for $\eta \in \alpha$, $\xi \in \beta$.
Both q_α and q_β are quotient maps.

By Theorem 3.3.6 we have $\mathbf{C} = \mathrm{CH}((C(\alpha)_\omega)_\alpha, (C(\beta)_\omega)_\beta)$. So we can use Lemma 1.5.6 characterizing the coreflective hull of a class of prime spaces.

The subset $M = \alpha \times \beta$ is precisely the set of isolated points of the prime space Y. For this set $\alpha \in \overline{M}$ holds.

Assume that $Y \in \mathbf{C}$. By Lemma 1.5.6 there exists a prime subspace B either of $(C(\alpha)_\omega)_\alpha$ or of $(C(\beta)_\omega)_\beta$ and a continuous map $f \colon B \to Y$ with $f[B \setminus \{a\}] \subseteq M$ and $f(a) = \alpha$, where a denotes the accumulation point of B.

First let us assume that B is a prime subspace of $(C(\alpha)_\omega)_\alpha$. The composite $f_\beta = q_\beta \circ f \colon B \to C(\beta)$ is a continuous map with the properties $f_\beta(\alpha) = \beta$, $B' := f_\beta[B \setminus \{\alpha\}] \subseteq \beta$ and $\beta \in \overline{B'}$.

We claim that the existence of such a map implies $C(\beta) \in \mathrm{CH}((C(\alpha)_\omega)_\alpha) = \mathrm{SCH}(C(\alpha))$.

To see this, just note that the subspace of $C(\beta)$ on the set $B' \cup \{\beta\}$ is homeomorphic to $C(\beta)$. So we obtain for each non-closed subset of $C(\beta)$ a map from B to $C(\beta)$ with the properties required in Lemma 1.5.6.

This implies $C(\beta) \in \mathrm{SCH}(C(\alpha)) \subseteq \mathbf{Top}(\alpha)$, a contradiction.

Analogously, if B is a prime subspace of $(C(\beta)_\omega)_\beta$, we can show that $C(\alpha) \in \mathrm{SCH}(C(\beta)) \subseteq \mathbf{Gen}(\beta)$, which again leads to a contradiction. $\square$

Corollary 4.1.4. *The subcategory* $\mathbf{C} = \mathrm{CH}(\mathrm{SCH}(C(\alpha)) \cup \mathrm{SCH}(C(\beta)))$ *does not have a hereditary coreflective kernel.*

Proof. Suppose $\mathbf{B} = \mathrm{HCK}(\mathbf{C})$, i.e., $\mathbf{B}$ is the largest hereditary coreflective subcategory of **Top**, which is contained in $\mathbf{C}$. Then $\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{B}$, $\mathrm{SCH}(C(\beta)) \subseteq \mathbf{B}$, consequently $\mathbf{B} = \mathbf{C}$. But $\mathbf{C}$ fails to be hereditary, a contradiction. □

4.2 Coreflective subcategories with $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$

The purpose of this section is to study the coreflective subcategories of **Top** with the property $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$. The main new result is that the class of all such subcategories is closed under non-empty finite joins.

In order to prove this result we first show that if $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{A} \cup \mathbf{B})$ for some coreflective subcategories $\mathbf{A}$, $\mathbf{B}$ of **Top**, then one of these subcategories contains $\mathrm{SCH}(C(\alpha))$. We show it separately for the case $\alpha = \omega_0$ and for $\alpha \geq \omega_1$.

We start with the case $\alpha = \omega_0$ where we can use some results presented in the paper [FR]. In this paper, the set TSS_γ of prime spaces is defined for each $\gamma < \omega_1$. As these sets do not coincide with the sets $TSS_\gamma(C(\omega_0))$ defined in Section 3.2 we denote the sets used in [FR] by $TSS'_\gamma(C(\omega_0))$.

The next lemma follows from [FR, Theorem 7.1], resp. [FR, Corollary 7.2].

Lemma 4.2.1. *The category* $\mathbf{SSeq} = \mathrm{SCH}(C(\omega_0))$ *of subsequential spaces is the coreflective hull of the set* $\bigcup_{\gamma<\omega_1} TSS'_\gamma(C(\omega_0))$.

As a consequence of [FR, Theorem 7.1] and [FR, Theorem 6.4] we obtain:

Lemma 4.2.2. *If* $\beta < \gamma < \omega_1$, *then* $TSS'_\beta(C(\omega_0)) \subseteq \mathrm{CH}(TSS'_\gamma(C(\omega_0)))$.

From [FR, Lemma 8.2, 8.4] and [FR, Theorem 8.6] we deduce:

Lemma 4.2.3. *If* $X \in TSS'_\beta(C(\omega_0))$ *then any prime subspace of* X *belongs to* $TSS'_\beta(C(\omega_0))$ *as well. Moreover,* $TSS'_\beta(C(\omega_0)) \subseteq \mathrm{CH}(X)$.

The following result concludes the part of this section concerning the subcategory $\mathrm{SCH}(C(\omega_0))$.

Proposition 4.2.4. *If* $\mathrm{SCH}(C(\omega_0)) \subseteq \mathrm{CH}(\bigcup_{i\in I} \mathbf{A}_i)$, $\mathbf{A}_i$ *is a coreflective subcategory of* **Top** *for every* $i \in I$ *and* $\operatorname{card} I \leq \aleph_0$, *then there exists* $i_0 \in I$ *such that* $\mathrm{SCH}(C(\omega_0)) \subseteq \mathbf{A}_{i_0}$.

Proof. Let X be an arbitrary space from $TSS'_\beta(C(\omega_0))$, $\beta < \omega_1$. By the assumption $X \in \mathrm{CH}(\bigcup_{i\in I} \mathbf{A}_i)$. We first show that this implies that some of the subcategories $\mathbf{A}_i$ contains a space belonging to $TSS'_\beta(C(\omega_0))$.

We have a quotient map $q\colon \coprod_{i\in I} A_i \to X$, where $A_i \in \mathbf{A}_i$ for each $i \in I$. Then for some $i \in I$ the subspace of X on the set $f[A_i]$ is a prime subspace of X. Thus it belongs to $TSS'_\beta(C(\omega_0))$ by Lemma 4.2.3. Clearly this subspace also belongs to $\mathbf{A}_i$. Consequently, $TSS'_\beta(C(\omega_0)) \subseteq \mathbf{A}_i$ (again by Lemma 4.2.3).

Put $\beta_i = \sup\{\beta : TSS'_\beta(C(\omega_0)) \subseteq \mathbf{A}_i\}$ for $i \in I$. Thus $\sup \beta_i = \omega_1$ and since ω_1 is a regular cardinal, there exists $i_0 \in I$ such that $\beta_{i_0} = \omega_1$. By Lemma 4.2.1 and Lemma 4.2.2 we get that the coreflective subcategory $\mathbf{A}_{i_0}$ contains the subcategory $\mathbf{SSeq} = \mathrm{SCH}(C(\omega_0))$. □

From Propositions 4.1.2 and 4.2.4 we obtain easily the following result.

Corollary 4.2.5. *Let* $\mathbf{A}_i$, $i \in I$, *be a non-empty countable system of coreflective subcategories of* **Top** *such that* $\mathrm{HCK}(\mathbf{A}_i) = \mathbf{FG}$ *and* $\mathbf{A}_i \cap \mathbf{Top}(\omega_1) = \mathbf{FG}$. *Then* $\mathrm{HCK}(\bigcup_{i\in I} \mathbf{A}_i) = \mathbf{FG}$.

Next we want to prove a result analogous to Proposition 4.2.4 for the space $C(\alpha)$, where $\alpha \geq \omega_1$ is a regular cardinal. In the case $\alpha \geq \omega_1$ the desired result holds only for non-empty finite joins of coreflective subcategories of **Top**.

We will use the construction of A_n from Chapter 3 for the space $A = C(\alpha)$. Recall that $C(\alpha)_1 = C(\alpha)$ and $C(\alpha)_{n+1} = \sum_{C(\alpha)} \langle C(\alpha)_n, \alpha\rangle$. According to Corollary 3.1.4 we obtain that $P(C(\alpha)_{n+1}) = P(\sum_{C(\alpha)} \langle P(C(\alpha)_n), \alpha\rangle)$ and it is easy to see that $\alpha^{n+1} \cup \{\alpha\}$ is the underlying set of the space $P(C(\alpha)_{n+1})$ and the subspace of $\sum P(C(\alpha)_n)$ on the set $\{\eta\} \cup (\{\eta\} \times \alpha^n)$ is homeomorphic to $P(C(\alpha)_n)$ for each $\eta < \alpha$. To simplify the notation we will write $C(\alpha)^-_n$ instead of $P(C(\alpha)_n)$.

The following result is just a special case of Proposition 3.3.8.

Proposition 4.2.6. $\mathrm{SCH}(C(\alpha)) = \mathrm{CH}(\{C(\alpha)^-_n; 0 < n < \omega_0\})$ *holds for any regular cardinal* $\alpha \geq \omega_1$.

The result corresponding in the case of $C(\omega_0)$ to the following lemma is [FR, Lemma 8.2].

Lemma 4.2.7. *Let* $\alpha \geq \omega_1$ *be a regular cardinal and* $0 < n < \omega_0$. *If* M *is a subset of* $C(\alpha)_n$ *such that* $\alpha \in \overline{M}$ *and* M *contains only isolated points of* $C(\alpha)_n$, *then there exists a subset* $M' \subseteq M$ *such that the subspace of* $C(\alpha)_n$ *on the set* $\overline{M'}$ *is homeomorphic to* $C(\alpha)_n$.

Proof. The case $n = 1$ is clear. Let the assertion be true for m. We show that it holds for $m + 1$ too. Denote the subspace of $C(\alpha)_{m+1} = \sum_{C(\alpha)} C(\alpha)_m$ on the set $\{\eta\} \cup (\{\eta\} \times (C(\alpha)_m \setminus \{\alpha\}))$, where $\eta < \alpha$, by $C(\alpha)^\eta_m$. (This is the η-th bristle of the $C(\alpha)$-sum.)

Put $B = \overline{M} \cap C(\alpha)$. Then B is a prime subspace of $C(\alpha)$, for each $\eta \in B \setminus \{\alpha\}$ all points of the set $M_\eta = M \cap C(\alpha)^\eta_m$ are isolated in the space $C(\alpha)^\eta_m$ and $\eta \in \overline{M_\eta}$ in $C(\alpha)^\eta_m$ (observe that $\overline{M_\eta}$ in $C(\alpha)^\eta_m$ coincides with $\overline{M_\eta}$ in $C(\alpha)_{m+1}$ because $C(\alpha)^\eta_m$ is closed in $C(\alpha)_{m+1}$). Since $C(\alpha)^\eta_m$ is homeomorphic to $C(\alpha)_m$ by the induction assumption we obtain that there exists a subset $M'_\eta \subseteq M_\eta$ such that $\eta \in \overline{M'_\eta}$ and the subspace $\overline{M'_\eta}$ of $C(\alpha)^\eta_m$ is homeomorphic to some space $C(\alpha)_m$.

Let $B' = B \setminus \{\alpha\}$ and $M' = \bigcup_{n\in B'} M'_\eta$. Clearly, $M' \subseteq M$, $\overline{M'} = \bigcup_{\eta\in B'} \overline{M'_\eta} \cup \{\alpha\}$ in S and $\overline{M'_\eta}$ is homeomorphic to $C(\alpha)_m$ for each $\eta \in B'$.

The subspace B of $C(\alpha)$ is homeomorphic to $C(\alpha)$ and it is easy to check that $\overline{M'}$ is homeomorphic to $\sum_{C(\alpha)} C(\alpha)_m = C(\alpha)_{m+1}$.

□

Corollary 4.2.8. *Let $\alpha \geq \omega_1$ be a regular cardinal and $0 < n < \omega_0$. Then every prime subspace T of $C(\alpha)_n^-$ is homeomorphic to $C(\alpha)_n^-$.*

Proof. Put $M = T \setminus \{\alpha\}$. Clearly, $\alpha \in \overline{M}$. According to Lemma 4.2.7 there exists a subset M' of M such that the subspace $M' \cup \{\alpha\}$ of $C(\alpha)_n^-$ is homeomorphic to $C(\alpha)_n^-$. It follows from the proof of Lemma 4.2.7 that $M \setminus M'$ is a discrete clopen subspace of $C(\alpha)_n^-$ with cardinality at most α. Hence, $T = M \cup \{\alpha\}$ is homeomorphic to $C(\alpha)_n^-$ as well. □

Proposition 4.2.9. *Let $\{\mathbf{A}_i; i \in I\}$ be a set of coreflective subcategories of* **Top** *and $\alpha \geq \omega_1$ be a regular cardinal and $0 < n < \omega_0$. If $C(\alpha)_n^- \in \mathrm{CH}(\bigcup_{i \in I} \mathbf{A}_i)$, then there exists $i_0 \in I$ such that $C(\alpha)_n^- \in \mathbf{A}_{i_0}$.*

Proof. The space $C(\alpha)_n^-$ is a prime space with the accumulation point α. If $C(\alpha)_n^- \in \mathrm{CH}(\bigcup_{i \in I} \mathbf{A}_i)$, then there exists a quotient map $f\colon \coprod_{i \in I} B_i \to C(\alpha)_n^-$, where B_i belongs to $\mathbf{A}_i$ for each $i \in I$. Put $f_i = f|_{B_i}$ and let A_i be the space on the set $f_i[B_i]$ endowed with the quotient topology with respect to f_i for each $i \in I$.

The topology of every space A_i is finer than the topology of the corresponding subspace of $C(\alpha)_n^-$ and it follows that A_i is either discrete or prime space. Clearly, a set $U \subseteq C(\alpha)_n^-$ is open in $C(\alpha)_n^-$ if and only if $U \cap A_i$ is open in A_i for each $i \in I$ and $A_i \in \mathbf{A}_i$. Hence there exists $i_0 \in I$ such that α is the accumulation point of A_{i_0} (otherwise α would be isolated in $C(\alpha)_n^-$).

We show that $C(\alpha)_n^- \in \mathrm{CH}(A_{i_0})$. Let M be a non-closed subset of $C(\alpha)_n^-$. By Lemma 1.5.6 it suffices to find a continuous map $f\colon A_{i_0} \to C(\alpha)_n^-$ such that $f[A_{i_0} \setminus \{\alpha\}] \subseteq M$ and $f(\alpha) = \alpha$. According to Corollary 4.2.8 the subspace on the set $M \cup \{\alpha\}$ is homeomorphic to $C(\alpha)_n^-$. Let us denote the homeomorphism from $C(\alpha)_n^-$ to $M \cup \{\alpha\}$ by g. Moreover, there is a continuous map $i\colon A_{i_0} \to C(\alpha)_n^-$ defined by $i(x) = x$ for each $x \in A_{i_0}$. The desired continuous map is $f = g \circ i$. □

If X and Y are prime spaces, then a continuous map $f\colon X \to Y$ is called a *prime map* if it maps only the accumulation point of X to the accumulation point of Y. (The term prime map was used in [FR].)

Lemma 4.2.10. *Let $\alpha \geq \omega_1$ be a regular cardinal and $0 < m < n < \omega_0$. There exists a quotient prime map $g\colon C(\alpha)_n^- \to C(\alpha)_m^-$.*

Proof. Obviously, it suffices to prove the lemma for $n = m + 1$. In this case $C(\alpha)_{m+1}^- = P(\sum C(\alpha)_m^-)$ is a topological space on the set $\{\alpha\} \cup \alpha^{m+1}$ and $C(\alpha)_m^-$ is a topological space on the set $\{\alpha\} \cup \alpha^m$. We define a map $g\colon C(\alpha)_{m+1}^- \to C(\alpha)_m^-$ by $g(\alpha) = \alpha$ and $g((\eta, x)) = x$ for all $\eta \in \alpha$ and $x \in \alpha^m$. It is easy to check that the map g is quotient. □

Corollary 4.2.11. *If $\alpha \geq \omega_1$ is a regular cardinal and $0 < m < n < \omega_0$, then $C(\alpha)^-_m \in \mathrm{CH}(C(\alpha)^-_n)$.*

Proposition 4.2.12. *If α is a regular cardinal and $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{A} \cup \mathbf{B})$, then $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{A})$ or $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{B})$.*

Proof. Since the case $\alpha = \omega_0$ follows immediately from Proposition 4.2.4 we may assume that $\alpha \geq \omega_1$.

By Proposition 4.2.9 for each n, $0 < n < \omega_0$, the space $C(\alpha)^-_n$ belongs either to $\mathbf{A}$ or to $\mathbf{B}$. By Lemma 4.2.10 we have a quotient map $f\colon C(\alpha)^-_n \to C(\alpha)^-_m$ for each $n > m$. Hence, one of these two coreflective categories contains all spaces $C(\alpha)^-_n$ and, consequently, it contains $\mathrm{SCH}(C(\alpha))$ by Proposition 4.2.6. □

Now we can state the main result of this section.

Theorem 4.2.13. *If $\mathbf{A}$, $\mathbf{B}$ are coreflective subcategories of the category* **Top** *and $\mathrm{HCK}(\mathbf{A}) = \mathrm{HCK}(\mathbf{B}) = \mathbf{FG}$, then $\mathrm{HCK}(\mathrm{CH}(\mathbf{A} \cup \mathbf{B})) = \mathbf{FG}$.*

Proof. Suppose the contrary. Then according to Proposition 4.1.2 there exists a regular cardinal α with $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{A} \cup \mathbf{B})$. Proposition 4.2.12 implies that $\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{A}$ or $\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{B}$, contradicting the assumption that the hereditary coreflective kernel of both these categories is $\mathbf{FG}$. □

Let $\mathcal{C}$ be the lattice of all coreflective subcategories of **Top**. Denote by $\mathcal{K}$ the conglomerate of all coreflective subcategories $\mathbf{A}$ of **Top** with $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$. The above theorem says that $\mathcal{K}$ is closed under the formation of non-empty finite joins in $\mathcal{C}$. We next show that $\mathcal{K}$ fails to be closed under the formation of infinite countable joins in $\mathcal{C}$. Namely, we prove that if $\alpha \geq \omega_1$ is a regular cardinal, then all categories $\mathrm{CH}(C(\alpha)^-_n)$ belong to $\mathcal{K}$. According to Proposition 4.2.6 the category $\mathrm{SCH}(C(\alpha))$ is the join of this family in $\mathcal{C}$ and, evidently, $\mathrm{SCH}(C(\alpha)) \notin \mathcal{K}$. The proof is divided into three lemmas. They correspond in a certain sense to [FR, Lemma 8.3, 8.4].

