DENSITY MEASURES

MARTIN SLEZIAK

Abstract. Density measures are finitely additive measures on $\mathbb{N}$ which extend asymptotic density. They can be considered as a way to assigning some kind of "size" to each subset of $\mathbb{N}$ and they are used in various areas, for example, social choice theory.

We will show that these measures can be characterized by the property that they are invariant w.r.t. Lévy group. We will discuss which values can density measures attain for a given set. We will also mention a correspondence between finitely additive measures on N and linear continuous functionals on ℓ^∞ and discuss, which functionals correspond to density measures.

This talk was presented at Séminaire ModMad in Saint-Etienne.

Most of the results in this talk are from papers [SZ1, SZ2]. The equality $\overline{\overline{d}}(A) = \overline{d_\infty}(A)$ is shown in [LMS].

1. Introduction

In this talk I would like to speak about density measures. Density measures are extensions of asymptotic density to the whole power set $\mathcal{P}(\mathbb{N})$.

Density measures have found applications in number theory and, more recently, in the theory of social choice (see e.g. Fey [Fe], Lauwers [L]).

We will start by showing that density measures exist and then discuss what possible values can density measures obtain for a given set $A \subseteq \mathbb{N}$.

Density measures can be characterized as those finitely additive measures on integers $\mathcal{G}$-invariant, where $\mathcal{G}$ denotes the Lévy group. (We will later see that this property characterizes the Lévy group as well.) We will also present a new characterization of the Lévy group via statistical convergence.

2. Preliminaries

2.1. $\mathcal{F}$-limits.

Definition 2.1. A subset $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is called a filter if $\mathcal{F} \neq \emptyset$, $\emptyset \notin \mathcal{F}$ and

(i) $A \in \mathcal{F}$, $A \subseteq B \Rightarrow B \in \mathcal{F}$ (i.e., $\mathcal{F}$ is closed under supersets);
(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$ (i.e., $\mathcal{F}$ is closed under intersections).

A filter which is maximal w.r.t. inclusion is called *ultrafilter.*

A filter is called free if $\bigcap \mathcal{F} = \emptyset$.

A filter $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is an ultrafilter if and only if

$$(\forall A \subseteq \mathbb{N}) A \in \mathcal{F} \vee (\mathbb{N} \setminus A) \in \mathcal{F},$$

i.e., for each $A \subseteq \mathbb{N}$, either A or complement of A belongs to $\mathcal{F}$.

Definition 2.2. If $\mathcal{F}$ is a filter on $\mathbb{N}$ and (a_n) is a sequence of real numbers then a number L is said to be the *$\mathcal{F}$-limit* of the sequence (a_n) if for each $\varepsilon > 0$.

$$\mathcal{F}\text{-}\lim a_n = L \Leftrightarrow (\forall\varepsilon > 0)\{n \in \mathbb{N}; |a_n - L| < \varepsilon\} \in \mathcal{F}.$$

Basic properties of $\mathcal{F}$-limits are summarized in the following result.

Proposition 2.3. *Let $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ be real sequences, $c \in \mathbb{R}$ and $\mathcal{F}$ be a filter on $\mathbb{N}$.*

(i) *If $\mathcal{F}\text{-}\lim x_n$ and $\mathcal{F}\text{-}\lim y_n$ exist, then $\mathcal{F}\text{-}\lim(x_n + y_n) = \mathcal{F}\text{-}\lim x_n + \mathcal{F}\text{-}\lim y_n$ and $\mathcal{F}\text{-}\lim(cx_n) = c\,\mathcal{F}\text{-}\lim x_n$.*

(ii) *If $\mathcal{F}\text{-}\lim x_n$ and $\mathcal{F}\text{-}\lim y_n$ exist, then $\mathcal{F}\text{-}\lim(x_n y_n) = \mathcal{F}\text{-}\lim x_n \cdot \mathcal{F}\text{-}\lim y_n$.*

(iii) *$A \in \mathcal{F}$ is an infinite set and the limit $\lim\limits_{n\in A} x_n = L$ exists, then the $\mathcal{F}$-limit has the same value $\mathcal{F}\text{-}\lim x_n = \lim\limits_{n\in A} x_n$.*

