Density measures

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Overview

Main topics:

- density measures
- existence of density measures
- possible values of density measures
- density measures and Lévy group
- density measures as functionals

Most results in this presentation were obtained in join works with Miloš Ziman: [SZ1, SZ2].

Asymptotic density

The asymptotic density of $A \subseteq \mathbb{N}$ is defined by

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

if this limit exists, where

$$A(n) = |A \cap \{1, 2, \dots, n\}|.$$

$\mathcal{D}$ = the set of all subsets of $\mathbb{N}$ having asymptotic density

Drawback: Some sets do not have asymptotic density.
Is it possible to extend d to a finitely additive measure?

Definition of density measures

We will call a finitely additive normalized measure on $\mathbb{N}$ briefly a *measure*.

Definition

A *density measure* is a finitely additive measure on $\mathbb{N}$ which extends the asymptotic density; i.e., it is a function $\mu\colon \mathcal{P}(\mathbb{N}) \to [0,1]$ satisfying the following conditions:

(a) $\mu(\mathbb{N}) = 1$;

(b) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$;

(c) $\mu|_{\mathcal{D}} = d$.

References for density measures

The term density measures was probably coined by Dorothy Maharam [M].

Density measures were studied by many authors, e.g.

- Blass, Frankiewicz, Plebanek and Ryll–Nardzewski [BFPRN]
- van Douwen [vD]
- Šalát and Tijdeman in [ŠT].

Density measures (and Lévy group) were used in theory of social choice [CK, Fe, L, T].

Existence of density measures

Existence of density measures is usually proved using either Hahn-Banach theorem or ultrafilters.

If $\mathcal{F}$ is any free ultrafilter on $\mathbb{N}$ then

$$\mu_{\mathcal{F}}(A) = \mathcal{F}\text{-}\lim \frac{A(n)}{n}$$

is a density measure

$\mathcal{F}\text{-}\lim a_n = L \Leftrightarrow \{n \in \mathbb{N}; |a_n - L| < \varepsilon\} \in \mathcal{F}$ for each $\varepsilon > 0$

Properties of $\mathcal{F}$-limits

$$\mathcal{F}\text{-}\lim a_n = L \Leftrightarrow \{n \in \mathbb{N}; |a_n - L| < \varepsilon\} \in \mathcal{F} \text{ for each } \varepsilon > 0$$

- If $\mathcal{F}$ is ultrafilter and (x_n) is bounded, then $\mathcal{F}\text{-}\lim x_n$ exists.
- $\mathcal{F}\text{-}\lim(x_n + y_n) = \mathcal{F}\text{-}\lim x_n + \mathcal{F}\text{-}\lim y_n$.
- If (x_n) is convergent and $\mathcal{F}$ is a free ultrafilter, then $\mathcal{F}\text{-}\lim x_n = \lim\limits_{n\to\infty} x_n$.
- $x_n \geq 0 \Rightarrow \mathcal{F}\text{-}\lim x_n \geq 0$

Density measures from $\mathcal{F}$-limits

$$\mu_{\mathcal{F}}(A)=\mathcal{F}\text{-lim}\,\frac{A(n)}{n}$$

$\mu_{\mathcal{F}}\colon \mathcal{P}(\mathbb{N})\to[0,1]$

- $A\cap B=\emptyset \Rightarrow \mu_{\mathcal{F}}(A\cup B)=\mathcal{F}\text{-lim}\,\frac{A\cup B(n)}{n}=\mathcal{F}\text{-lim}\,\frac{A(n)}{n}+\mathcal{F}\text{-lim}\,\frac{B(n)}{n}=\mu_{\mathcal{F}}(A)+\mu_{\mathcal{F}}(B)$
- $\frac{A(n)}{n}\geq 0 \Rightarrow \mu_{\mathcal{F}}(A)=\mathcal{F}\text{-lim}\,\frac{A(n)}{n}\geq 0$
- $A\in\mathcal{D}\Rightarrow \mu_{\mathcal{F}}(A)=\mathcal{F}\text{-lim}\,\frac{A(n)}{n}=\lim\limits_{n\to\infty}\frac{A(n)}{n}=d(A)$

AC and density measures

Some form of Axiom of Choice is needed in order to get a density measure.
There exists a model of ZF constructed by Pincus and Solovay [PS] in which there are no nonprincipal finitely additive measures on $\mathbb{N}$.

Values of density measures

What are possible values of density measures for a given set $A \subseteq \mathbb{N}$?

$$\{\mu(A); \mu \text{ is a density measure}\}$$

This set is convex.
sup $=$?, max $=$?, inf $=$?, min $=$?

Extremal values of density measures

$$\underline{\underline{d}}(A) = \sup\{d(B);\ B \subseteq A,\ B \in \mathcal{D}\},$$
$$\overline{\overline{d}}(A) = \inf\{d(C);\ C \supseteq A,\ C \in \mathcal{D}\}.$$

$$\underline{\underline{d}}(A) \le \mu(A) \le \overline{\overline{d}}(A)$$

Theorem

Let $A \subseteq \mathbb{N}$. There exists a density measure μ such that $\mu(A) = x$ if and only if $x \in [\underline{\underline{d}}(A), \overline{\overline{d}}(A)]$.