Lemma 4.2.14. *Let $\alpha \geq \omega_1$ be a regular cardinal and $2 \leq n < \omega_0$. If there exists a prime map $f\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$, then there exists a prime map $f'\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$ such that $f'[\{\xi\} \times \alpha^{n-1}] \cap (\bigcup_{\eta<\xi}\{\eta\} \times \alpha^n) = \emptyset$ for each $\xi < \alpha$.*

Proof. Let $f\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$ be a prime map. Denote by B_ξ the subspace of $\sum C(\alpha)^-_{n-1}$ on the set $\{\xi\} \cup (\{\xi\} \times \alpha^{n-1})$ where $\xi < \alpha$. The subspace B_ξ is homeomorphic to $C(\alpha)^-_{n-1}$.

For each $\xi < \alpha$ the set $f^{-1}(\{\alpha\} \cup (\bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n))$ is open in $C(\alpha)^-_n$, therefore there exists an ordinal $\gamma < \alpha$ such that for each $\gamma' > \gamma$ the set $\{\gamma'\} \cup (f^{-1}(\bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n) \cap B_{\gamma'})$ is open in $B_{\gamma'}$. Hence, we can define an increasing sequence $(\gamma_\xi)_{\xi<\alpha}$ such that $C_\xi := \{\gamma_\xi\} \cup (f^{-1}(\bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n) \cap B_{\gamma_\xi})$ is open in B_{γ_ξ}. Clearly, $f[C_\xi \setminus \{\gamma_\xi\}] \subseteq \bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n$.

According to Corollary 4.2.8 the subspace of B_{γ_ξ} on the set C_ξ is homeomorphic to $C(\alpha)^-_{n-1}$. Hence, for each $\xi < \alpha$ we can define an embedding $h_\xi\colon C(\alpha)^-_{n-1} \hookrightarrow \sum C(\alpha)^-_{n-1}$ such that $h_\xi[C(\alpha)^-_{n-1}] = C_\xi$. It is easy to see that

the map $h\colon \sum C(\alpha)^-_{n-1} \to \sum C(\alpha)^-_{n-1}$ given by $h(\xi) = \gamma_\xi$ for each $\xi < \alpha$, $h(\alpha) = \alpha$ and $h(\xi, x) = h_\xi(x)$ for each $\xi < \alpha$ and $x \in \alpha^{n-1}$ is also an embedding. Put $A_\xi = \{\xi\} \times \alpha^{n-1}$ ($A_\xi \subseteq B_\xi$). Then $h[A_\xi] \subseteq h_\xi[C(\alpha)^-_{n-1}] = C_\xi$ and $f[h[A_\xi]] \subseteq f[C_\xi \setminus \{\gamma_\xi\}] \subseteq \bigcup_{\eta \geq \xi} \{\eta\} \times \alpha^n$. Consequently, $f \circ h[A_\xi] \cap (\bigcup_{\eta<\xi} \{\eta\} \times \alpha^n) = \emptyset$ and the prime map $f' = f \circ (h|_{C(\alpha)^-_n})\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$ is a prime map satisfying the required condition. □

Lemma 4.2.15. *Let $\alpha \geq \omega_1$ be a regular cardinal and $0 < n < \omega_0$. Then there exists no prime map from $C(\alpha)^-_n$ to $C(\alpha)^-_{n+1}$.*

Proof. First let $n = 1$. Suppose that $f\colon C(\alpha) \to C(\alpha)^-_2$ is a prime map. For each $\gamma < \alpha$ the set $\{\gamma\} \times \alpha$ is closed in $C(\alpha)^-_2$. Consequently, $f^{-1}(\{\gamma\} \times \alpha)$ is closed in $C(\alpha)$, hence it contains less than α points and there exists a set $U_\gamma \subseteq \alpha$ with $\mathrm{card}(\alpha \setminus U_\gamma) < \alpha$ such that $(\{\gamma\} \times U_\gamma) \cap f[C(\alpha)] = \emptyset$. Thus, $W = \{\alpha\} \cup \left(\bigcup_{\gamma<\alpha} \{\gamma\} \times U_\gamma\right)$ is an open neighborhood of α in $C(\alpha)^-_2$ such that $f^{-1}(W) = \{\alpha\}$ and this contradicts the continuity of f.

Let $n > 1$ and the lemma hold for $n - 1$. Suppose that there exists a prime map $f\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$. By Lemma 4.2.14 we can assume w.l.o.g. that $f[\{\xi\} \times \alpha^{n-1}] \cap (\bigcup_{\eta<\xi} \{\eta\} \times \alpha^n) = \emptyset$ for each $\xi < \alpha$.

Recall the definition of the quotient prime map $g\colon C(\alpha)^-_n \to C(\alpha)^-_{n-1}$ from Lemma 4.2.10. The map g is defined by $g(\alpha) = \alpha$ and $g(\eta, x) = x$ for $\eta < \alpha$, $x \in \alpha^{n-1}$.

Put $A_\xi = \{\xi\} \times \alpha^{n-1}$. Let us denote the subspace of $\sum C(\alpha)^-_{n-1}$ on the set $\{\xi\} \cup A_\xi = \{\xi\} \cup (\{\xi\} \times \alpha^{n-1})$ by B_ξ for each $\xi < \alpha$. Clearly, B_ξ is homeomorphic to $C(\alpha)^-_{n-1}$. We define a map $f_\xi\colon B_\xi \to C(\alpha)^-_{n+1}$ by $f_\xi(\xi) = \alpha$ and $f_\xi(\xi, x) = f(\xi, x)$ for each $x \in \alpha^{n-1}$.

The map $g \circ f_\xi\colon B_\xi \to C(\alpha)^-_n$ cannot be continuous, otherwise we would get a prime map from a space homeomorphic to $C(\alpha)^-_{n-1}$ to the space $C(\alpha)^-_n$. Therefore there exists an open subset of $C(\alpha)^-_n$ such that inverse image of this set is not open in B_ξ. This set can be written in the form $U_\xi \cup \{\alpha\}$, where $\alpha \notin U_\xi$, and we get that the set

$$f_\xi^{-1}(g^{-1}(U_\xi \cup \{\alpha\})) = f_\xi^{-1}(\{\alpha\} \cup (\bigcup_{\eta<\alpha} (\{\eta\} \times U_\xi))) = \{\xi\} \cup (B_\xi \cap f^{-1}(\bigcup_{\eta<\alpha} \{\eta\} \times U_\xi))$$

is not open in B_ξ.

Put $V_\xi = \bigcap_{\eta \leq \xi} U_\eta$ for $\xi < \alpha$. The family V_ξ is non-increasing and it has the same properties as the family U_ξ. Each $V_\xi \cup \{\alpha\}$ is open in $C(\alpha)^-_n$, because $C(\alpha)^-_n$ belongs to $\mathbf{Top}(\alpha)$ ($\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{Top}(\alpha)$). The set $\{\xi\} \cup (B_\xi \cap f^{-1}(\bigcup_{\eta<\alpha} \{\eta\} \times V_\xi))$ is not open in B_ξ, since B_ξ is a prime space with the accumulation point ξ (and $\{\xi\} \cup (B_\xi \cap f^{-1}(\bigcup_{\eta<\alpha} \{\eta\} \times V_\xi)) \subseteq \{\xi\} \cup (B_\xi \cap f^{-1}(\bigcup_{\eta<\alpha} \{\eta\} \times U_\xi))$).

Finally let us put $W = \bigcup_{\xi<\alpha} \{\xi\} \times V_\xi$. The set $W \cup \{\alpha\}$ is open in $C(\alpha)^-_{n+1}$. We claim that $f^{-1}(\{\alpha\} \cup W)$ is not open in $C(\alpha)^-_n$. It suffices to show that $\{\xi\} \cup (f^{-1}(\{\alpha\} \cup W) \cap B_\xi)$ is not open in B_ξ for each $\xi < \alpha$.

Clearly, $B_\xi = A_\xi \cup \{\xi\}$ and we get $\{\xi\} \cup (f^{-1}(\{\alpha\} \cup W) \cap B_\xi) = \{\xi\} \cup (f^{-1}(\bigcup_{\eta<\alpha}\{\eta\} \times V_\eta) \cap A_\xi)$. We have $f[A_\xi] \cap (\bigcup_{\eta<\xi}\{\eta\} \times \alpha^{n-1}) = \emptyset$, hence $f^{-1}(\bigcup_{\eta<\alpha}\{\eta\} \times V_\eta) \cap A_\xi = f^{-1}(\bigcup_{\eta\geq\xi}\{\eta\} \times V_\eta) \cap A_\xi$ and we obtain

$$\{\xi\} \cup (f^{-1}(\{\alpha\} \cup W) \cap B_\xi) = \{\xi\} \cup (f^{-1}(\bigcup_{\eta\geq\xi}\{\eta\} \times V_\eta) \cap B_\xi) \subseteq$$
$$\subseteq \{\xi\} \cup (f^{-1}(\bigcup_{\eta\geq\xi}\{\eta\} \times V_\xi) \cap B_\xi) \subseteq \{\xi\} \cup (f^{-1}(\bigcup_{\eta<\alpha}\{\eta\} \times V_\xi) \cap B_\xi).$$

The latter set is not open in B_ξ therefore $\{\xi\} \cup (f^{-1}(\{\alpha\} \cup W) \cap B_\xi)$ is not open in B_ξ as well. □

Lemma 4.2.16. *Let $\alpha \geq \omega_1$ be a regular cardinal and $0 < n < \omega_0$. Then* $\mathrm{HCK}(\mathrm{CH}(C(\alpha)_n^-)) = \mathbf{FG}$.

Proof. Recall that if $\gamma > \delta$, then $\mathbf{Top}(\gamma) \cap \mathbf{Gen}(\delta) = \mathbf{FG}$ (see the note after Example 1.5.33). For $\beta < \alpha$ we have $\mathrm{SCH}(C(\beta)) \subseteq \mathbf{Gen}(\beta)$ and $C(\alpha)_n^- \in \mathbf{Top}(\alpha)$, hence $\mathrm{SCH}(C(\beta)) \nsubseteq \mathrm{CH}(C(\alpha)_n^-)$. On the other hand, if $\beta > \alpha$, then $\mathrm{SCH}(C(\beta)) \subseteq \mathbf{Top}(\beta)$ and $C(\alpha)_n^- \in \mathbf{Gen}(\alpha)$. Thus, $\mathrm{SCH}(C(\beta)) \nsubseteq \mathrm{CH}(C(\alpha)_n^-)$.

By Lemma 4.2.15 and Lemma 1.5.6 $C(\alpha)_{n+1}^- \notin \mathrm{CH}(C(\alpha)_n^-)$ (every prime subspace of $C(\alpha)_n^-$ is homeomorphic to $C(\alpha)_n^-$) and $C(\alpha)_{n+1}^- \in \mathrm{SCH}(C(\alpha))$, therefore $\mathrm{SCH}(C(\alpha)) \nsubseteq \mathrm{CH}(C(\alpha)_{n+1}^-)$ as well. □

Denote by $\mathcal{L}$ the collection of all coreflective subcategories $\mathbf{A}$ of $\mathbf{Top}$ such that $\mathrm{S}\mathbf{A} = \mathbf{Top}$ and $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$. In Corollary 2.2.11 we have shown that $\mathcal{L}$ has the smallest element $\mathbf{A}_0 = \mathrm{CH}(\{S^\alpha; \alpha \in \mathrm{Cn}\})$ and $\mathcal{L}$ is closed under the formation of arbitrary non-empty intersections (Corollary 2.2.10). This together with Theorem 4.2.13 yields:

Theorem 4.2.17. *The collection $\mathcal{L}$ is closed under the formation of non-empty intersections, non-empty finite joins in $\mathcal{C}$ and has the smallest element.*

Proposition 4.2.18. *There is no maximal coreflective subcategory* $\mathbf{A}$ *of* $\mathbf{Top}$ *such that* $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$. *Consequently, the collection $\mathcal{L}$ has no maximal element.*

Proof. Suppose that $\mathbf{A}$ is maximal coreflective subcategory of $\mathbf{Top}$ with the property $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$. Let $\alpha \geq \omega_1$ be a regular cardinal. According to Lemma 4.2.16 and Theorem 4.2.13 $\mathrm{HCK}(\mathrm{CH}(\mathbf{A} \cup \{C(\alpha)_n^-\})) = \mathbf{FG}$ for each n, $0 < n < \omega_0$. Thus, we get $C(\alpha)_n^- \in \mathbf{A}$ for each n and by Proposition 4.2.6 $\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{A}$, a contradiction.

The proof that $\mathcal{L}$ has no maximal elements is analogous. □

The family $\mathrm{CH}(\mathbf{A}_0 \cup \{C(\alpha)_n^-\})$, $0 < n < \omega_0$, where $\alpha \geq \omega_1$ is a regular cardinal, is an example of a countable family of elements of $\mathcal{L}$ such that its join does not belong to $\mathcal{L}$.

Chapter 5

Epireflective subcategories of Top

Throughout this chapter we will assume that $\mathbf{A}$ is an epireflective subcategory of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$. Theorem 2.1.9 was generalized to this situation by J. Činčura in [Č4]. He showed that a coreflective subcategory $\mathbf{C}$ of $\mathbf{A}$ is hereditary if and only if it is closed under prime factors.

We will include also some results concerning the coreflective hulls in $\mathbf{A}$, but we will deal in this chapter mostly with another possible generalization. We will study the subclasses of $\mathbf{A}$ which are additive (closed under the topological sums) and divisible (closed under quotient spaces) in $\mathbf{A}$. Such classes we call AD-classes. The AD-classes include as a special case the coreflective subcategories of $\mathbf{A}$ as well.

So our main goal is to determine whether also for AD-classes in $\mathbf{A}$ heredity is equivalent to the closedness under prime factors. Unfortunately, we were able neither to prove this in full generality nor to find a counterexample. We show that this holds whenever the AD-class $\mathbf{B}$ contains at least one prime space. Many HAD-classes are shown to contain one. We also show that this characterization is true if $\mathbf{A} \subseteq \mathbf{Haus}$.

In Chapter 6 we show how one can avoid the condition $I_2 \notin \mathbf{A}$, which was motivated in [Č4] by the fact that precisely the proper epireflective subcategories of $\mathbf{Top}$ closed under prime factors fulfill this condition.

The results presented in this and the following chapter have been included in the paper [Sl4].

Convention: In this chapter $\mathbf{A}$ will always mean an epireflective subcategory of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$.

5.1 HAD-hulls and coreflective hulls

There are two notions which correspond to the notion of coreflective subcategory of **Top** - coreflective subcategories of **A** and AD-classes in **A**. In both cases one can ask when they are hereditary too. As we have already mentioned, for the coreflective subcategories of **A** this was solved in [Č4]. We will give a partial answer for the AD-classes.

In this section we start by providing the necessary definitions and then we will study heredity of the coreflective hulls in **A**. Analogous question for AD-classes will be studied in the next section.

We will show in Proposition 5.1.11 that the coreflective subcategories of **A** are precisely the classes of topological spaces which are closed under **A**-extremal epimorphisms and topological sums.

Definition 5.1.1. A class **B** is said to be *additive*, if it is closed under topological sums. We say that **B** is *divisible in* **A** if for every quotient map $q\colon X \to Y$ with $X \in \mathbf{B}$ and $Y \in \mathbf{A}$ we have $Y \in \mathbf{B}$.

If **A** is quotient reflective then **B** is coreflective in **A** if and only if it is additive and divisible in **A**. This fact is relatively well-known, but we will reprove it in Corollary 5.1.19.

Definition 5.1.2. A class **B** which is simultaneously additive and divisible in **A** will be called briefly *AD-class* in **A**. If **B** is moreover hereditary, we say that it is an *HAD-class* in **A**.

We define *AD-hull* (*HAD-hull*) of $\mathbf{B} \subseteq \mathbf{A}$ as the smallest (hereditary) AD-class in **A** containing **B**. It will be denoted by $\mathrm{AD}_{\mathbf{A}}(\mathbf{B})$, resp. $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})$.

If $\mathbf{A} = \mathbf{Top}$ then the notion of AD-class (HAD-class) coincides with the notion of (hereditary) coreflective subcategory. We have seen in Proposition 2.1.5 and Corollary 2.1.6 that the hereditary coreflective hull of a class **B** in **Top** is $\mathrm{SCH}(\mathbf{B})$.

In this section we will study the (hereditary) coreflective subcategories of **A**.

5.1.1 Hereditary coreflective subcategories in A

We now recall some auxiliary results concerning **A**-extremal epimorphisms and quotient maps and the characterization of coreflective subcategories of **A** as the subcategories which are closed under topological sums and **A**-extremal epimorphisms.

This characterization implies that every coreflective subcategory **B** of **A** is an AD-class in **A**. (We provide a counterexample to the opposite implication in Example 5.1.22.) The aim of this section is to describe the relationship between the hereditary coreflective hull in **A** and the HAD-hull in **A**. Corollary 5.1.15 implies that if the coreflective hull of **D** in **A** is hereditary, it is at the same time the HAD-hull of **D** in **A**.