(iv) *If $\mathcal{F}$ is a free filter and $(x_n)_{n=1}^\infty$ is a convergent sequence then*
$$\mathcal{F}\text{-}\lim x_n = \lim_{n\to\infty} x_n.$$

(v) *If $(x_n)_{n=1}^\infty$ is a bounded sequence and $\mathcal{F}$ is an ultrafilter, then $\mathcal{F}\text{-}\lim x_n$ exists.*

(vi) *If $\mathcal{F}$ is a free ultrafilter, then $\mathcal{F}\text{-}\lim x_n$ is a limit point of the sequence $(x_n)_{n=1}^\infty$. Conversely, for each limit point L of the sequence $(x_n)_{n=1}^\infty$ there exists a free ultrafilter such that $\mathcal{F}\text{-}\lim x_n = L$.*

(vii) *If $x_n \geq y_n$ for each n, then $\mathcal{F}\text{-}\lim x_n \geq \mathcal{F}\text{-}\lim y_n$. In particular, $x_n \geq 0$ implies $\mathcal{F}\text{-}\lim x_n \geq 0$.*

2.2. **Lévy group.** We will also use a group of permutations of $\mathbb{N}$ which is related to the asymptotic density.

Definition 2.4. The *Lévy group* $\mathcal{G}$ is the group of all permutations π of $\mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{\big|\{k;\, k \leq n < \pi(k)\}\big|}{n} = 0. \tag{2.1}$$

We will use the following characterizations of Lévy group [B, Lemma 2].

A permutation $\pi \in \mathcal{G}$ if and only if

$$\lim_{n\to\infty} \frac{A(n) - (\pi A)(n)}{n} = 0 \tag{2.2}$$

for all $A \subseteq \mathbb{N}$.

We have found an interesting connection between the Lévy group and statistical convergence. This new characterization of $\mathcal{G}$ has proved to be useful in the proof of the main theorem.

Let us first recall the definition of statistical convergence.

We say that $\operatorname*{limstat}\limits_{n\to\infty} x_n = L$ iff for every $\varepsilon > 0$ the set

$$A_\varepsilon = \{n;\, |x_n - L| \geq \varepsilon\}$$

has zero asymptotic density ($d(A_\varepsilon) = 0$).

The statistical limit is in fact $\mathcal{F}$-limit for the filter $\mathcal{F}$ consisting of all sets with $d(A) = 1$. (This filter is not an ultrafilter, hence there exist sequences without statistical limit. Of course, the statistical convergence can be formulated using ideals an $\mathcal{I}$-convergence too.)

Theorem 2.5. *A permutation $\pi : \mathbb{N} \to \mathbb{N}$ belongs to $\mathcal{G}$ if and only if*

$$\operatorname*{limstat}_{n\to\infty} \frac{\pi(n)}{n} = 1. \tag{2.3}$$

3. Density measures

3.1. **Definition of density measures.** Recall that the asymptotic density of $A \subseteq \mathbb{N}$ is defined by

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

if this limit exists. The asymptotic density is one of standard tools for measuring the size of subsets of $\mathbb{N}$. However, the drawback of the asymptotic density is that not every subset of $\mathbb{N}$ has the asymptotic density. (The above limit need not exist.) Therefore it is very natural to ask whether it is possible to extend the asymptotic density to a finitely additive measure on $\mathbb{N}$. The set of all subsets of $\mathbb{N}$ having asymptotic density will be denoted by $\mathcal{D}$.

Definition 3.1. A *density measure* is a finitely additive measure on $\mathbb{N}$ which extends the asymptotic density; i.e., it is a function $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ satisfying the following conditions:

(a) $\mu(\mathbb{N}) = 1$;
(b) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$;
(c) $\mu|_{\mathcal{D}} = d$.

(For the sake of brevity we will call the functions $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ fulfilling the conditions (a) and (b) from the preceding definition *measures*.)

The term density measures was probably coined by Dorothy Maharam [Ma]. They were studied (among many others) by Blass, Frankiewicz, Plebanek and Ryll–Nardzewski in [BFPRN], van Douwen in [vD] or Šalát and Tijdeman in [ŠT].