Pólya's result

The following expression for $\overline{\overline{d}}(A)$ was obtained by Pólya [P].

$$\overline{\overline{d}}(A) = \lim_{\theta \to 1^-} \limsup_{n\to\infty} \frac{A(n)-A(\theta n)}{n-\theta n}$$

α-densities

Upper and lower α-density for $\alpha \geq -1$ (see [GAGM]):

$$\underline{d_\alpha}(A) = \liminf_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)};$$

$$\overline{d_\alpha}(A) = \limsup_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}.$$

where $A_\alpha(n) = \sum_{k=1}^{n} \chi_A(k)k^\alpha$.
It is known that for $-1 \leq \alpha \leq \beta$

$$\underline{d_\beta}(A) \leq \underline{d_\alpha}(A) \leq \overline{d_\alpha}(A) \leq \overline{d_\beta}(A).$$

If $A \in \mathcal{D}$, then $\underline{d_\alpha}(A) = \overline{d_\alpha}(A) = d(A)$ for $\alpha \geq -1$.

Density measures from α-densities

If $\alpha > -1$ and $\mathcal{F}$ is a free ultrafilter, then

$$\mu_\alpha^{\mathcal{F}}(A) = \mathcal{F}\text{-lim}\,\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}.$$

is a density measure.

$$\underline{d_\infty}(A) = \lim_{\alpha\to\infty} \underline{d_\alpha}(A) = \inf_{\alpha\geq -1} \underline{d_\alpha}(A);$$
$$\overline{d_\infty}(A) = \lim_{\alpha\to\infty} \overline{d_\alpha}(A) = \sup_{\alpha\geq -1} \overline{d_\alpha}(A).$$

Clearly

$$\underline{\underline{d}}(A) \leq \underline{d_\infty}(A) \leq \overline{d_\infty}(A) \leq \overline{\overline{d}}(A).$$

Density measures from α-densities

In fact, the equalities hold:

$$\overline{\overline{d}}(A) = \overline{d_\infty}(A) = \lim_{\alpha\to\infty} \overline{d_\alpha}(A),$$

see [LMS].

Summary

$$
\begin{aligned}
\overline{\overline{d}}(A) &= \inf\{d(C);\ C \supseteq A,\ C \in \mathcal{D}\} \\
&= \lim_{\theta \to 1^-} \limsup_{n \to \infty} \frac{A(n) - A(\theta n)}{n - \theta n} \\
&= \lim_{\alpha \to \infty} \overline{d_\alpha}(A)
\end{aligned}
$$

Lévy group

Definition

The *Lévy group* $\mathcal{G}$ is the group of all permutations π of $\mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{\left|\{k;\ k \leq n < \pi(k)\}\right|}{n} = 0. \tag{3.1}$$

$$\pi \in \mathcal{G} \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{A(n) - (\pi A)(n)}{n} = 0 \text{ for all } A \subseteq \mathbb{N} \tag{3.2}$$

Equivalent characterization of $\mathcal{G}$

$$\pi \in \mathcal{G} \ \Leftrightarrow\ \operatorname*{limstat}_{n\to\infty} \frac{\pi(n)}{n} = 1 \tag{3.3}$$

Recall that $\operatorname*{limstat}_{n\to\infty} x_n = L$ iff for every $\varepsilon > 0$ the set

$$A_\varepsilon = \{n;\ |x_n - L| \geq \varepsilon\}$$

has zero asymptotic density ($d(A_\varepsilon) = 0$).

$\mathcal{F}$-lim for $\mathcal{F} = \{A \subseteq \mathbb{N}; d(A) = 1\}$

$\mathcal{G}$-invariance

Theorem

A measure μ on $\mathbb{N}$ is a density measure if and only if it is $\mathcal{G}$-invariant, i.e., $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi \in \mathcal{G}$.

$\mathcal{G}$-invariance

We use van Douwen's result [vD, Theorem 1.12]:

Theorem

A measure μ on $\mathbb{N}$ is a density measure if and only if $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{\pi(n)}{n} = 1. \tag{3.4}$$

$$(3.4) \Rightarrow (3.3)$$

$\mathcal{G}$-invariant $\Rightarrow$ density measure

This implication follows also from a result of Blümlinger and Obata [BO, Theorem 2].

$\mathcal{G}$-invariance

The proof of the opposite implication uses the following result (Fridy [Fr, Theorem 1] or Šalát [Š, Lemma 1.1]):

Theorem

A sequence (x_n) is statistically convergent to $L \in \mathbb{R}$ if and only if there exists a set A such that $d(A) = 1$ and the sequence x_n converges to L along the set A, i.e., L is limit of the subsequence $(x_n)_{n \in A}$.