Clearly, every quotient map between spaces from **A** is an extremal epimorphism in **A**. (The converse implication is not true in general, see Example

5.1.4.) On the other hand, **A**-extremal epimorphisms are characterized using (topological) quotients and bijective **A**-reflection arrows in Proposition 5.1.6.

Lemma 5.1.3. *Let* **A** *be an epireflective subcategory of* **Top**. *If* $q\colon X \to Y$ *is quotient in* **Top** *then* Rq *is a regular epimorphism in* **A** (*hence it is an* **A**-*extremal epimorphism*).

Proof. Coadjoint functors preserve regular epimorphisms. The epireflector R is coadjoint, therefore Rq is a regular epimorphism in **A** (thus it is an **A**-extremal epimorphism). □

Example 5.1.4. Let X be the topological space on $\mathbb{R}$ in which precisely the sets of the form $U \setminus B$ are open, where U is open in $\mathbb{R}$ and $B \subseteq \{\frac{1}{n}; n = 1, 2, \ldots\}$. (This topology is sometimes called the Smirnov topology, see [SS, Example 64]). The space X is not regular, since 0 and the closed set $\{\frac{1}{n}; n = 1, 2, \ldots\}$ cannot be separated by open sets. It is first countable, thus it is sequential.

Since X is sequential, there is a quotient map q from a topological sum of spaces $C(\omega_0)$ to X. The map Rq, where R denotes the **Reg**-reflector, is an extremal epimorphism in **Reg**. But $RX \neq X$, so Rq is not quotient.

Lemma 5.1.5. *Let* **A** *be an epireflective subcategory of* **Top**. *Let* $m\colon X \to A$ *be a monomorphism in* **Top** (*i.e.,* m *is injective*) *with* $A \in \mathbf{A}$. *Then the* **A**-*reflection arrow* $r_X\colon X \to RX$ *of* X *is bijective and the mapping* $Rm\colon RX \to A$ *is an* **A**-*monomorphism* (*i.e., injection*).

Proof.

$$\begin{tikzcd} X \arrow[rr, "m"] \arrow[dr, "r_X"'] & & A \\ & RX \arrow[ur, "Rm"'] & \end{tikzcd}$$

In the situation described in the lemma, $m = Rm \circ r_X$. Since $m = Rm \circ r_X$ is injective, r_X is injective as well. The map r_X is surjective at the same time (**A** is epireflective), thus it is a bijection.

From the fact that $m = Rm \circ r_X$ is injective, with r_X being bijection, we can deduce that Rm is injective too. Thus Rm is an **A**-monomorphism. □

Proposition 5.1.6. *If* **A** *is an epireflective subcategory of* **Top** *then each* **A**-*extremal epimorphism* e *is a composite of an* **A**-*reflection arrow and a quotient map* (*in* **Top**), $e = r_X \circ q$. *Moreover* r_X *is bijective.*

Proof. Let $e\colon A \to B$ be an **A**-extremal epimorphism and $e = m \circ q$ be its (quotient map, mono)-factorization in **Top**, $q\colon A \to X$, $m\colon X \to B$. If $r_X\colon X \to$

RX is the $\mathbf{A}$-reflection arrow of X then the following diagram commutes.

$$\begin{tikzcd}
A \arrow[rr, "e"] \arrow[dr, "Rq"] \arrow[ddr, "q"'] & & B \\
& RX \arrow[ur, "Rm"] & \\
& X \arrow[u, "r_X"] \arrow[uur, "m"'] &
\end{tikzcd}$$

Since Rm is an $\mathbf{A}$-monomorphism (by Lemma 5.1.5) and e is an $\mathbf{A}$-extremal epimorphism, we obtain that Rm is an $\mathbf{A}$-isomorphism and m is an $\mathbf{A}$-reflection arrow of X.

The morphism m is simultaneously an epimorphism (since $\mathbf{A}$ is epireflective) and a monomorphism, thus it is a bijection. □

The following Corollary follows easily from Proposition 5.1.6. It can be considered as a partial converse of Lemma 5.1.3.

Corollary 5.1.7. *If* $\mathbf{A}$ *is epireflective in* **Top** *and* e *is an* $\mathbf{A}$*-extremal epimorphism, then* $e = Rq$ *for some quotient map* q *in* **Top**.

If the bijective reflection arrow r_X from Proposition 5.1.6 is a quotient map, it is an isomorphisms. Therefore

Corollary 5.1.8. *If* $\mathbf{A}$ *is a quotient reflective subcategory of* **Top** *then* e *is an* $\mathbf{A}$*-extremal epimorphism if and only if* e *is a quotient map.*

The above characterization of $\mathbf{A}$-extremal epimorphisms helps us to describe the coreflective subcategories of $\mathbf{A}$. We first include another fairly general characterization of coreflective subcategories. This characterization is the dual claim to the well-known theorem which can be found e.g. in [HS3, Theorem 37.1, Corollary 37.6] or [P, Theorems 2.2.4 and 2.3.2].

Theorem 5.1.9. *If* $\mathbf{A}$ *is a full and isomorphism-closed subcategory of a wellpowered, (extremal epi, mono)-category* $\mathbf{B}$ *that has coproducts, then the following are equivalent:*

(i) $\mathbf{A}$ *is monocoreflective in* $\mathbf{B}$.

(ii) $\mathbf{A}$ *is closed under the formation of coproducts and extremal quotients in* $\mathbf{B}$.

The monocoreflective hull of a subcategory $\mathbf{B}$ *of* $\mathbf{A}$ *consists precisely of all* $\mathbf{B}$*-extremal quotients of coproducts of* $\mathbf{A}$*-objects.*

We want to show that any epireflective subcategory $\mathbf{A}$ of **Top** fulfills the assumptions of Theorem 5.1.9. Note that a map between two spaces from $\mathbf{A}$ is an $\mathbf{A}$-monomorphism if and only if it is injective. This implies immediately

that **A** is wellpowered. It is well-known that coproducts in an epireflective subcategory of **Top** are obtained by applying the reflector on the coproducts in **Top**, thus **A** has coproducts. In the following theorem we show that every epireflective subcategory of **Top** is an (extremal epi, mono)-category.

Lemma 5.1.10. *If* **A** *is an epireflective subcategory of* **Top***, then* **A** *is* (*extremal epi, mono*)*-category.*

Proof. We know that **Top** is (extremal epi, mono)-factorizable. By Lemmas 5.1.3 and 5.1.5, if we are given an **A**-morphism, the reflector maps its (extremal epi, mono)-factorization in **Top** to an (extremal epi, mono)-factorization in **A**. To verify that **A** is an (extremal epi, mono)-category it remains to show the uniqueness of these factorizations.

Let $m_1 \circ e_1 = m_2 \circ e_2$ be two (extremal epi, mono)-factorizations of the same morphism in **A**. The extremal epimorphisms are of the form $e_i = r_i \circ q_i$, where q_i is a quotient map and r_i is a bijective reflection arrow.

Since $(m_i \circ r_i) \circ q_i$ are the (extremal epi, mono)-factorizations in **Top**, there exists an isomorphism d which makes the following diagram commute.

$$\begin{tikzcd}
 & \cdot \arrow[r, "r_1"] \arrow[dd, "d"] \arrow[dd, bend right=60, "Rd"'] & \cdot \arrow[dr, "m_1"] & \\
\cdot \arrow[ur, "q_1"] \arrow[dr, "q_2"'] & & & \cdot \\
 & \cdot \arrow[r, "r_2"'] & \cdot \arrow[ur, "m_2"'] &
\end{tikzcd}$$

Then Rd is the required isomorphism for the factorization in **A**. □

If $D_2 \in \mathbf{A}$, then the **A**-sums are precisely the topological sums. Note that if $I_2 \notin \mathbf{A}$ then $D_2 \in \mathbf{A}$ (with the exception of the trivial case that **A** contains only one-points spaces and the empty space).

It follows from Proposition 1.3.10 that every coreflective subcategory in **A** is monocoreflective. Hence Theorem 5.1.9 and Lemma 5.1.10 yield the following characterization of coreflective subcategories in **A**.

Proposition 5.1.11. *Let* **A** *be an epireflective subcategory of* **Top** *with* $D_2 \in \mathbf{A}$*. A subcategory* **C** *of* **A** *is coreflective in* **A** *if and only if it is closed under the formation of topological sums and* **A***-extremal quotients.*

From Theorem 5.1.9 we also see that the coreflective hull $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ of **D** in **A** can be formed by taking all **A**-extremal quotients of topological sums of spaces from **D**.

The following facts about AD-hulls in **A** and coreflective hulls in **A** are now quite obvious.

Lemma 5.1.12. *Let* **A** *be an epireflective subcategory of* **Top** *with* $I_2 \notin \mathbf{A}$*. Then* $\mathrm{AD}_{\mathbf{A}}(\mathbf{B}) = \mathrm{CH}(\mathbf{B}) \cap \mathbf{A}$ *and* $\mathrm{S}(\mathrm{AD}_{\mathbf{A}}(\mathbf{B})) \subseteq \mathrm{HAD}_{\mathbf{A}}(\mathbf{B}) \subseteq \mathrm{SCH}(\mathbf{B}) \cap \mathbf{A}$.

Proposition 5.1.13. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{D} \subseteq \mathbf{A}$*. Then the prime* T_2*-spaces contained in* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ *and the prime* T_2*-spaces contained in* $\mathrm{CH}(\mathbf{D})$ *are the same. I.e.,* $\{P \in \mathrm{CH}_{\mathbf{A}}(\mathbf{D}), P$ *is a prime* T_2*-space*$\} = \{P \in \mathrm{CH}(\mathbf{D}), P$ *is a prime* T_2*-space*$\}$*.*

In the case $\mathbf{A} = \mathbf{Top}_0$ *we moreover get* $\{P \in \mathrm{CH}_{\mathbf{A}}(\mathbf{D}), P$ *is prime*$\} = \{P \in \mathrm{CH}(\mathbf{D}), P$ *is prime*$\}$*.*

Proof. Let P be a prime T_2-space belonging to $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$. There is an $\mathbf{A}$-extremal epimorphism $e\colon A \to P$, where A is a sum of spaces from $\mathbf{D}$, which can be factorized as $r \circ q$ with q a quotient map and r a bijective reflection arrow.

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[dr, "e"'] & X \arrow[d, "r"] \\ & P \end{tikzcd}$$

Since there is a continuous bijection $r\colon X \to P$, the topology of X is finer than the topology of P and X is either discrete or a prime T_2-space. Thus $X \in \mathbf{ZD_0} \subseteq \mathbf{A}$. Since both X and P belong to $\mathbf{A}$, the $\mathbf{A}$-reflection arrow $r\colon X \to P$ is an isomorphism. Hence $P \in \mathrm{CH}(\mathbf{D})$.

The second part follows from the equality $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{CH}(\mathbf{D}) \cap \mathbf{A}$ which is valid for the subcategory $\mathbf{A} = \mathbf{Top}_0$. □

Theorem 5.1.14. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{D} \subseteq \mathbf{A}$*. If* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ *is hereditary then* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{CH}(\mathbf{D}) \cap \mathbf{A}$*.*

Proof. The inclusion $\mathrm{CH}(\mathbf{D}) \cap \mathbf{A} \subseteq \mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ holds for any $\mathbf{D} \subseteq \mathbf{A}$. We show the opposite inclusion.

If $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ is a hereditary coreflective subcategory of $\mathbf{A}$, then it is closed under the formation of prime factors (see [Č4, Theorem 1]). If $\mathbf{A} \subseteq \mathbf{Top}_1$ then any prime factor Y_a of Y is T_2. All of them belong to $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$. According to Proposition 5.1.13 prime T_2-spaces in $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ and $\mathrm{CH}(\mathbf{D})$ are the same. As Y is a quotient of spaces Y_a belonging to $\mathrm{CH}(\mathbf{D})$, we get $Y \in \mathrm{CH}(\mathbf{D})$.

If $\mathbf{A} = \mathbf{Top}_0$ then $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{CH}(\mathbf{D}) \cap \mathbf{A}$ holds for any class $\mathbf{D} \subseteq \mathbf{A}$. □

Corollary 5.1.15. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{D} \subseteq \mathbf{A}$*. If* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ *is hereditary then* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{AD}_{\mathbf{A}}(\mathbf{D}) = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D})$*.*

This corollary implies that some of the results we will prove about HAD-hulls in $\mathbf{A}$ can be applied in the case of hereditary coreflective hulls in $\mathbf{A}$ as well.

5.1.2 Coreflective hulls and epireflections

We next provide a characterization of coreflective hulls in $\mathbf{A}$ using the $\mathbf{A}$-reflection.

Lemma 5.1.16. *If* $\mathbf{A}$ *is epireflective in* $\mathbf{Top}$*,* $D_2 \in \mathbf{A}$ *and* $\mathbf{C}$ *is coreflective in* $\mathbf{Top}$*, then* $\mathbf{D} = \{RC : C \in \mathbf{C}\}$ *is coreflective in* $\mathbf{A}$*. (*R *denotes the* $\mathbf{A}$*-reflector.)*

Proof. Let $D \in \mathbf{D}$, i.e., there is $C \in \mathbf{C}$ such that D is the $\mathbf{A}$-reflection of C. Let $e\colon D \to D'$ be an $\mathbf{A}$-extremal epimorphism and $m \circ q = e \circ r_C$ be the (extremal epi, mono)-factorization of $e \circ r_C$ in **Top**.

$$\begin{tikzcd}
C \arrow[rr, "q"] \arrow[dd, "r_C"'] & & C' \arrow[dl, "r_{C'}"'] \arrow[dd, "m"] \\
& RC' \arrow[dr, "Rm"] & \\
D \arrow[ur, "Rq"] \arrow[rr, "e"'] & & D'
\end{tikzcd}$$

By an easy computation we get:

$$Rq \circ r_C = r_{C'} \circ q,$$
$$Rm \circ Rq \circ r_C = Rm \circ r_{C'} \circ q = m \circ q = e \circ r_C$$

and, since r_C is an epimorphism,

$$e = Rm \circ Rq.$$

Clearly $C' \in \mathbf{C}$ and $RC' \in \mathbf{D}$. By Lemma 5.1.5 Rm is an $\mathbf{A}$-monomorphism. Since $e = Rm \circ Rq$ is an $\mathbf{A}$-extremal epimorphism, Rm is an $\mathbf{A}$-isomorphism and $D' \in \mathbf{D}$.

We have shown that $\mathbf{D}$ is closed under $\mathbf{A}$-extremal epimorphisms. The closedness under sums follows form the fact that R preserves sums. Therefore $\mathbf{D}$ is coreflective. □

Lemma 5.1.17. *If* $\mathbf{A}$ *is epireflective in* **Top**, $D_2 \in \mathbf{A}$ *and* $\mathbf{C}$ *is coreflective in* **Top**, *then* $\mathbf{D} = \{RC : C \in \mathbf{C}$ *and the* $\mathbf{A}$*-reflection arrow* $r_C\colon C \to RC$ *is bijective*$\}$ *is coreflective in* $\mathbf{A}$. (R *denotes the* $\mathbf{A}$*-reflector.*)

Proof. Note that in the proof of the last lemma $m = Rm \circ r_{C'}$ is a monomorphism, hence $r_{C'}$ is a bijection. □

Theorem 5.1.18. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **Top**, $D_2 \in \mathbf{A}$ *and* $\mathbf{D} \subseteq \mathbf{A}$. *Then* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \{RC : C \in \mathrm{CH}(\mathbf{D})\} = \{RC : C \in \mathrm{CH}(\mathbf{D})$ *and the* $\mathbf{A}$*-reflection arrow* $r_C\colon C \to RC$ *is bijective*$\}$.

Proof. Let $\mathbf{B} = \{RC : C \in \mathrm{CH}(\mathbf{D})\}$ and $\mathbf{C} = \{RC : C \in \mathrm{CH}(\mathbf{D})$ and the $\mathbf{A}$-reflection arrow $r_C\colon C \to RC$ is bijective$\}$.

Since $\mathbf{C}$ is coreflective in $\mathbf{A}$, we have $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) \subseteq \mathbf{C}$. Obviously, $\mathbf{C} \subseteq \mathbf{B}$.

We next show $\mathbf{B} \subseteq \mathrm{CH}_{\mathbf{A}}(\mathbf{D})$. If B is the $\mathbf{A}$-reflection of $C \in \mathrm{CH}(\mathbf{D})$, then we have a quotient $q\colon \coprod D_i \to C$ with $D_i \in \mathbf{D}$. Then $r_C \circ q = Rq$ is an $\mathbf{A}$-extremal epimorphism by Lemma 5.1.3.

We get $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) \subseteq \mathbf{C} \subseteq \mathbf{B} \subseteq \mathrm{CH}_{\mathbf{A}}(\mathbf{D})$, hence all these categories are equal to each other. □

Note that if $\mathbf{A}$ is quotient reflective, then bijective reflection arrows are isomorphisms. Thus from Theorem 5.1.18 we get that in quotient reflective subcategories coreflective hull and AD-hull coincide.

Corollary 5.1.19. *If* $\mathbf{A}$ *is a quotient reflective in* **Top**, *then* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{CH}(\mathbf{D}) \cap \mathbf{A} = \mathrm{AD}_{\mathbf{A}}(\mathbf{D})$.

A topological space X is a *k_R-space* if a function $f\colon X \to \mathbb{R}$ is continuous whenever the restriction $f|_K$ is continuous for every compact subset $K \subseteq X$. For more information on k_R-spaces see e.g. [Hu1] or [L]. (Most authors require a k_R-space to be moreover completely regular.)

We next show that the completely regular k_R-spaces coincide with the coreflective hull of compact Hausdorff spaces in the epireflective subcategory **CReg**. This will be used to get an example of an AD-class in **CReg** which is not coreflective in **CReg**.