Recently the density measures and related concept of Lévy group were also used in the theory of social choice [CK, Fe, L, T].

3.2. **Existence of density measures.** The existence of density measures is usually proved either using Hahn-Banach theorem or using ultrafilters. (We will also mention use of Banach-Alaoglu Theorem in Section 6.) We will use the following approach several times:

For any ultrafilter $\mathcal{F} \in \beta\mathbb{N}^*$ the function

$$\mu_{\mathcal{F}}(A) = \mathcal{F}\text{-lim} \frac{A(n)}{n} \tag{3.1}$$

is a density measure (see e.g. [BŠ, Theorem 8.33], [HJ, p.207]).

Indeed, if $A \cap B = \emptyset$, then we have

$$\mu_{\mathcal{F}}(A \cup B) = \mathcal{F}\text{-lim} \frac{A \cup B(n)}{n} = \mathcal{F}\text{-lim} \frac{A(n)}{n} + \mathcal{F}\text{-lim} \frac{B(n)}{n} = \mu_{\mathcal{F}}(A) + \mu_{\mathcal{F}}(B).$$

We also get that $\mu_{\mathcal{F}}(A) \geq 0$ (since $A(n)/n \geq 0$ for each n.) And if $A \in \mathcal{D}$, then

$$\mu_{\mathcal{F}}(A) = \mathcal{F}\text{-lim} \frac{A(n)}{n} = \lim_{n\to\infty} \frac{A(n)}{n} = d(A).$$

Let us note that at least some form of axiom of choice is needed in the construction of finitely additive measures on $\mathbb{N}$, since there exists a model of ZF constructed by Pincus and Solovay [PS] in which there are no nonprincipal finitely additive measures on $\mathbb{N}$, see also [HR]. (See also [V] for the version of this result for ℓ_∞^*, which can be identified with the space of all finitely additive measures.)

4. Possible values of density measures for a given set

One of natural questions, which one can ask about density measures is: If we are given a set A, what possible values can be obtained as $\mu(A)$ for some density measure μ. In the other words, what can be said about the following set:

$$\{\mu(A); \mu \text{ is a density measure}\}.$$

Since the set of density measures is convex, we are basically asking about supremum and infimum of this set (or maximum and minimum, if they exist).

Since every density measure is monotone (in the sense that $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$), we immediately get

$$\underline{\underline{d}}(A) \leq \mu(A) \leq \overline{\overline{d}}(A),$$

where

$$\underline{\underline{d}}(A) = \sup\{d(B);\, B \subseteq A,\, B \in \mathcal{D}\},$$
$$\overline{\overline{d}}(A) = \inf\{d(C);\, C \supseteq A,\, C \in \mathcal{D}\}.$$

It is interesting that this rather trivial estimate is, in fact, best possible.

Theorem 4.1. *Let $A \subseteq \mathbb{N}$. There exists a density measure μ such that $\mu(A) = x$ if and only if $x \in [\underline{\underline{d}}(A), \overline{\overline{d}}(A)]$.*

This result is shown in [SZ2] using some more general results about partial measures from [BRBR] (together with some facts that can be shown about asymptotic density and the quantities $\underline{\underline{d}}(A)$ and $\overline{\overline{d}}(A)$).

Another expression for this value was given by Pólya [P, Satz VIII]. The setting of the paper [P] is more general, but for densities on $\mathbb{N}$ it says that

$$\overline{\overline{d}}(A) = \lim_{\theta\to 1^-} \limsup_{n\to\infty} \frac{A(n) - A(\theta n)}{n - \theta n}.$$

For $A \subseteq \mathbb{N}$ and $\alpha > -1$ we can define $A_\alpha(n) = \sum\limits_{k=1}^{n} \chi_A(k) k^\alpha$.

We can obtain another class of density measures in a way similar to (3.1) but using the fraction $A_\alpha(n)/\mathbb{N}_\alpha(n)$ instead of $A(n)/n$. The fractions $A_\alpha(n)/\mathbb{N}_\alpha(n)$ are used in the definition of α-densities

$$\underline{d_\alpha}(A) = \liminf_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)};$$
$$\overline{d_\alpha}(A) = \limsup_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}.$$

The α-densities were studied in [GAGM].