Basic idea of the proof

If π fulfills (3.3)

$$\pi \in \mathcal{G} \Leftrightarrow \operatorname*{limstat}_{n\to\infty} \frac{\pi(n)}{n} = 1$$

it can be modified to ψ fulfilling (3.4)

$$\lim_{n\to\infty} \frac{\psi(n)}{n} = 1$$

and πA and ψA differ only in a set of zero density.

$$\mu(A) = \mu(\psi A) = \mu(\pi A)$$

Lévy group and invariance of density measures

Proposition

If π is a permutation such that every density measure is π-invariant, i.e., $\mu(\pi A) = \mu(A)$ for every $A \subseteq \mathbb{N}$ and every density measure μ, then $\pi \in \mathcal{G}$.

Finitely additive measure and ℓ_∞^*

Finitely additive signed measure is a function $\mu\colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ such that

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever $A \cap B = \emptyset$.

If $f \in \ell_\infty^*$ then

$$\mu(A) = f(\chi_A)$$

μ is a finitely additive signed measure.

Finitely additive measure and ℓ_∞^*

For every finitely additive measure there exists precisely one $f \in \ell_\infty^*$ such that

$$\mu(A) = f(\chi_A).$$

positive measures = positive functionals ($f \geq 0$)
probabilistic measures = normed functionals $\|f\| = 1$
density measures = ?

Functionals corresponding to density measures

If for a bounded sequence (x_n) exists the limit

$$C(x) = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n}$$

then $C(x)$ is called Cesàro mean of the sequence x.

Theorem

Let μ be a measure and $f \in \ell_\infty^*$ be the corresponding functional. The measure μ is a density measure if and only if f extends Cesàro mean.

Existence of density measures using Banach-Alaoglu theorem

Theorem (Banach-Alaoglu)

If X is a normed linear space, then the unit ball of X^* endowed with weak*-topology is compact.

Weak*-topology is

- initial topology w.r.t. the evaluation maps $\varphi \mapsto \varphi(x)$ for $x \in X$ (i.e., the coarsest topology which makes all these maps continuous);
- the topology of pointwise convergence (a net $(\varphi_d)_{d\in D}$ converges to φ if and only if $\varphi_d(x) \to \varphi(x)$ for each $x \in X$);
- the subspace topology inherited from the product topology on $\mathbb{R}^X$.

Existence of density measures using Banach-Alaoglu theorem

$$C_n(x) = \frac{x_1 + \dots + x_n}{n}.$$

Each C_n is a positive functional which belongs to the unit ball of ℓ^*_∞.
There exists a convergent subnet: $f = \lim\limits_{d\in D} C_{n_d}$.

$$f(x) = \lim_{d\in D} C_{n_d}(x) = \lim_{n\to\infty} C_n(x) = C(x)$$

f is a positive functional such that $\|f\| = 1$ and f extends Cesàro mean.

An interesting density measure

Blümlinger [B]:
$2\mathcal{F} = \{B \subseteq \mathbb{N}; B \supseteq 2A \text{ for some } A \in \mathcal{F}\}$
(the ultrafilter given by the base $\{2A; A \in \mathcal{F}\}$)

$$\mu(A) = 2\,(2\mathcal{F})\text{-lim}\,\frac{A(n)}{n} - \mathcal{F}\text{-lim}\,\frac{A(n)}{n}$$

is a density measure
Let $A = \bigcup_{i=1}^{\infty}\{2^{2^i}, 2^{2^i}+1, \ldots, 2.2^{2^i}-1\}$ and $\{2^{2^i}; i \in \mathbb{N}\} \in \mathcal{F}$. Then $\mu(A) = 1$ and $\overline{d}(A) = \frac{1}{2}$.

An interesting density measure

A negative answer van Douwen [vD, Question 7A.1]:
Does $\mu(A) \le \overline{d}(A)$ hold for every density measure?
Counterexample to the following claim of Lauwers [L, p.46]:
Every density measure can be expressed in the form

$$\mu_\varphi(A) = \int_{\beta\mathbb{N}^*} \mathcal{F}\text{-lim}\, \frac{A(n)}{n}\, \mathrm{d}\varphi(\mathcal{F}), \qquad A \subseteq \mathbb{N} \tag{5.1}$$

for some probability Borel measure φ on the set of all free ultrafilters $\beta\mathbb{N}^*$.

An interesting density measure

Šalát and Tijdeman [ŠT]: Has every density measure the following properties?
a) If $A(n) \leq B(n)$ for all $n \in \mathbb{N}$ then $\mu(A) \leq \mu(B)$ (where $A, B \subseteq \mathbb{N}$).
b) If $\lim\limits_{n\to\infty} \frac{A(n)}{B(tn)} = 1$ then $\mu(A) = t\mu(B)$ (where $A, B \subseteq \mathbb{N}$ and $t \in \mathbb{R}$).
Answer to both these questions is negative.

a) If $\mu(A) > \overline{d}(A)$ and $d(B) \in (\overline{d}(A), \mu(A))$ then $B(n) > A(n)$ for $n > n_0$, but $\mu(A) > d(B) = \mu(B)$.
b) In the preceding example we have $\mu(A) = 1$ and $\mu(2A) = 0$.

Thanks for your attention!

The preprints of [SZ1, SZ2] presented here, as well as the text of this talk and these slides can be found at:
http://thales.doa.fmph.uniba.sk/sleziak/papers/

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