Lemma 5.1.20. *Let R be the* **CReg**-*reflector and C be the* **CGen**-*coreflector. If X is a completely regular k_R-space then $X = RCX$.*

Proof. We want to show that $id\colon CX \to X$ is the **CReg**-reflection arrow of CX. Since $\mathbf{CReg} = \mathrm{EH}(\mathbb{R})$, it suffices to show that $C(X, \mathbb{R}) = C(CX, \mathbb{R})$.

Recall that the compact subspaces of X and CX are the same (and the relative topologies on the compact subspaces coincide as well). Hence if $f \in C(CX, \mathbb{R})$ then $f|_K$ is continuous for every compact subspace K of X. Since X is a k_R-space, this implies $f \in C(X, \mathbb{R})$. □

Proposition 5.1.21 ([L, Theorem 2.1(ii)], [Wy1, Theorem 1.6]). *Let* $\mathbf{A} = \mathbf{CReg}$ *and* $\mathbf{D}$ *be the class of all compact Hausdorff spaces. Then* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ *is precisely the class of all completely regular k_R-spaces.*

Proof. By Theorem 5.1.18 $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ consists precisely of spaces RX, where $X \in \mathrm{CH}(\mathbf{D})$ and the $\mathbf{A}$-reflection arrow $X \to RX$ is bijective. (Here R means again the **CReg**-reflector.)

$\boxed{k_R \subseteq \mathrm{CH}_{\mathbf{A}}(\mathbf{D})}$ Follows directly from Lemma 5.1.20 and Theorem 5.1.18.

$\boxed{\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) \subseteq k_R}$ Let $X = RY$ for some compactly generated space Y with the bijective $\mathbf{A}$-reflection arrow $r\colon Y \to X$. Let f be a map from X to $\mathbb{R}$. Suppose that $f|_K$ is continuous for each compact subset $K \subseteq X$. We want to show that f is continuous.

Let L be a compact subspace of Y. Then the subspace $r[L]$ of X is compact as well. Thus $f \circ r|_L$ is continuous for every compact subspace of Y. This implies the continuity of $f \circ r$ (Y is a k_R-space) and, in turn, the continuity of f (X is the **CReg**-reflection of Y). □

Example 5.1.22. E. Michael constructed in [Mi] an example of a normal k_R-space which is not compactly generated. This shows that for $\mathbf{A}$ and $\mathbf{D}$ with the same meaning as in the above lemma $\mathrm{AD}_{\mathbf{A}}(\mathbf{D}) = \mathrm{CH}(\mathbf{D}) \cap \mathbf{A} \subsetneq \mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ holds. Thus $\mathrm{AD}_{\mathbf{A}}(\mathbf{D})$ is an AD-class in $\mathbf{A}$, which is not coreflective in $\mathbf{A}$.

5.2 Heredity and prime factors

As it was already mentioned, it was shown in [Č4] that a coreflective subcategory **C** of **A** is hereditary if and only if it is closed under prime factors. The objective of this section is to extend this result to the case of AD-class **B** in **A**. We show that it holds under some assumptions on **A** or on **B**.

5.2.1 Closedness under prime factors

We first show that also in this case closedness under prime factors implies heredity. The proof follows the proof of [Č4, Theorem 2, Theorem 7].

Lemma 5.2.1. *Let* **B** *be additive and divisible in* **A**, **A** *being an epireflective subcategory of* **Top** *with* $I_2 \notin \mathbf{A}$. *If* **B** *is closed under prime factors then it is hereditary.*

Proof. Let X be an arbitrary space from **B** and Y be its nonempty subspace. We want to show that $Y \in \mathbf{B}$. Since $Y \in \mathbf{A}$, it suffices to show that $Y \in \mathrm{CH}(\mathbf{B})$. We show that any non-discrete prime factor of Y belongs to $\mathrm{CH}(\mathbf{B})$. Any prime factor Y_a is a subspace of the corresponding prime factor X_a of X. Since **B** is closed under prime factors, every X_a belongs to **B** and by Lemma 1.2.3 we obtain that all prime factors of Y belong to **B** as well. The space Y is a quotient of a topological sum of its prime factors, thus $Y \in \mathbf{B}$. □

The opposite direction is more difficult, hence also more interesting. We obtained only some partial results. To present them we need to recall the definition of the operation $\triangle$ from [Č4].

Definition 5.2.2. If X and Y are topological spaces and $b \in Y$ be a point such that the set $\{b\}$ is closed, then we denote by $X \triangle_b Y$ the topological space on the set $X \times Y$ which has the final topology w.r.t the family of maps $\{f, g_a; a \in X\}$, where $f\colon X \to X \times Y$, $f(x) = (x, b)$ and $g_a\colon Y \to X \times Y$, $g_a(y) = (a, y)$.

In other words, $X \triangle_b Y$ is the quotient of $X \sqcup (\coprod_{a \in X} Y)$ with respect to the map obtained as the combination of the maps f and g_a, $a \in X$. Since $X \triangle_b Y$ is constructed from X and Y using only topological sums and quotient maps, any divisible and additive class containing X and Y contains $X \triangle_b Y$, too.

A local base for the topology of $X \triangle_b Y$ at a point (a, y), $y \neq b$ is $\{\{a\} \times V; V$ is an open neighborhood of y in Y, $b \notin V\}$. A base at (a, b) consists of all sets of the form $\bigcup_{x \in U} \{x\} \times V_x$ where U is an open neighborhood of a in X each V_x is an open neighborhood of b in Y.

Figure 5.1 depicts the space $X \triangle_b Y$ by showing typical sets from the neighborhood basis.

We denote by $X^a_{(Y,b)}$ the subspace of $X \triangle_b Y$ on the set $\{(a, b)\} \cup (X \setminus \{a\}) \times (Y \setminus \{b\})$. The following lemma shows why this operation is useful. The easy proof is omitted.

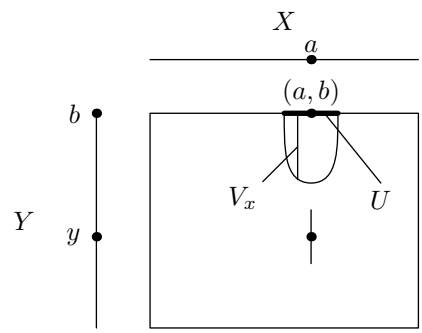


Figure 5.1: The space $X \triangle_b Y$

Lemma 5.2.3 ([Č4]). *Let X, Y be topological spaces and $b \in Y$ be a point such that the set $\{b\}$ is closed and not open in Y. The map $q\colon X^a_{(Y,b)} \to X_a$ given by $q(x,y) = x$ is quotient.*

This yields the following proposition:

Proposition 5.2.4. *Let $\mathbf{B}$ be an HAD-class in an epireflective subcategory $\mathbf{A}$ of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$. Let for any $X \in \mathbf{B}$ there exist $Y \in \mathbf{B}$ and a non-isolated point $b \in Y$ with $\{b\}$ being closed in Y such that $X \triangle_b Y$ belongs to $\mathbf{A}$. Then $\mathbf{B}$ is closed under the formation of prime factors.*

Proof. Let $X \in \mathbf{B}$ and $a \in X$. We want to show that $X_a \in \mathbf{B}$. By the assumption there exist $Y \in \mathbf{B}$ and $b \in Y$ such that $\{b\}$ is closed but not open and $X \triangle_b Y \in \mathbf{A}$.

Since $X \triangle_b Y$ is constructed using quotients and sums, we get $X \triangle_b Y \in \mathbf{B}$. Thus also its subspace $X^a_{(Y,b)}$ belongs to $\mathbf{B}$ and X_a is a quotient of this space. □

Definition 5.2.5. We say that a subcategory $\mathbf{A}$ of $\mathbf{Top}$ is closed under $\triangle$ if $X \triangle_b Y \in \mathbf{A}$ whenever $X, Y \in \mathbf{A}$ and $b \in Y$.

Proposition 5.2.6. *Let $\mathbf{A}$ be an epireflective subcategory of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$. If $\mathbf{A}$ is closed under $\triangle$ and $\mathbf{B}$ is hereditary, additive and divisible in $\mathbf{A}$, then $\mathbf{B}$ is closed under prime factors.*

Proof. Let $X \in \mathbf{B}$ and $a \in X$. We want to show that $X_a \in \mathbf{B}$.

Choose any space $Y \in \mathbf{B}$ and a non-isolated point $b \in Y$ such that $\{b\}$ is closed. (We can w.l.o.g. assume that $\mathbf{B}$ contains a non-discrete space, since discrete spaces are closed under prime factors trivially. If $\mathbf{A} = \mathbf{Top}_0$, then $\mathbf{A}$ is a quotient reflective subcategory of $\mathbf{Top}$ and the smallest coreflective subcategory of $\mathbf{A}$ containing some non-discrete space contains $\mathbf{FG} \cap \mathbf{A}$. Thus we can take for Y the Sierpiński space S. If $\mathbf{A} \neq \mathbf{Top}_0$ then $\mathbf{A} \subseteq \mathbf{Top}_1$ and all one-point sets are closed in spaces belonging to $\mathbf{A}$. Thus in this case it suffices to take any non-discrete space for Y.)

The result follows now from Proposition 5.2.4. □

It is relatively easy to show that many well-known epireflective subcategories of **Top** are closed under the operation $\triangle$, e.g. $\mathbf{Top}_0$, $\mathbf{Top}_1$, **Haus**, **Reg**, **Tych**, $\mathbf{ZD_0}$. By [Č4, Proposition 1] every quotient reflective subcategory of **Top** is closed under $\triangle$. This leads to the following observation:

Corollary 5.2.7. *If* $\mathbf{A}$ *is a quotient reflective subcategory of* **Top** (*in particular, if* $\mathbf{A} = \mathbf{Top}_{0,1,2}$) *and* $\mathbf{B}$ *is an HAD-class in* $\mathbf{A}$*, then* $\mathbf{B}$ *is closed under prime factors.*

Since in a quotient reflective subcategory $\mathbf{A}$ HAD-classes are precisely the hereditary coreflective subcategories, Corollary 5.2.7 follows also from [Č4, Theorem 1].

In Example 5.4.7 we will show that epireflective subcategories of **Top** need not be closed under $\triangle$ in general. Therefore it could be interesting to show the above result under some less restrictive conditions on $\mathbf{A}$. We will show that we can obtain it by putting some additional conditions on $\mathbf{B}$.

We first describe a relationship between the operations $\bigvee$ (see Definition 1.5.8) and $\triangle$. Namely, we show that $X \triangle_b Y$ can be understood as a subspace of $X \times (\bigvee_X Y)$. This embedding is illustrated by Figure 5.2 – thick lines in the picture correspond to the points of the subspace which is homeomorphic to $X \triangle_b Y$.

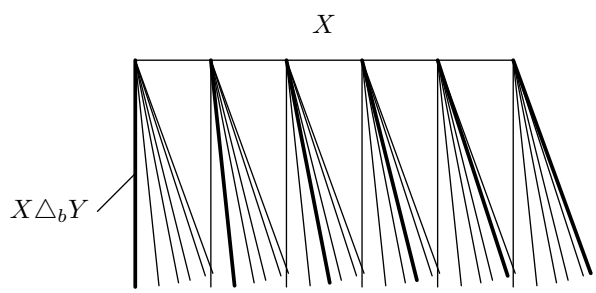


Figure 5.2: $X \triangle_b Y$ as a subspace of $X \times (X \triangle_b Y)$

Lemma 5.2.8. *Let* X, Y *be topological spaces and* $b \in Y$ *be a point such that the set* $\{b\}$ *is closed. The space* $X \triangle_b Y$ *can be embedded into* $X \times (\bigvee_X Y)$.

Proof. Denote $Z := \bigvee_X Y$. The underlying set of Z is $\{0\} \cup \bigcup_{x \in X} \{x\} \times (Y \setminus \{b\})$ and Z is quotient with respect to the map $q\colon \coprod_{x \in X} Y \to Z$, $q(x,y) = (x,y)$ if $y \neq b$ and $q(x,y) = 0$ otherwise. (We can assume that the underlying set of $\coprod_{x \in X} Y$ is $X \times Y$.)

We define $f\colon X \triangle_b Y \to X \times (\coprod_{x \in X} Y)$ by $f(x,y) = (x,x,y)$. Let us denote $q' := id_X \times q$. We claim that $e = q' \circ f$ is an embedding.

$$\begin{tikzcd} X \triangle_b Y \arrow[r, hook, "e"] \arrow[dr, "f"'] & X \times Z \\ & X \times (\coprod_{x \in X} Y) \arrow[u, "q'"'] \end{tikzcd}$$

It is easy to check that e is injective. Standard basis for the product topology $X \times Z$ induces basis for the subspace on the subset $e[X \triangle_b Y]$ which consists of the sets of the form $q'(\{(x,x)\} \times V)$, where V is open in Y and $b \notin V$, and sets of the form $\bigcup_{x \in U} q'(\{(x,x)\} \times V_x)$ where U is an open set in X and V_x is a neighborhood of b in Y for each x. This base is in a one-to-one correspondence with the base for $X \triangle_b Y$ described above. $\square$

Definition 5.2.9. We say that a subcategory $\mathbf{A}$ of $\mathbf{Top}$ is closed under $\bigvee$ if for any $A \in \mathbf{A}$ and any set I the wedge sum $\bigvee_I A$ belongs again to $\mathbf{A}$.

Corollary 5.2.10. *If an epireflective subcategory* $\mathbf{A}$ *of* $\mathbf{Top}$ *is closed under* $\bigvee$*, then it is closed under* $\triangle$*.*

Now we introduce a (sufficient) condition on a space Y and $b \in Y$, under which every wedge sum $\bigvee_I (Y, b)$ belongs to $\mathbf{A}$ (and, consequently, $X \triangle_b Y \in \mathbf{A}$ for each $X \in \mathbf{A}$).

Definition 5.2.11. Let Y, Z be topological spaces and $b \in Y$ be a point such that the set $\{b\}$ is closed. We say that $P(b, Y, Z)$ holds if there exists an open local base $\mathcal{B}$ at b in Y, a point $a \in Z$ and an open neighborhood U_0 of a in Z such that for any $V \in \mathcal{B}$ there exists a continuous map $f\colon Y \to Z$ with $f(b) = a$, $f^{-1}(U_0) = V$.

We first present some simple examples of spaces with this property. Let $I = \langle 0, 1 \rangle$. We claim that $P(b, I, I)$ holds for any $b \in I$. If $b = 0$ we take $\mathcal{B} = \{\langle 0, \varepsilon); 0 < \varepsilon < 1\}$, symmetrically $\mathcal{B} = \{(1 - \varepsilon, 1\rangle; 0 < \varepsilon < 1\}$ if $b = 1$, otherwise $\mathcal{B} = \{(b - \varepsilon, b + \varepsilon) \cap (0, 1); \varepsilon > 0\}$. (For a and U_0 we can take e.g. 0 and $\langle 0, \frac{1}{2})$, 1 and $(\frac{1}{2}, 1\rangle$, b and $(\frac{b}{2}, \frac{1+b}{2})$ respectively.)

Another example: Let Y be any zero-dimensional space, D_2 be the 2-point discrete space. Any point has a clopen local base, therefore $P(b, Y, D_2)$ for any $b \in Y$.

Example 5.2.12. Let X be any infinite space with the cofinite topology and $b \in X$. Then $P(b, X, X)$.

To show this, we take all neighborhoods of b for $\mathcal{B}$. Let $a = b$, $u \neq b$ and $U_0 = X \setminus \{u\}$.

Any $V \in \mathcal{B}$ has the form $V = X \setminus F$, where F is a finite set and $b \notin F$. Then there exists a bijection $h\colon V \to U_0$ such that $h(b) = b$. We can define a map $f\colon X \to X$ by $f|_V = h$ and $f[F] = \{u\}$. Clearly, f is a continuous map.

Lemma 5.2.13. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$*. Let* $Y, Z \in \mathbf{A}$*,* $b \in Y$ *be a non-isolated point such that the set* $\{b\}$ *is closed in* Y *and* $P(b, Y, Z)$ *holds. Then any wedge sum* $\bigvee_I (Y, b)$ *belongs to* $\mathbf{A}$*.*

Proof. It suffices to show that there exists an initial monosource with domain $\bigvee_I Y$ and codomain in $\mathbf{A}$.

Let $\mathcal{B}$ be the local base at b from the definition of $P(b, Y, Z)$. Then the sets $\{0\} \cup (\bigcup_{i \in I} (V_i \setminus \{b\}))$, $V_i \in \mathcal{B}$, form a local base for $\bigvee_I Y$ at 0. Let us denote this local base by $\mathcal{B}'$. For each $W = \{0\} \cup (\bigcup_{i \in I} (V_i \setminus \{b\})) \in \mathcal{B}'$ and each $i \in I$

there is a map $f_i\colon Y \to Z$ with $f_i^{-1}(U_0) = V_i$, where U_0 is the open set from Definition 5.2.11. Let $f_W\colon \bigvee_I Y \to Z$ be the map determined by the maps f_i. Then we have $W = \{0\} \cup (\bigcup_{i\in I}(V_i \setminus \{b\})) = f_W^{-1}(U_0)$.

For any $a \in Y$, $a \neq b$, the point (i, a) has a neighborhood base consisting of sets $\{i\} \times U$, where U is an open neighborhood of a in Y with $b \notin U$. For a set of the form $\{i\} \times U$ with $b \notin U$ the equality $\{i\} \times U = g_i^{-1}(U)$ holds, where $g_i\colon \bigvee_I Y \to Y$ is given by $g_i(i, y) = y$ and $g_i(j, y) = b$ for $j \neq i$. This map is clearly continuous.

The family g_i, $i \in I$, is obviously a monosource. Let us take together with them also all maps f_W, $W \in \mathcal{B}'$.

All these maps are continuous with respect to the topology of $\bigvee_I Y$. So $\bigvee_I Y$ is finer than the initial topology with respect to those maps. On the other hand, for every set from the basis $\mathcal{B}'$ of $\bigvee_I Y$ described above we have found a map to a space from $\mathbf{A}$ such that this basic set is a preimage of some open set.

Together we get that the initial topology with respect to this family of maps is precisely the topology of $\bigvee_I Y$. So we have obtained the desired initial monosource. □

Corollary 5.2.14. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **Top**. *Let* $X, Y, Z \in \mathbf{A}$, $b \in Y$ *be a non-isolated point such that* $\{b\}$ *is closed in* Y *and* $P(b, Y, Z)$ *holds. Then* $X \triangle_b Y \in \mathbf{A}$.

Theorem 5.2.15. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **Top** *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{B}$ *be an HAD-class in* $\mathbf{A}$. *If* $\mathbf{B}$ *contains a space* Y *with* $P(b, Y, Z)$ *for some* $Z \in \mathbf{A}$ *and a non-isolated point* $b \in Y$ *such that* $\{b\}$ *is closed in* Y, *then* $\mathbf{B}$ *is closed under prime factors.*

Proof. The claim follows easily from Proposition 5.2.4 and Corollary 5.2.14. □

Corollary 5.2.16. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **Top** *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{B}$ *be an HAD-class in* $\mathbf{A}$. *If* $\mathbf{B}$ *contains an infinite space with the cofinite topology or it contains a non-discrete zero-dimensional space* (*in particular a prime* T_2*-space*), *then* $\mathbf{B}$ *is closed under prime factors.*

Corollary 5.2.17. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **Top** *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{B}$ *be an HAD-class in* $\mathbf{A}$. *If* $\mathbf{B}$ *contains a prime space, then* $\mathbf{B}$ *is closed under prime factors.*

Proof. If $\mathbf{A}$ contains a prime T_2-space then the claim follows from Corollary 5.2.16.

Now assume that $\mathbf{A}$ contains a non-Hausdorff prime space P. To resolve this case we provide an argument which will be used several more times in this thesis (see the next subsection).

Since P is not T_2, there exists a point b which cannot be separated from the non-isolated point a of P. The subspace on the set $\{a, b\}$ is homeomorphic to the Sierpiński space S. Thus we get $S \in \mathbf{A}$, $\mathbf{A} = \mathbf{Top}_0$ and the result now follows from Corollary 5.2.7. □

5.2.2 The prime T_2-spaces (usually) suffice

We have already seen several times, that our situation is usually better if we deal only with prime T_2-spaces. The aim of this section is to convince the reader that only in a few exceptional situations we need to take all prime spaces into account.

We start by a few trivial observations. Let P be a prime space, which is not T_2. Let us denote by U the intersection of all neighborhoods of the accumulation point a. Since P is not Hausdorff, $U \setminus \{a\} \neq \emptyset$.

Any 2-point subspace of U containing a is homeomorphic to the Sierpiński space S. So if an epireflective subcategory $\mathbf{A}$ contains P, then $\mathrm{EH(S)} = \mathbf{Top}_0 \subseteq \mathbf{A}$. Together with the assumption $I_2 \notin \mathbf{A}$ we get $\mathbf{A} = \mathbf{Top}_0$.

So we have observed that the prime spaces which are not T_2 are important only for $\mathbf{A} = \mathbf{Top}_0$ and from Corollary 5.2.7 we know, that in this case the characterization of heredity using prime factors holds. (Later in Chapter 6 we will extend our results to bireflective subcategories of **Top** too. The bireflective subcategory corresponding to $\mathbf{Top}_0$ is **Top**. For this subcategory again everything works fine.)

Even for the subcategory $\mathbf{Top}_0$ the prime T_2-spaces are usually sufficient. By Lemma 1.5.15 if P is not finitely generated then there exists a prime T_2-space P_1 with $\mathrm{CH}(P_1) = \mathrm{CH}(P)$. So only in the case that all prime spaces in $\mathbf{B}$ (where $\mathbf{B}$ is an AD-class in $\mathbf{A}$ or a coreflective subcategory of $\mathbf{A}$ which is generated by a class of prime spaces) are finitely generated, $\mathbf{B}$ is not generated by prime T_2-spaces.

To sum up briefly our observations: with the exception of the cases $\mathbf{A} = \mathbf{Top}_0$ and $\mathbf{B} \subseteq \mathbf{FG}$ we do not need to use prime spaces which are not T_2.

5.3 HAD-classes containing a prime space

We have shown in the foregoing section that if an AD-class $\mathbf{B}$ in $\mathbf{A}$ contains a prime space, then heredity of $\mathbf{B}$ is equivalent to the closedness under prime factors. Clearly a class, which is closed under prime factors, must contain a prime space. It is natural to ask, whether all HAD-classes (in some epireflective subcategory $\mathbf{A}$) must contain a prime space.

We were not able to solve this question completely, but we will show that if an HAD-class $\mathbf{B}$ in $\mathbf{A}$ contains a space which is not locally connected or a Hausdorff non-discrete space, then it contains a prime space. As a consequence we get that the mentioned equivalence is valid whenever $\mathbf{A} \subseteq \mathbf{Haus}$. By the results of the last section, the same is true if $\mathbf{A}$ is closed under $\triangle$.

5.3.1 AD-classes containing $C(\alpha)$

In this part we obtain the result that an AD-class contains a prime space if and only if it contains some $C(\alpha)$. We first state two lemmas needed in the proof.

Recall that if α is a regular cardinal, then a subset $V \subseteq C(\alpha)$ is closed if and only if $\alpha \in V$ or $\operatorname{card} V < \alpha$. Note that this implies that every injective map $f\colon C(\alpha) \to C(\alpha)$ such that $f(\alpha) = \alpha$ is continuous.

Lemma 5.3.1. *Let α be an infinite regular cardinal. If $Y \prec C(\alpha)$ is a prime space (with the accumulation point α), then $C(\alpha) \in \mathrm{CH}(Y)$.*

Proof. Let f_i, $i \in I$, be the family of all injective mappings $f_i\colon \alpha \cup \{\alpha\} \to \alpha \cup \{\alpha\}$ such that $f_i(\alpha) = \alpha$. Let us denote by X the quotient space with respect to the mapping $[f_i]\colon \coprod_{i \in I} Y \to \alpha \cup \{\alpha\}$. (In other words, X has the final topology with respect to this family of mappings.) We claim that $X = C(\alpha)$.
One of the maps f_i is the identity, hence $Y \prec X$ and α is non-isolated in X.
Since $Y \prec C(\alpha)$ and all f_i's are continuous as maps from $C(\alpha)$ to $C(\alpha)$, we get $X \prec C(\alpha)$.
To verify that $C(\alpha) \prec X$ we show that any set which is not closed in $C(\alpha)$ is not closed in X.

$X \prec C(\alpha)$ implies that X is a prime space, therefore we only need to compare the sets not containing its accumulation point α. So let V be a subset of α with the cardinality α and $\alpha \notin V$. Then there exists an $i \in I$ such that f_i maps bijectively the set α to V. Since the subset α is not closed in Y (the point α is not isolated), we get that V is not closed in X. □

Lemma 5.3.2. *Let α be any infinite cardinal. If $Y \prec C(\alpha)$ is a prime space (with the accumulation point α), then there exists a regular cardinal β with $C(\beta) \in \mathrm{CH}(Y)$. (More precisely β is the cofinality of α.)*

Proof. Let β be the cofinality of α. There exists a quotient map $q\colon C(\alpha) \to C(\beta)$ which maps only the point α to β. Let Y' be the quotient of Y with respect to the same map q. Then Y' is a prime space, since $q^{-1}(\beta) = \{\alpha\}$ and α is not isolated in Y. Moreover $Y' \prec C(\beta)$ and β is a regular cardinal, thus $C(\beta) \in \mathrm{CH}(Y') \subseteq \mathrm{CH}(Y)$ by Lemma 5.3.1. □

Corollary 5.3.3. *If an AD-class* **B** *in an epireflective subcategory* **A** *such that $D_2 \in$* **A** *contains a prime T_2-space then* **B** *contains $C(\alpha)$ for some regular cardinal number α.*

Proof. Let P be a prime space with the accumulation point a. Denote by α the smallest cardinality of a non-closed subset of $P \setminus \{a\}$. Let C be some such subset. (Note that since X is T_2, all finite subsets of $P \setminus \{a\}$ are closed, hence α must be an infinite cardinal.)

If V is any subset of C with cardinality smaller than α then it is closed (since α was chosen as the smallest cardinality of a non-closed set). Therefore $C \cup \{a\}$ is a prime subspace of P and it is finer than $C(\alpha)$. (In the case that α is regular it is even homeomorphic to $C(\alpha)$, but in either case complements of all basic neighborhoods B_β of α are closed.) □

Since every prime T_2-space is zero-dimensional, Corollary 5.3.3 could be also deduced from Proposition 5.3.12. But the proof presented here is more straightforward.

5.3.2 How to obtain a prime space

We now turn our attention to some conditions which are sufficient to enforce that **B** contains a prime space.

Let us denote by **Con** the class of all connected spaces. The coreflective hull $\mathrm{CH}(\mathbf{Con}) = \mathbf{Con}_{\mathrm{sum}}$ consists precisely of sums of connected spaces (Example 1.5.29). An equivalent characterization is that $X \in \mathrm{CH}(\mathbf{Con})$ if and only if each point has a connected open neighborhood (see Proposition 1.5.28).

Proposition 5.3.4. *If X is not a sum of connected spaces then there exists a quotient map $f\colon X \to P$, where P is a prime T_2-space and $P \prec C(\alpha)$.*

Proof. Since X does not belong to $\mathbf{Con}_{\mathrm{sum}}$, there exists $a \in X$ such that no open neighborhood of a is connected. This means for any open neighborhood U of a there exist disjoint open proper subsets V, W of U such that $V \cup W = U$. Using this fact we construct by transfinite induction a decreasing family U_α of open neighborhoods of a.

We put $U_0 = X$. For any β the neighborhood U_β can be divided into two disjoint open non-empty sets. Let us denote by $U_{\beta+1}$ that one which contains the point a.

Now suppose that β is a limit ordinal and U_γ is already defined for each $\gamma < \beta$. We put $U_\beta := \bigcap_{\gamma<\beta} U_\gamma$ if this set is open. If not, we stop the process and put $\alpha := \beta$. (We must stop at some ordinal β, otherwise there would be a proper class of open sets in X.)

Thus we get a system $(U_\beta)_{\beta<\alpha}$ of open neighborhoods of a with the following properties: $U_\beta \subsetneqq U_\gamma$ holds whenever $\beta > \gamma$. For any limit ordinal $\beta < \alpha$ the equality $U_\beta = \bigcap_{\gamma<\beta} U_\gamma$ holds. The set $U_\beta \setminus U_{\beta+1}$ is open for any $\beta < \alpha$, but $\bigcap_{\beta<\alpha} U_\beta$ is not open.

Define $f\colon X \to \alpha \cup \{\alpha\}$ by

$$f(x) = \sup\{\beta \in \alpha : x \in U_\beta\}.$$

We have $f^{-1}(B_\beta) = U_\beta$ and $f^{-1}(\beta) = U_\beta \setminus U_{\beta+1}$ for any $\beta < \alpha$ and $f^{-1}(\alpha) = \bigcap_{\beta<\alpha} U_\beta$. Thus the quotient space on $\alpha \cup \{\alpha\}$ is finer than $C(\alpha)$ and the point α is not isolated in it. Hence it is a prime T_2-space. □

Recall that a topological space X is totally disconnected if all components of X are singletons. Totally disconnected spaces form a quotient reflective subcategory **TD** of **Top**.

If a space X is totally disconnected, then the only non-empty connected subsets of X are the one-point sets. So if a totally disconnected space X is a sum of connected spaces, then X is clearly discrete.

Corollary 5.3.5. *If X is non-discrete and totally disconnected then there exists a quotient map from X to a prime T_2-space.*

All zero-dimensional spaces T_0-spaces are totally-disconnected, so the above corollary applies to the class $\mathbf{ZD_0}$ as well. We will see in Proposition 5.3.12 that in the case of zero-dimensional spaces this result can be reformulated a little

bit sharper, which leads to the description of atoms above **Disc** in the lattice of coreflective subcategories of the category $\mathbf{ZD_0}$.

Lemma 5.3.6. *Let X be a topological space. If X is not locally connected then there exists an open subspace V of X such that V is not a sum of connected spaces.*

Proof. If X is not locally connected then there exist a point x and an open neighborhood V of x such that no open neighborhood U of x with $U \subseteq V$ is connected. So x has no open connected neighborhood in the subspace V and $V \notin \mathrm{CH}(\mathbf{Con})$. □

Corollary 5.3.7. *Let $\mathbf{A}$ be an epireflective subcategory of* **Top** *with $I_2 \notin \mathbf{A}$. If $\mathbf{B}$ is an HAD-class in $\mathbf{A}$ and $\mathbf{B}$ contains at least one space which is not locally-connected, then $\mathbf{B}$ is closed under prime factors.*

Proposition 5.3.8. *If X is Hausdorff and not discrete. Then X contains a subspace Y, such that there exists a prime T_2-space P which is quotient of Y.*

Proof. Let a be any non-isolated point in X. We would like to get a subspace Y in which a is again non-isolated and which contains enough disjoint open subsets.

By transfinite induction we construct a non-empty open set U_β for each $\beta < \alpha$, such that for each $\beta, \gamma < \alpha$ the following hold:
(1) If $\beta \neq \gamma$ then $U_\beta \cap U_\gamma = \emptyset$;
(2) $a \in V_\beta = X \setminus \overline{\bigcup\limits_{\eta \leq \beta} U_\eta} = \mathrm{Int}(X \setminus \bigcup\limits_{\eta \leq \beta} U_\eta)$;
(3) if $\gamma < \beta$ then $U_\beta \subseteq V_\gamma$;
(4) $a \in \overline{\bigcup\limits_{\eta < \alpha} U_\eta}$.

$\beta = 0$: Since a is not-isolated, there exists $b \neq a$ in X. By Hausdorffness we have non-empty open sets U, V with $U \cap V = \emptyset$, $a \in V$, $b \in U$. We put $U_0 := U$. Since $a \in V \subseteq X \setminus U$ and V is open, the condition $a \in V_0 = \mathrm{Int}(X \setminus U_0)$ is fulfilled. The conditions (1) and (3) are vacuously true in this step of induction.

Now suppose that U_γ for $\gamma < \beta$ have already been defined. There are two possibilities. Either $a \in \overline{\bigcup\limits_{\eta < \beta} U_\eta}$ and we can stop the process (putting $\alpha := \beta$) or $a \notin \overline{\bigcup\limits_{\eta < \beta} U_\eta}$.

In the latter case the set $W := X \setminus \overline{\bigcup\limits_{\eta < \beta} U_\eta}$ is an open neighborhood of a such that $W \cap (\bigcup\limits_{\eta < \beta} U_\eta) = \emptyset$. Since a is not isolated, there exists $b \in W$, $b \neq a$. Again, by T_2-axiom, there exist open sets U, V such that $U \cap V = \emptyset$, $a \in V$, $b \in U$. We put $U_\beta := U \cap W$.

Since $U_\beta \subseteq W$ and $W \cap (\bigcup\limits_{\eta < \beta} U_\eta) = \emptyset$, we do not violate (1).

The point a belongs to the open set $V \cap W$ and $(V \cap W) \cap U_\gamma = \emptyset$ for every $\gamma \leq \beta$, we get $a \in \mathrm{Int}(X \setminus \bigcup\limits_{\eta \leq \beta} U_\eta)$, so (2) is fulfilled as well.

For $\gamma < \beta$ we have $W \subseteq V_\gamma = X \setminus \overline{\bigcup_{\eta \leq \gamma} U_\eta}$, thus $U_\beta \subseteq V_\gamma$ and (3) holds.

The condition (4) does make sense only at the end of induction, when we have finished the process and said, what α is. Note, that this procedure must stop at some ordinal α, otherwise we would obtain a proper class of open subsets of X.

Now we put $Y := \{a\} \cup (\bigcup_{\beta<\alpha} U_\beta)$. The prime space P will be obtained as the space on the set $\alpha \cup \{\alpha\}$ which is quotient with respect to $q\colon Y \to P$ defined by $q(a) = \alpha$ and $q[U_\beta] = \{\beta\}$ for any $\beta < \alpha$. By (1) and (2) the map q is well-defined.

From $q^{-1}(\{\beta\}) = U_\beta$ we get that each $\beta < \alpha$ is isolated. By (4) and $q^{-1}(\alpha) = \{a\}$, the point α is not isolated. Hence P is a prime space.

Since the set $q^{-1}(\{\gamma \in \alpha \cup \{\alpha\}; \gamma > \beta\}) = V_\beta \cap Y$ is open in Y for each $\beta < \alpha$, the prime space P is T_2 (every isolated point can be separated from the accumulation point).

We could note here, that we in fact proved $P \prec C(\alpha)$. □

From Proposition 5.3.8 and Corollary 5.2.16 we get

Theorem 5.3.9. *If* $\mathbf{A}$ *is an epireflective subcategory of* **Top** *such that* $\mathbf{A} \subseteq$ **Haus** *and* $\mathbf{B}$ *is an HAD-class in* $\mathbf{A}$*, then* $\mathbf{B}$ *is closed under prime factors.*

Corollary 5.3.10. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **Top** *such that* $\mathbf{A} \subseteq$ **Haus**. *For every HAD-class* $\mathbf{B}$ *in* $\mathbf{A}$ *there exists a class* $\mathbf{S}$ *of prime spaces such that* $\mathbf{B} = \mathrm{AD}_{\mathbf{A}}(\mathbf{S})$.

5.3.3 Lattices of coreflective subcategories

We have already mentioned some facts about the lattice $\mathcal{C}$ of all coreflective subcategories of **Top** in subsection 1.5.5. Here we prove some new results about this lattice and about the lattice of all coreflective subcategories of **ZD**.

We describe the minimal subcategories in $\mathcal{C}$ containing a space which does not belong to CH(**Con**) and the atoms in the lattice of all coreflective subcategories of **ZD** above the subcategory **Disc**.

Proposition 5.3.11. *If* $\mathbf{C}$ *is a subcategory of* **Top** *with* $\mathbf{C} \not\subseteq \mathrm{CH}(\mathbf{Con})$*, then there exists a regular cardinal* α *such that* $\mathrm{CH}(C(\alpha)) \subseteq \mathbf{C}$.

Proof. If we have $X \in \mathbf{C}$, where $\mathbf{C}$ is coreflective and $X \notin \mathrm{CH}(\mathbf{Con})$, then by Proposition 5.3.4 and Lemma 5.3.2 we get $C(\alpha) \in \mathbf{C}$ for some cardinal α. □

Recall (see notes after Definition 1.5.32) that $\mathbf{B}_\alpha = \mathrm{CH}(B(\alpha))$ is the smallest coreflective subcategory of **Top** such that in each space $X \in \mathbf{B}_\alpha$ any intersection of less than α open sets is open, and there exists a space $Y \in \mathbf{B}_\alpha$ and a system of α open sets in Y with non-open intersection.

We show that if we have a zero-dimensional space with similar properties then we can obtain a prime space from it. Thus the atoms in the lattice of coreflective subcategories of **ZD** have a similar description. The proof is similar to the proof of Proposition 2.2.12.

Proposition 5.3.12. *Let X be a zero-dimensional space and α be the smallest cardinal number such that there exists a system U_β, $\beta < \alpha$, of open subsets of X with non-open intersection $\bigcap_{\beta<\alpha} U_\beta$, but every intersection of less than α open subsets of X is open. Then there exists a prime space $Y \prec C(\alpha)$ and a quotient map $q\colon X \to Y$.*

Proof. Recall that the space $C(\alpha)$ is the prime space on the set $\alpha \cup \{\alpha\}$ in which basic neighborhoods of the accumulation point α are the sets $B_\beta = \{\gamma \in \alpha \cup \{\alpha\}; \gamma \geq \beta\}$, $\beta < \alpha$.

Denote by $\{U_\beta; \beta < \alpha\}$ a system of α open sets whose intersection is not open. We can assume w.l.o.g. that this system is decreasing and all sets U_β are clopen. (From an arbitrary decreasing system of open sets we obtain a system of clopen sets by choosing any non-interior point $a \in \bigcap_{\gamma<\alpha} U_\gamma \setminus \mathrm{Int}(\bigcap_{\gamma<\alpha} U_\gamma)$ of the intersection and choosing a basic clopen neighborhood U'_β with $a \in U'_\beta \subseteq U_\beta$ for each $\beta < \alpha$.) If necessary, we can modify this system in such a way that $U_0 = X$ and $U_\beta = \bigcap_{\gamma<\beta} U_\gamma$ for any limit ordinal $\beta < \alpha$.

Now we again define $f\colon X \to \alpha \cup \{\alpha\}$ by

$$f(x) = \sup\{\beta \in \alpha : x \in U_\beta\}.$$

Let Y be the quotient space with respect to f.

We have $f^{-1}(B_\beta) = U_\beta$ for any $\beta < \alpha$. Since each U_β is clopen, we see that B_β and its complement are open in the quotient topology.

The set $f^{-1}(\alpha) = \bigcap_{\beta<\alpha} U_\beta$ is not open in X, therefore $\{\alpha\}$ is not open in Y. Since the sets U_β are clopen, all sets $\{\beta\} = f^{-1}(U_\beta \setminus U_{\beta+1})$ are open in Y. Thus Y is a prime space and $Y \prec C(\alpha)$. □

Theorem 5.3.13. *Let $\mathbf{A} = \mathbf{ZD}$ or $\mathbf{A} = \mathbf{ZD_0}$. Let $\mathbf{C}$ be a coreflective subcategory (an AD-class) in $\mathbf{A}$ such that $\mathbf{C} \not\subseteq \mathbf{FG} \cap \mathbf{A}$. Then there exists a regular cardinal α such that $\mathrm{CH}_{\mathbf{A}}(C(\alpha)) \subseteq \mathbf{C}$ (resp. $\mathrm{AD}_{\mathbf{A}}(C(\alpha)) \subseteq \mathbf{C}$). Namely, α is the smallest cardinal such that there exists a space $X \in \mathbf{C}$ and a system U_β, $\beta < \alpha$, of open sets in X whose intersection $\bigcap_{\beta<\alpha} U_\beta$ is not open.*

5.4 Some applications and examples

5.4.1 The coreflective hull of an HAD-class

In this section we show that for an HAD-class $\mathbf{B}$, that contains at least one prime space, its coreflective hull $\mathrm{CH}(\mathbf{B})$ in **Top** is hereditary.

We first observe that the space $(A_\omega)_a$ has also for the HAD-hulls of prime spaces in $\mathbf{A}$ a similar function as for the hereditary coreflective hulls of prime spaces in **Top**.

Lemma 5.4.1. *Let $\mathbf{A}$ be an epireflective subcategory of* **Top** *with $I_2 \notin \mathbf{A}$. If A is a prime space and $A \in \mathbf{A}$, then $\mathrm{HAD}_{\mathbf{A}}(A) = \mathrm{AD}_{\mathbf{A}}((A_\omega)_a) = \mathrm{SCH(A)} \cap \mathbf{A}$. Moreover $\mathrm{card}(A_\omega)_a = \mathrm{card}\, A$.*

Proof. We first observe that $\mathbf{A}$ contains A_ω. If A is T_2 then this is true by Proposition 3.3.11. If A is not T_2, then $\mathbf{A} = \mathbf{Top}_0$ and A_ω is clearly a T_0-space.

According to Lemma 5.1.12 and Theorem 3.3.6 $\mathrm{HAD}_{\mathbf{A}}(A) \subseteq \mathrm{SCH(A)} \cap \mathbf{A} = \mathrm{CH}((A_\omega)_a) \cap \mathbf{A} = \mathrm{AD}_{\mathbf{A}}((A_\omega)_a)$.

On the other hand, $A_\omega \in \mathbf{A}$ implies $A_\omega \in \mathrm{HAD}_{\mathbf{A}}(A)$. The HAD-class $\mathrm{HAD}_{\mathbf{A}}(A)$ contains the prime space A. So by Corollary 5.2.17 it is closed under prime factors and$(A_\omega)_a \in \mathrm{HAD}_{\mathbf{A}}(A)$, which proves the opposite inclusion. □

Lemma 5.4.2. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *such that* $I_2 \notin \mathbf{A}$. *If* $\mathbf{B} = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D})$, *where* $\mathbf{D}$ *is any set of spaces and* $\mathbf{B}$ *contains at least one prime space, then there exists a prime space* $B \in \mathbf{A}$ *such that* $\mathbf{B} = \mathrm{AD}_{\mathbf{A}}(B)$. *Moreover* $\mathrm{CH}(B) = \mathrm{SCH}(B)$ *is hereditary.*

Proof. Let us denote by $\mathbf{D}'$ the set of all non-discrete prime factors of spaces from $\mathbf{D}$. Let A be the prime space obtained as the wedge sum of all spaces from $\mathbf{D}$. By Proposition 1.5.10 $\mathrm{CH}(A) = \mathrm{CH}(\mathbf{D}')$. We consider two cases. If $\mathbf{A} = \mathbf{Top}_0$ then clearly $A \in \mathbf{A}$. If $\mathbf{A} \subseteq \mathbf{Top}_1$ then all spaces in $\mathbf{D}'$ are T_2 and A is T_2 as well. Therefore in both cases $A \in \mathbf{A}$ and $\mathrm{HAD}_{\mathbf{A}}(A) = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D}')$.

Any space from $\mathbf{D}$ can be obtained as a quotient of the sum of its prime factors and consequently $\mathrm{AD}_{\mathbf{A}}(\mathbf{D}') = \mathrm{CH}(\mathbf{D}') \cap \mathbf{A}$ contains the whole $\mathbf{D}$ and $\mathbf{B} = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D}) = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D}') = \mathrm{HAD}_{\mathbf{A}}(A)$.

Using Lemma 5.4.1 we obtain that the claim holds for $B = (A_\omega)_a$. □

If $\mathbf{B} = \mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ is hereditary, then by Corollary 5.1.15 it fulfills the assumptions of Lemma 5.4.2 and we get $\mathbf{B} = \mathrm{CH}_{\mathbf{A}}(B)$ in this case.

Using Lemma 5.4.2 we can prove, using very similar methods as in [Č4, Proposition 4], that

Theorem 5.4.3. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *such that* $I_2 \notin \mathbf{A}$. *If* $\mathbf{B}$ *is an HAD-class in* $\mathbf{A}$ *and* $\mathbf{B}$ *contains at least one prime space, then the coreflective hull* $\mathrm{CH}(\mathbf{B})$ *of* $\mathbf{B}$ *in* $\mathbf{Top}$ *is hereditary.*

Proof. We first represent $\mathbf{B}$ as a union of an ascending chain of HAD-classes $\mathbf{B}_\alpha$ in $\mathbf{A}$, such that each of them is generated by a single space (as an AD-class).

Let us denote by $\mathbf{B}_\alpha$ the HAD-hull of all spaces from $\mathbf{B}$ with cardinality at most α. Clearly, $\mathbf{B} = \bigcup_{\alpha \in \mathrm{Cn}} \mathbf{B}_\alpha$ and the system $\mathbf{B}_\alpha$ is nondecreasing.

Since $\mathbf{B}$ contains a prime space, there exists the smallest α_0 such that $\mathbf{B}_{\alpha_0}$ contains a prime space. Then $\mathbf{B} = \bigcup_{\alpha \geq \alpha_0} \mathbf{B}_\alpha$ and for each $\alpha \geq \alpha_0$ the class $\mathbf{B}_\alpha$ is a HAD-hull of a set of spaces and it contains a prime space. So we can use Lemma 5.4.2 and we get that for any cardinal number $\alpha \geq \alpha_0$ there exists a prime space $B_\alpha \in \mathbf{A}$ such that $\mathbf{B}_\alpha = \mathrm{AD}_{\mathbf{A}}(B_\alpha) \subseteq \mathrm{CH}(B_\alpha)$.

It is easy to see, that $\mathrm{CH}(\mathbf{B})$ consists of quotients of spaces from $\mathbf{B}$. Thus if $Y \in \mathrm{CH}(\mathbf{B})$, then Y is quotient of some space $X \in \mathbf{B}$ and there exists $\alpha \geq \alpha_0$ such that $X \in \mathbf{B}_\alpha$. Consequently we get $Y \in \mathrm{CH}(B_\alpha)$.

Any subspace of Y belongs to $\mathrm{SCH}(B_\alpha) = \mathrm{CH}(B_\alpha) \subseteq \mathrm{CH}(\mathbf{B})$. Thus $\mathrm{CH}(\mathbf{B})$ is closed under the formation of subspaces. □

Using the above theorem we can prove the result corresponding to [Č4, Corollary 1].

Corollary 5.4.4. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **Top** *such that* $I_2 \notin \mathbf{A}$. *Let* $\mathbf{B} \subseteq \mathbf{A}$ *and* $\mathbf{B}$ *contains at least one prime space. Then* $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B}) = \mathrm{SCH}(\mathbf{B}) \cap \mathbf{A}$.

Proof. By Lemma 5.1.12 we have $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B}) \subseteq \mathrm{SCH}(\mathbf{B}) \cap \mathbf{A}$.

To obtain the opposite inclusion, we use Theorem 5.4.3 for the HAD-class $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})$. We get that $\mathrm{CH}(\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})) = \mathrm{SCH}(\mathrm{HAD}_{\mathbf{A}}(\mathbf{B}))$. This implies $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B}) \supseteq \mathrm{CH}(\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})) \cap \mathbf{A} = \mathrm{SCH}(\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})) \cap \mathbf{A} \supseteq \mathrm{SCH}(\mathbf{B}) \cap \mathbf{A}$. □

Corollary 5.4.5. *For any epireflective subcategory* $\mathbf{A}$ *of* **Top** *such that* $\mathbf{A} \subseteq$ **Haus** *the assignment given by* $\mathbf{C} \mapsto \mathbf{C} \cap \mathbf{A}$ *yields a bijection between the hereditary coreflective subcategories of* **Top** *with* $\mathbf{C} \supseteq \mathbf{FG}$ *and HAD-classes in* $\mathbf{A}$.

Proof. If $\mathbf{C}$ is a hereditary coreflective subcategory of **Top** then the class $\mathbf{C} \cap \mathbf{A}$ is an intersection of two hereditary classes, thus it is hereditary as well. It is clearly an AD-class in $\mathbf{A}$.

Let us denote by F the assignment defined in the claim. We will show that G given by $G(\mathbf{Disc}) = \mathbf{FG}$ and $G(\mathbf{B}) = \mathrm{CH}(\mathbf{B})$ for $\mathbf{B} \neq \mathbf{Disc}$ is inverse to F.

First, observe that if $\mathbf{B} \neq \mathbf{Disc}$ is an HAD-class in $\mathbf{A}$ then by Proposition 5.3.8 it contains a prime space and from Theorem 5.4.3 we get that $\mathrm{CH}(\mathbf{B})$ is hereditary.

Let $\mathbf{C} \supsetneq \mathbf{FG}$ be a hereditary coreflective subcategory of **Top**. Then we have $G(F(\mathbf{C})) = \mathrm{CH}(\mathbf{C} \cap \mathbf{A})$. Since $\mathbf{C} \cap \mathbf{A}$ contains all prime T_2-spaces from $\mathbf{C}$, we get $\mathrm{CH}(\mathbf{C} \cap \mathbf{A}) = \mathbf{C}$ by Lemma 1.5.15. The equality $G(F(\mathbf{FG})) = \mathbf{FG}$ is clear.

On the other hand, if $\mathbf{B}$ is an HAD-class in $\mathbf{A}$ such that $\mathbf{B} \neq \mathbf{Disc}$, then $F(G(\mathbf{B})) = \mathrm{CH}(\mathbf{B}) \cap \mathbf{A} = \mathrm{AD}_{\mathbf{A}}(\mathbf{B}) = \mathbf{B}$. □

5.4.2 An epireflective subcategory not closed under $\triangle$

We have observed in Proposition 5.2.6 that if $\mathbf{A}$ is closed under $\triangle$ then every HAD-class in $\mathbf{A}$ is closed under prime factors. We have also noticed that for many familiar epireflective subcategories of **Top** this condition is fulfilled. Now we provide an example showing that it does not hold in general.

Recall (see Definition 1.1.5) that a topological space X is strongly rigid if the family $C(X, X)$ contains only the constant maps and the identity map id_X.

We will show that for a strongly rigid space X which is not "too trivial" $X \triangle_b X \notin \mathrm{EH(X)}$.

Lemma 5.4.6. *Let* X *be a topological space such and* $b \in X$. *If* X *is a strongly rigid space and* $X \triangle_b X \in \mathrm{EH(X)}$, *then for* $a \neq b$ *the set* $\{U \times V; U$ *is an open neighborhood of* a *and* V *is an open neighborhood of* $b\}$ *is a local base for the topology of* $X \triangle_b X$ *at* (a, b), *i.e., at any point* (a, b) *with* $a \neq b$ *the topology has the same local base as the product topology.*

Proof. Recall, that the underlying set of $X \triangle_b X$ is $X \times X$ (see Definition 5.2.2). The subspaces on the sets $\{a\} \times X$ for any $a \in X$ and $X \times \{b\}$ are homeomorphic to X. To be more precise, the homeomorphisms are given by $h_a(a, x) = x$ (between the subspace $\{a\} \times X$ and X) and $h(x, b) = x$ (between the subspace $X \times \{b\}$ and X). Thus for any map $f \in C(X \triangle_b X, X)$ the restrictions of f to these subspaces are either constant or coincidental with h_a or h. We next investigate in detail all maps f in $C(X \triangle_b X, X)$.

First, assume that $f|_{X \times \{b\}}$ is not constant. Then $f(x, b) = x$ for any $x \in X$. Thus for $a \neq b$ we get $f(a, b) = a \neq b = h_a(a, b)$. Therefore the restriction of f to the subspace $\{a\} \times X$, $a \neq b$, is the constant map $f(a, x) = a$. For the subspace $\{b\} \times X$ we have two possibilities: h_b or a constant map. In this case we obtain two continuous maps: f_1 such that $f_1(x, y) = x$ and f_2 given by $f_2(x, y) = x$ for $x \neq b$ and $f_2(b, y) = y$.

The second possibility remains: $f(x, b) = a_0$ for any $x \in X$. If $a_0 \neq b$ then for any $a \in X$ we have $f(a, b) = a_0 \neq b = h_a(a, b)$ and f is a constant map. Thus the only interesting case is $a_0 = b$. In this case some restrictions are equal to h_a's and some are constant. I.e., every such map corresponds to a subset A of X in the following way: $f_A(x, y) = y$ if $x \in A$ and $f_A(x, y) = b$ otherwise.

We showed that the family $C(X \triangle_b X, X)$ consists precisely of all constant maps, the maps f_1, f_2 and the maps of the form f_A, $A \subseteq X$. If $X \triangle_b X \in \mathbf{A}$, then it must have the initial topology with respect to this family (Proposition 1.3.4). Therefore $\{f_1^{-1}(U), f_2^{-1}(U), f_A^{-1}(U); U \text{ is open in } X\}$ is a subbase for the topology of $X \triangle_b X$. Each of the subbasic neighborhoods of (a, b) contains a set of the form $U \times V$, where U, V are open neighborhoods of a resp. b in X. (Note that for $b \in U$ we have $f_1^{-1}(U) = U \times X$, $f_2^{-1}(U) = (\{b\} \times U) \cup ((U \setminus \{b\}) \times X)$ and $f_A^{-1}(U) = (A \times U) \cup ((X \setminus A) \times X)$.) The sets $U \times V = f_X^{-1}(V) \cap f_1^{-1}(U)$ are open in the topology of $X \triangle_b X$. Thus they form a local base at (a, b). $\square$

Example 5.4.7. If X is a strongly rigid space and $\bigcap_{x \in X} U_x$ is an intersection of open neighborhoods of a point b in X which fails to be a neighborhood of b, then the set $\bigcup_{x \in X} \{x\} \times U_x$ is an open neighborhood of any $(a, b) \in X \triangle_b X$ such that $a \neq b$. But this set does not contain any subset from the local basis described in the above lemma. Therefore in such case $X \triangle_b X \notin \mathrm{EH(X)}$.

This means that to obtain a counterexample, it suffices to have a strongly rigid space with a non-isolated point b such that at the same time the set $\{b\}$ is closed and it is an intersection of a family U_i, $i \in I$, of open sets, with $\operatorname{card} I \leq \operatorname{card} X$. Any of the examples of strongly rigid T_2-spaces constructed in [KR], [dG] or [C] satisfies this condition.

5.4.3 Heredity of the coreflective hulls in Top

In this section we turn our attention to another question. Relatively little is known about conditions on a class of spaces which ensure the heredity of the coreflective hull (AD-hull) of this class. V. Kannan has a result saying that if $\mathbf{B}$ is a hereditary family closed under the formation of spaces with finer topologies

then the coreflective hull CH(**B**) of **B** in **Top** is hereditary as well ([K4, Remark 2.4.4(6)]). Our Theorem 5.4.3 yields a kind of such condition, too.

For any infinite cardinal α let $\mathbf{G}_\alpha$ be the class of all spaces with cardinality at most α. The coreflective hull of each $\mathbf{G}_\alpha$ is hereditary - it is precisely the subcategory $\mathbf{Gen}(\alpha)$ from Example 1.5.24. Note that each of classes $\mathbf{G}_\alpha$ is hereditary, map-invariant and closed under the formation of prime factors.

On the other hand, let **B** be a class of topological spaces which is map-invariant and closed under prime factors. If **B** consists of finite spaces only, then CH(**B**) is either **FG** or **Disc**. If **B** contains an infinite space, then either $\mathbf{B} = \mathbf{Top}$ or $\mathbf{B} = \mathbf{G}_\alpha$ for some cardinal α. (To see this, just note that if $\operatorname{card} X = \alpha$ and $X \in \mathbf{B}$, then **B** contains the discrete space of cardinality α which can be obtained as $(X_a)_b$, where $b \neq a$, $a, b \in X$. Every topological space of cardinality at most α is a continuous image of this discrete space.)

It is natural to ask whether we can somehow weaken the above mentioned properties of the classes $\mathbf{G}_\alpha$ in such a way, that for every class **B** with these properties the coreflective hull CH(**B**) of **B** in **Top** is hereditary.

One possible weakening is replacing the condition that **B** is map-invariant by divisibility. We can construct easily an example showing that for such a class CH(**B**) need not be hereditary in general.

Example 5.4.8. Let **B** consist of all quotients of the space $C(\omega_0)$ and of all discrete (at most) countable spaces. This class is divisible. Each space in **B** is prime or discrete, hence **B** is closed under the formation of prime factors. A subspace of a prime space P is either a discrete space or a quotient of P, thus **B** is hereditary. The coreflective hull $\mathrm{CH}(\mathbf{B}) = \mathrm{CH}(C(\omega_0)) = \mathbf{Seq}$ is not hereditary.

Another possible weakening is omitting the closedness under prime factors. We show in the rest of this section that there exists a class **B** which is hereditary and map-invariant but CH(**B**) is not hereditary.

We start with two easy examples.

Example 5.4.9. Let **B** be the class of all continuous images of the space $B(\omega_0)$ and of all finite spaces. Note that any infinite subspace of $B(\omega_0)$ containing ω_0 is homeomorphic to $B(\omega_0)$. If X is a continuous image of $B(\omega_0)$, i.e., $f\colon B(\omega_0) \to X$ is surjective, then there exists a subspace Y of $B(\omega_0)$ such that $f|_Y$ is bijection. If Y is infinite, then it we have $B(\omega_0) \prec X$. Thus the infinite spaces in the class **B** are precisely the spaces which have coarser topology than **B**. All these spaces belong to $\mathrm{CH}(B(\omega_0))$ since they can be obtained as quotients of sum of $B(\omega_0)$ and several copies of I_2. (For each set B_β, $\beta < \omega_0$, which is not open in X with $B(\omega_0) \prec X$, we use the indiscrete space on the subset $\{\beta - 1, \beta\}$.)

We can also see that this class is hereditary – if $B(\omega_0) \prec X$ and Y is an infinite subspace of X, then Y has coarser topology than the subspace of $B(\omega_0)$ on the set Y, which is homeomorphic to $B(\omega_0)$. It is clear directly from the definition of **B**, that it is map-invariant.

The coreflective hull $\mathrm{CH}(\mathbf{B}) = \mathrm{CH}(B(\omega_0))$ is not hereditary. This follows from the fact that the prime factor $(B(\omega_0))_{\omega_0}$ is $C(\omega_0)$ and $C(\omega_0) \notin \mathrm{CH}(B(\omega_0))$.

Recall (see Proposition 1.5.22) that if $\mathbf{A}$ is a map-invariant class of topological spaces then $\mathrm{CH}(\mathbf{A})$ is the subcategory $\mathbf{A}_{\mathrm{gen}}$ of all $\mathbf{A}$-generated spaces. We will study the map-invariant class $\mathbf{A}_\alpha$ of all spaces with cardinality of topology $o(X) < 2^\alpha$ and its coreflective hull – the subcategory $\mathbf{C}_\alpha$ of all $\mathbf{A}_\alpha$-generated spaces (see Example 1.5.26).

Note that $o(B(\omega_0)) = \aleph_0$, thus $\mathrm{CH}(\mathbf{B}) \subseteq \mathbf{C}_{\aleph_0}$ holds for the category $\mathbf{B}$ from Example 5.4.9.

Example 5.4.10. We show that $\mathbf{A}_{\aleph_0}$ is not closed under the formation of prime factors and consequently it is not hereditary.

Let X be a countable topological space with the cofinite topology. Clearly, $o(X) = \aleph_0$, thus $X \in \mathbf{A}_{\aleph_0}$. But the prime factor X_a of X is homeomorphic to $C(\omega_0)$. Only the finite subspaces of $C(\omega_0)$ belong to $\mathbf{A}_{\aleph_0}$. Thus the point ω_0 is isolated in each subspace belonging to $\mathbf{A}_{\aleph_0}$ and $C(\omega_0) \notin \mathbf{C}_{\aleph_0}$.

Note that by Proposition 5.3.8 in every non-discrete Hausdorff space X we have infinitely many disjoint open subsets in the subspace Y constructed in the proof of this proposition. Therefore $o(X) \geq \mathfrak{c}$. This implies $\mathbf{A}_{\aleph_0} \cap \mathbf{Haus} \subseteq \mathbf{Disc}$ and, consequently, $\mathbf{C}_{\aleph_0} \cap \mathbf{Haus} = \mathbf{Disc}$.

Note that, since the space constructed in the above example is T_1, we also obtain that $\mathrm{CH}_{\mathbf{A}}(\mathbf{A}_\alpha \cap \mathbf{A})$ is not hereditary for $\mathbf{A} = \mathbf{Top}_{0,1}$.

It is quite natural to look for a Hausdorff example after we have constructed a T_1-space with the required properties. We have already seen that there is no such example in the subcategory $\mathbf{C}_{\aleph_0}$. We were able to construct a Hausdorff example only under the assumption $2^{\aleph_1} = 2^{\mathfrak{c}}$ (which holds under CH). It is $\mathbb{R}$ with the countable complement extension topology ([SS, Example 63]).

Example 5.4.11 ($2^{\aleph_1} = 2^{\mathfrak{c}}$). Suppose $2^{\aleph_1} = 2^{\mathfrak{c}}$. Let X be the topological space on the set $\mathbb{R}$ with the topology $\mathcal{T} = \{U \setminus A; U \text{ is open in } \mathbb{R} \text{ and } \operatorname{card} A \leq \aleph_0\}$. Clearly, $o(X) = o(\mathbb{R}).\operatorname{card}\{A \subseteq \mathbb{R}; A \text{ is countable}\} = \mathfrak{c}.\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. Thus $X \in \mathbf{A}_{\mathfrak{c}}$.

We claim that, for any $a \in X$, the prime factor X_a does not belong to $\mathbf{C}_{\mathfrak{c}}$. Indeed, if $a \in V$ and V is a subspace of X_a such that $V \in \mathbf{A}_{\mathfrak{c}}$, then $\operatorname{card} V = \aleph_0$ (otherwise V contains a discrete subspace $V \setminus \{a\}$ of cardinality $\aleph_1$ and $o(V) = 2^{\aleph_1} = 2^{\mathfrak{c}}$). At the same time $a \notin \overline{V \setminus \{a\}}$ (since $\{a\} \cup (\mathbb{R} \setminus V)$ is a neighborhood of a). We see that a is isolated in all subspaces of X_a belonging to $\mathbf{A}_{\mathfrak{c}}$, but a is not isolated in X_a, thus $X_a \notin \mathbf{C}_{\mathfrak{c}}$.

Example 5.4.12. After we have shown that $\mathbf{C}_\alpha$ is not hereditary for some α, we can be interested in finding a concrete example of a space from $\mathbf{C}_\alpha$ and its subspace which is not in $\mathbf{C}_\alpha$. Such an example can be found with the help of the operation $\triangle$.

Suppose that $X \in \mathbf{C}_\alpha$ is such a space that $X_a \notin \mathbf{C}_\alpha$. Let $Y := X \triangle_a X$. Clearly, $Y \in \mathbf{C}_\alpha$. Recall that $X^a_{(X,a)}$ is the subspace on the set $\{(a,a)\} \cup (X \setminus \{a\}) \times (X \setminus \{a\})$. Since $X_a \notin \mathbf{C}_\alpha$ and X_a is a quotient of $X^a_{(X,a)}$, we get that the subspace $X^a_{(X,a)} \notin \mathbf{C}_\alpha$ as well.

Note that, since the subcategories $\mathbf{Top}_1$, $\mathbf{Haus}$ are closed under $\triangle$, if we start with the space X from Example 5.4.10 (or Example 5.4.11), the resulting space Y will be T_1 (resp. Hausdorff) as well.

Chapter 6

Bireflective subcategories of Top

In the last chapter we have only dealt with the epireflective subcategories of **Top** fulfilling the condition $I_2 \notin \mathbf{A}$. This is equivalent to the condition that $\mathbf{A}$ is not bireflective (Proposition 1.3.7). We would like to extend somehow our results to the case of bireflective categories of **Top**.

For this purpose the correspondence between bireflective and non-bireflective subcategories of **Top** comes handy. This correspondence was described e.g. in [Ma] or [Na]. We will show that this correspondence preserves AD-classes and HAD-classes of epireflective subcategories too.

Since the mentioned correspondence is introduced using the T_0-reflector R_0, we start by showing some useful properties of this functor.

6.1 T_0-reflection

In many situations it seems that there is in fact no difference (from the topological point of view) if we restrict ourselves only to the class of T_0-spaces. Many properties of a topological space remain valid after applying the T_0-reflector. (See [HS4, p.302] for the historical account on this fact.)

This is true in our situation as well. It turns out that instead of the epireflective subcategory of **Top** we can study the subcategory obtained by taking all T_0-reflections of spaces from the original subcategory. To show this we will use some properties of the T_0-reflection. This section is devoted to the proofs of these properties.

Of course, all of them were well-known before. E.g. some of them (as well as some which we have not mentioned) can be found in [CM, Proposition 2.1], [Hof1, Theorem 1.4], [Sa] or [Wy2]. In spite of this we find it useful to put them here in detail with the proofs.

The T_0-reflection of a space X is given by the following equivalence relation: $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$ (see e.g. [He2, Beispiel 8.3(2)]). This condition

is equivalent to $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Another equivalent formulation is that the open neighborhoods of x and of y are the same.

We will denote the T_0-reflector by R_0.

The definition of $\sim$ implies that every open set can be divided into equivalence classes. (All points of an equivalence class must lie in the same open sets.) This can be reformulated as:

Lemma 6.1.1. *Let X be a topological space and $q\colon X \to R_0X$ be the T_0-reflection arrow of X. For any open set $U \subseteq X$ the equality $q^{-1}(q[U]) = U$ holds and the set $q[U]$ is open in R_0X. (I.e., q is an open map.)*

Corollary 6.1.2. *If $q\colon X \to R_0X$ is the T_0-reflection arrow of X, then X has the initial topology with respect to q.*

Proof. If U is open in X, then $U = q^{-1}(q[U])$ is a preimage of an open set. □

We now provide another construction of the T_0-reflection, which realizes the space R_0X as a subspace (moreover a retract) of the original space X.

Proposition 6.1.3. *Let X be a topological space. Then its T_0-reflection R_0X is a subspace of X and the T_0-reflection arrow $q\colon X \to R_0X$ is a retraction.*

Proof. Since $q\colon X \to R_0X$ is surjective, there exists an injective map $e\colon R_0X \to X$ with $q \circ e = id_{R_0X}$.

First we show that e is continuous. Indeed, if U is open in X, then $U = q^{-1}(q[U])$ by Lemma 6.1.1 and $e^{-1}(U) = e^{-1}(q^{-1}(q[U])) = (q \circ e)^{-1}(q[U]) = q[U]$. This set is open by Lemma 6.1.1.

Since we have two continuous maps with the property $q \circ e = id_{R_0X}$, q is a retraction and e an embedding. □

The above proposition provides an alternative construction of the T_0-reflection. It suffices to pick up one point from each equivalence class of $\sim$. The resulting space does not depend on the choice of the points in individual equivalence classes.

Note that, since the T_0-reflection arrow is a retraction, it is a hereditary quotient map as well.

Corollary 6.1.4. *If $\mathbf{A}$ is any epireflective subcategory of* **Top** *and $A \in \mathbf{A}$, then $R_0A \in \mathbf{A}$.*

Corollary 6.1.5. $A \in \mathrm{EH}(\{\mathrm{R_0A}, \mathrm{I_2}\}) = \mathrm{BH}(\{R_0A\})$ *for any space A.*

Proof. There exists the initial source consisting of the single map $q\colon A \to R_0A$. So by Proposition 1.3.5 $A \in \mathrm{BH}(\{R_0A\})$. □

Corollary 6.1.6. *R_0A is a dense subspace of A.*

Proof. For any $a \in A$ there exists some $b \in R_0A$ with $a \sim b$, i.e., $a \in \overline{\{b\}} \subseteq \overline{R_0A}$. □

We have shown that the spaces A and R_0A are contained in the same bireflective subcategories of **Top**. The following lemma provides a similar claim for the coreflective subcategories of **Top**.

Lemma 6.1.7. *$A \in \mathrm{CH}(\{R_0A, I_2\})$ for any topological space A.*

Proof. Since the equivalence classes of the relation $\sim$ are indiscrete subspaces of the original space A and the points of the same equivalence class are contained in the same open sets, we can obtain the space A from R_0A by attaching an indiscrete space (with the same cardinality as the corresponding equivalence class) to each point of R_0A. Hence, A is a quotient of sum of R_0A and several indiscrete spaces, i.e., $A \in \mathrm{CH}(R_0A, I_2)$. □

We next show that the T_0-reflector behaves well with respect to subspaces and initial topologies.

Proposition 6.1.8. *If $q\colon X \to R_0X$ is the T_0-reflection arrow of X and Y is a subspace of X, then $q|_Y\colon Y \to q[Y]$ is the T_0-reflection arrow of Y. The T_0-reflection R_0Y of Y is a subspace of R_0X and the T_0-reflector R_0 preserves embeddings.*

Proof. The T_0-reflection arrow q' of Y is given by the equivalence relation $\sim_Y$ defined by: $x \sim_Y y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$ in Y. Obviously, $x \sim_Y y \Leftrightarrow x \sim_X y$ for $x, y \in Y$. It means that the same pairs of points of Y are identified by both relations.

By the alternative construction of the T_0-reflection, we can obtain R_0X by choosing one point from each equivalence class. If we moreover choose from each class which intersects Y a point belonging to Y, the subspace $R_0X \cap Y$ is the T_0-reflection of Y at the same time. In this case $q|_Y$ is the T_0-reflection arrow of Y.

Of course, this is a subspace of R_0X at the same time. □

Proposition 6.1.9. *The T_0-reflector R_0 preserves initial sources.*

Proof. Let $(f_i\colon A \to B_i)_{i\in I}$ be an initial source. Denote $r_A\colon A \to R_0A$ and $r_{B_i}\colon B_i \to R_0B_i$ the T_0-reflection arrows. By Proposition 6.1.3 we have an embedding $e\colon R_0A \to A$ with $r_A \circ e = id_{R_0A}$.

$$\begin{tikzcd}
A \arrow[r, "f_i"] \arrow[d, shift left, "r_A"] & B_i \arrow[d, "r_{B_i}"] \\
R_0A \arrow[r, "R_0 f_i"'] \arrow[u, shift left, hook, "e"] & R_0B_i
\end{tikzcd}$$

From $R_0f_i \circ r_A = r_{B_i} \circ f_i$ we get $R_0f_i = R_0f_i \circ r_A \circ e = r_{B_i} \circ f_i \circ e$. Thus the source $(R_0f_i)_{i\in I}$ is the composite of three initial sources and it is itself initial. □

Let us note without proof that the T_0-reflector preserves the topological products as well.

6.2 Epireflective and bireflective subcategories

Using the properties of the T_0-reflection, which we have shown in the foregoing section, we can introduce a one-to-one correspondence between bireflective and non-bireflective subcategories of **Top** and show that it behaves well with respect to AD-classes and HAD-classes in these categories. This correspondence was described in the setting of arbitrary topological category in [Ma] and for **Top** in [Na]. It is presented in those papers analogously to our maps F and G.

We will ignore two trivial cases: $\mathbf{A} = \mathbf{Ind}$ and $\mathbf{A}$ is the category consisting of the empty space and all one-point spaces. The reason for excluding these two categories is the fact, that for all other epireflective subcategories we have $D_2 \in \mathbf{A}$, hence such subcategories are closed under topological sums. (If $\mathbf{A}$ is not closed under topological sums, we cannot speak about additive classes in $\mathbf{A}$. In both these cases the only coreflective subcategory of $\mathbf{A}$ is the whole category $\mathbf{A}$.)

Let $\mathcal{L}'$ be the lattice of all bireflective subcategories of **Top** such that $\mathbf{A} \neq \mathbf{Ind}$ and $\mathcal{L}$ be the lattice of all epireflective subcategories of **Top**, which are contained in $\mathbf{Top}_0$ and contain at least one 2-point space. We define the following maps:
$F\colon \mathcal{L} \to \mathcal{L}'$; $\mathbf{A} \mapsto \mathrm{BH}(\mathbf{A})$,
$G\colon \mathcal{L}' \to \mathcal{L}$; $\mathbf{A}' \mapsto \mathbf{A}' \cap \mathbf{Top}_0$,
$G'\colon \mathcal{L}' \to \mathcal{L}$; $\mathbf{A}' \mapsto \{R_0A; A \in \mathbf{A}'\}$.

We first observe that

Lemma 6.2.1. *For any $\mathbf{A}' \in \mathcal{L}'$ we have $G(\mathbf{A}') = G'(\mathbf{A}')$.*

Proof. Let $A \in \mathbf{A}'$, $R_0A \in G'(\mathbf{A}')$. By Proposition 6.1.3 R_0A is a subspace of A, thus $R_0A \in \mathbf{A}'$. Of course, $R_0A \in \mathbf{Top}_0$, thus we get $R_0A \in G(\mathbf{A}')$ and $G'(\mathbf{A}') \subseteq G(\mathbf{A}')$.

Conversely, if $A \in G(\mathbf{A}')$, then $A = R_0A \in G'(\mathbf{A}')$. □

Now we prove that the maps F and G form indeed a bijective correspondence between $\mathcal{L}$ and $\mathcal{L}'$.

Corollary 6.2.2. *The maps F and G are inverse to each other. Consequently, they are bijective.*

Proof. $G(F(\mathbf{A})) = \mathbf{A}$ holds for any $\mathbf{A} \in \mathcal{L}$ by Lemma 1.3.6.

It remains to show $F(G'(\mathbf{A}')) = \mathbf{A}'$ for $\mathbf{A}' \in \mathcal{L}'$. Since $\mathbf{A}'$ is bireflective and contains $G(\mathbf{A}')$, we get $F(G(\mathbf{A}')) \subseteq \mathbf{A}'$.

Now let $A \in \mathbf{A}'$. By Corollary 6.1.5 $A \in \mathrm{BH}(R_0A) \subseteq F(G'(\mathbf{A}'))$. □

This yields a one-to-one correspondence between the epireflective subcategories which are not bireflective and the bireflective subcategories of **Top**.

We provide also another description of the map F. Let us define
$F'\colon \mathcal{L} \to \mathcal{L}'$; $\mathbf{A} \mapsto \{A \in \mathbf{Top}; R_0A \in \mathbf{A}\}$.

Lemma 6.2.3. *For any $\mathbf{A} \in \mathcal{L}$ the equality $F(\mathbf{A}) = F'(\mathbf{A})$ holds.*

Proof. If $A \in F'(\mathbf{A})$ then we get from Corollary 6.1.5 $A \in \mathrm{BH}(R_0A) \subseteq \mathrm{BH}(\mathbf{A}) = F(\mathbf{A})$.

On other hand, if $A \in F(\mathbf{A}) = \mathrm{BH}(\mathbf{A})$, then $R_0A \in G'(F(\mathbf{A})) = \mathbf{A}$ and $A \in F'(\mathbf{A})$. □

6.3 The correspondence for AD-classes

In the foregoing section we introduced a correspondence between bireflective and non-bireflective epireflective subcategories of **Top**. Next we show that by almost identical constructions we can find similar correspondence between AD-classes (HAD-classes) in an epireflective subcategory $\mathbf{A} \in \mathcal{L}$ and in $F(\mathbf{A})$.

The following lemma says that in the collection of all AD-classes in an epireflective subcategory $\mathbf{A}$ of **Top** with $I_2 \in \mathbf{A}$ all AD-classes except from **Disc** contain I_2. Lemma 6.3.2 shows why it is useful for us when an AD-class in a bireflective subcategory contains I_2.

Lemma 6.3.1. *Let $\mathbf{C}$ be an AD-class in an epireflective subcategory $\mathbf{A}$ with $I_2, D_2 \in \mathbf{A}$. If $\mathbf{C}$ contains a non-discrete space then $I_2 \in \mathbf{C}$.*

Proof. Let $C \in \mathbf{C}$ be a non-discrete space and $c \in C$ be non-isolated. Let us define $f, g\colon C \to I_2$ by $f(c) = 0$, $f[C \setminus \{c\}] = \{1\}$ and $g(c) = 1$, $g[C \setminus \{c\}] = \{0\}$. One can easily verify that $[f, g]\colon C \sqcup C \to I_2$ is quotient. □

Lemma 6.3.2. *Let $\mathbf{C}$ be an AD-class in an epireflective subcategory $\mathbf{A}$, $D_2 \in \mathbf{A}$ and $I_2 \in \mathbf{C}$. Then $A \in \mathbf{C}$ if and only if $R_0A \in \mathbf{C}$.*

Proof. $\boxed{\Rightarrow}$ By Corollary 6.1.4 $R_0A \in \mathbf{A}$. The T_0-reflection arrow $A \to R_0A$ is quotient, therefore $R_0A \in \mathbf{C}$.

$\boxed{\Leftarrow}$ By Corollary 6.1.5 $A \in \mathrm{EH}(\{\mathrm{R_0A}, \mathrm{I_2}\}) = \mathrm{BH}(R_0A) \subseteq \mathbf{A}$. Lemma 6.1.7 implies $A \in \mathrm{CH}(\{R_0A, I_2\}) \cap \mathbf{A} \subseteq \mathbf{C}$. □

Suppose we are given an epireflective subcategory $\mathbf{A} \in \mathcal{L}$ and let $\mathbf{B} = F(\mathbf{A})$ be the corresponding bireflective subcategory. (In particular, $\mathbf{A} \subseteq \mathbf{B}$ holds.) Then we can define the following maps between the subclasses of $\mathbf{A}$ and the subclasses of $\mathbf{B}$:

For any $\mathbf{C} \subseteq \mathbf{A}$ we put
$F_{\mathbf{A}}(\mathbf{C}) = \{B \in \mathbf{B}; R_0B \in \mathbf{C}\}$.

For any $\mathbf{D} \subseteq \mathbf{B}$ we put
$G_{\mathbf{B}}(\mathbf{D}) = \mathbf{D} \cap \mathbf{Top}_0$ and
$G'_{\mathbf{B}}(\mathbf{D}) = \{R_0A; A \in \mathbf{D}\}$.

It means that the maps $F_{\mathbf{A}}$ and $G_{\mathbf{B}}$ are nearly the same as F and G, but their domain and codomain have changed. (We could say that $F_{\mathbf{Top}_0} = F$ and $G_{\mathbf{Top}} = G$.)

A few lemmas, showing that these maps lead to a one-to-one correspondence, follow. Their proofs mostly mimic the proofs of analogical claims about the maps G and F. In all these lemmas we assume the above situation.

Lemma 6.3.3. *For any* $\mathbf{D} \subseteq \mathbf{B}$ *we have* $G_{\mathbf{B}}(\mathbf{D}) \subseteq G'_{\mathbf{B}}(\mathbf{D})$.
If $\mathbf{D}$ *is hereditary then* $G_{\mathbf{B}}(\mathbf{D}) = G'_{\mathbf{B}}(\mathbf{D})$.
If $\mathbf{D}$ *is divisible in* $\mathbf{B}$ *then* $G_{\mathbf{B}}(\mathbf{D}) = G'_{\mathbf{B}}(\mathbf{D})$.

Proof. The first part is clear. The second part follows from the fact that R_0A is a subspace and a quotient space of A. □

We will need the fact that these maps behave well with respect to HAD-classes and AD-classes.

Lemma 6.3.4. *Let* $\mathbf{C} \subseteq \mathbf{A}$.

(i) *If* $\mathbf{C}$ *is hereditary, then* $F_{\mathbf{A}}(\mathbf{C})$ *is hereditary.*

(ii) *If* $\mathbf{C}$ *is additive, then* $F_{\mathbf{A}}(\mathbf{C})$ *is additive.*

(iii) *If* $\mathbf{C}$ *is divisible in* $\mathbf{A}$*, then* $F_{\mathbf{A}}(\mathbf{C})$ *is divisible in* $\mathbf{B}$.

Proof. (i): If A is a subspace of B (we write this briefly as $A \hookrightarrow B$) and $R_0B \in \mathbf{C}$ then $R_0A \hookrightarrow R_0B$, since R_0 preserves embeddings (Proposition 6.1.8). Thus $R_0A \in \mathbf{C}$ and $A \in F_{\mathbf{A}}(\mathbf{C})$.

(ii): R_0 preserves topological sums as well.

(iii): Since $\mathbf{Top}_0$ is quotient reflective, the extremal epimorphisms in $\mathbf{Top}_0$ are precisely the quotient maps and R_0q is quotient for every quotient map. If $R_0A \in \mathbf{C}$ and $q\colon A \to B$ is quotient, then $R_0q\colon R_0A \to R_0B$ is quotient as well and $R_0B \in \mathbf{C}$. □

Lemma 6.3.5. *Let* $\mathbf{D} \subseteq \mathbf{B}$.

(i) *If* $\mathbf{D}$ *is hereditary, then* $G_{\mathbf{B}}(\mathbf{D})$ *is hereditary.*

(ii) *If* $\mathbf{D}$ *is additive, then* $G_{\mathbf{B}}(\mathbf{D})$ *is additive.*

(iii) *If* $\mathbf{D}$ *is divisible in* $\mathbf{B}$*, then* $G_{\mathbf{B}}(\mathbf{D})$ *is divisible in* $\mathbf{A}$.

Proof. The (i) and (ii) follows simply from the fact that $G_{\mathbf{B}}(\mathbf{D}) = \mathbf{D} \cap \mathbf{Top}_0$ is an intersection of two hereditary (resp. additive) subcategories of $\mathbf{Top}$ (namely $\mathbf{Top}_0$ and $\mathbf{D}$).

(iii): If $q\colon R_0B \to C$ is quotient, then $R_0q = r \circ q\colon B \to C$ is quotient as well. (Here $r\colon C \to R_0C$ is the T_0-reflection arrow.) □

Now we can show that the maps $F_{\mathbf{A}}$ and $G_{\mathbf{B}}$ yield a bijective correspondence between AD-classes in $\mathbf{A}$ and in M. Moreover, HAD-classes in one-category correspond to HAD-classes in the other one.

Lemma 6.3.6. *For any* $\mathbf{C} \subseteq \mathbf{A}$ *we have* $G'_{\mathbf{B}}(F_{\mathbf{A}}(\mathbf{C})) = \mathbf{C}$.
If $\mathbf{D}$ *is an AD-class in* $\mathbf{B}$ *and* $I_2 \in \mathbf{D}$*, then* $F_{\mathbf{A}}(G'_{\mathbf{B}}(\mathbf{D})) = \mathbf{D}$.

Proof. The first part of the lemma says, in other words, that $R_0[R_0^{-1}(\mathbf{C})] = \mathbf{C}$. This is valid if every $C \in \mathbf{C}$ has the form R_0B from some $B \in \mathbf{B}$. Clearly, $R_0C = C$ and $C \in \mathbf{A} \subseteq \mathbf{B}$.

Now let $\mathbf{D}$ be an AD-class in $\mathbf{B}$ and $I_2 \in \mathbf{D}$. Clearly $\mathbf{D} \subseteq F_{\mathbf{A}}(G'_{\mathbf{B}}(\mathbf{D}))$. On the other hand, if $D \in F_{\mathbf{A}}(G'_{\mathbf{B}}(\mathbf{D}))$ then $R_0D = R_0C$ for some $C \in \mathbf{D}$. By Lemma 6.3.2 we get $R_0C = R_0D \in \mathbf{D}$ and $D \in \mathbf{D}$. □

We have obtained a one-to-one correspondence between HAD-classes (AD-classes) in $\mathbf{B}$ and HAD-classes (AD-classes) in $F(\mathbf{B}) = \mathbf{B} \cap \mathbf{Top}_0$. This correspondence leads to

Theorem 6.3.7. *Let $\mathbf{B}$ be an epireflective subcategory with $\mathbf{B} \neq \mathbf{Ind}$ and $\mathbf{A} = G(\mathbf{B}) = \mathbf{B} \cap \mathbf{Top}_0$. Then $\mathbf{C}$ is an HAD-class in $\mathbf{B}$ if and only if $\mathbf{C} \cap \mathbf{Top}_0$ is an HAD-class in $\mathbf{A}$.*

Proof. The correspondence obtained above implies that the subcategory $\mathbf{C}$ is an HAD-class if and only if $G_{\mathbf{B}}(\mathbf{C}) = \mathbf{C} \cap \mathbf{Top}_0$ is HAD. □

Thus for any AD-class $\mathbf{C}$ in $\mathbf{B}$ the correspondence defined above shifts the question whether $\mathbf{C}$ is hereditary to the same question about the corresponding AD-class in $\mathbf{B} \cap \mathbf{Top}_0$. This is precisely the situation which we have studied in Chapter 5.

We get easily from Theorem 6.3.7 the following corollary

Corollary 6.3.8. *Let $\mathbf{C}$ be an AD-class in an epireflective subcategory $\mathbf{B} \neq \mathbf{Ind}$ with $I_2 \in \mathbf{B}$. Then $\mathbf{C}$ is hereditary if and only if $R_0\mathbf{C} = \mathbf{C} \cap \mathbf{Top}_0$ is hereditary.*

We close this section with reformulating some of our results to the case of bireflective subcategories of **Top**.

Lemma 6.3.9. *If $f\colon X \to Y$ is a surjective initial map, $b \in Y$ and $f^{-1}(b) = \{a\}$, then $X_a \in \mathrm{CH}(Y_b)$.*

Proof. Let $g\colon Y_b \to X_a$ be any map such that $f(g(x)) = x$ for any $x \in X_a$ (in particular, $g(b) = a$). We first show that g is continuous.

If $a \in U$ and U is open in X_a, then there exists an open set $U' \subseteq Y$ with $a \in f^{-1}(U') \subseteq U$. Then $g^{-1}(U) \supseteq g^{-1}(f^{-1}(U')) = U' \ni b$, hence $g^{-1}(U)$ is open in Y_b.

We have continuous maps f, g such that $f \circ g = id_{X_a}$. So f is a retraction, thus it is a quotient map and $X_a \in \mathrm{CH}(Y_b)$. □

Corollary 6.3.10. *Let X be a topological space, $a \in X$ be a point such that X_a is T_2 and $r\colon X \to RX_0$ be the T_0-reflection arrow of X. Then $X_a \in \mathrm{CH}((R_0X)_b)$, where $b = r(a)$.*

Proof. Since $\overline{\{a\}} = \{a\}$, the equivalence class of the point a consists of this single point. Therefore $r^{-1}(b) = \{a\}$ holds for the $\mathbf{Top}_0$-reflection arrow r of X. The claim follows now from Lemma 6.3.9. □

Note that, if $\mathbf{A}$ is none of the categories $\mathbf{Top}$, $\mathbf{Top}_0$, then all prime factors belonging to $\mathbf{A}$ are Hausdorff. So we see from the above corollary that, if $\mathbf{C}$ is an AD-class in an epireflective subcategory $\mathbf{A} \neq \mathbf{Top}, \mathbf{Top}_0$ and $F_{\mathbf{A}}(\mathbf{C}) = R_0\mathbf{C}$ is closed under the formation of prime factors, then $\mathbf{C}$ is closed under the formation of prime factors too.

So our results for arbitrary epireflective subcategories can be subsumed as follows:

Proposition 6.3.11. *Let* $\mathbf{B} \neq \mathbf{Ind}$ *be an epireflective subcategory of* $\mathbf{Top}$ *and* $\mathbf{C}$ *be an AD-class in* $\mathbf{B}$*. If* $\mathbf{B} \cap \mathbf{Top}_0 \subseteq \mathbf{Haus}$ *or the subcategories* $\mathbf{B} \cap \mathbf{Top}_0$ *and* $\mathbf{C} \cap \mathbf{Top}_0$ *fulfill the assumptions of Theorem 5.2.15 or those of Corollary 5.3.7, then* $\mathbf{C}$ *is hereditary if and only if it is closed under the formation of prime factors which belong to* $\mathbf{A}$*.*

In particular we get that $\mathbf{B}$ is closed under the formation of prime factors which are Hausdorff.

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