It is known that if $A \in \mathcal{D}$, then $\underline{d_\alpha}(A) = \overline{d_\alpha}(A) = d(A)$ for $\alpha > -1$.

If $\alpha > -1$ and $\mathcal{F}$ is a free ultrafilter, then

$$\mu_\alpha^{\mathcal{F}}(A) = \mathcal{F}\text{-}\lim \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} = \underline{d_\alpha}(A).$$

is a density measure.

If we denote

$$\underline{d_\infty}(A) = \lim_{\alpha\to\infty} \underline{d_\alpha}(A) = \inf_{\alpha\geq -1} \underline{d_\alpha}(A);$$
$$\overline{d_\infty}(A) = \lim_{\alpha\to\infty} \overline{d_\alpha}(A) = \sup_{\alpha\geq -1} \overline{d_\alpha}(A).$$

then we clearly have

$$\underline{\underline{d}}(A) \leq \underline{d_\infty}(A) \leq \overline{d_\infty}(A) \leq \overline{\overline{d}}(A).$$

If can be shown that in fact

$$\overline{\overline{d}}(A) = \overline{d_\infty}(A)$$

and $\underline{\underline{d}}(A) = \underline{d_\infty}(A)$ (see [LMS]).

5. Density measures and Lévy group

5.1. **$\mathcal{G}$-invariance.** The main result of this [SZ1] is the following theorem.

Theorem 5.1. *A measure μ on $\mathbb{N}$ is a density measure if and only if it is $\mathcal{G}$-invariant, i.e., $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi \in \mathcal{G}$.*

The proof employs van Douwen's result [vD, Theorem 1.12]:

Theorem 5.2. *A measure μ on $\mathbb{N}$ is a density measure if and only if $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi : \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{\pi(n)}{n} = 1. \tag{5.1}$$

With this result at hand, one half of Theorem 5.1 is easy. Clearly, if a permutation π fulfills (5.1), then it fulfills also (2.3). This yields the implication:
π is $\mathcal{G}$-invariant $\Rightarrow$ π is a density measure.

Let us note that this result can also be deduced from Blümlinger and Obata [BO, Theorem 2], where it was proved by different means. This theorem deals with linear functionals on the space $\mathcal{D}$ of Césaro summable sequences, but it can be applied to our situation. They have shown that every $\mathcal{G}$-invariant linear functional on $\mathcal{D}$ is a multiple Césaro mean.

Every measure assigns some value to characteristic sequences of subsets of $\mathbb{N}$. We can extend it to a linear functional on ℓ_∞ (see also Section 6) without violating the $\mathcal{G}$-invariance and then restrict this functional to $\mathcal{D}$. Since this functional is normalized, the restriction is precisely the Césaro mean. In terms of measures, any $\mathcal{G}$-invariant extends the density.

In this short talk we will not go into details of the proof of the opposite implications. We just note that main components of this proof are our characterization of the Lévy group using the statistical convergence (Theorem 2.5) and the following result (see Fridy [Fr, Theorem 1] or Šalát [Š, Lemma 1.1]):

Theorem 5.3. *A sequence (x_n) is statistically convergent to $L \in \mathbb{R}$ if and only if there exists a set A such that $d(A) = 1$ and the sequence x_n converges to L along the set A, i.e., L is limit of the subsequence $(x_n)_{n\in A}$.*

The basic idea of the proof is that if a permutation π fulfills (2.3) then it can be modified to a new permutation ψ fulfilling (5.1) in such a manner that $\pi(A)$ and $\psi(A)$ differ only in a set having zero density. By the van Douwen's result (Theorem 5.2) the permutation ψ preserves density measure and we can use it to show that the measure of the set A will be preserved by the permutation π as well.

5.2. **Characterization of Lévy group.** By theorem 5.1 every density measure is π-invariant for permutations $\pi \in \mathcal{G}$. It is natural to ask whether there are other permutations with this property. Proposition 5.4 states that this property characterizes Lévy group.

Proposition 5.4. *If π is a permutation such that every density measure is π-invariant, i.e., $\mu(\pi A) = \mu(A)$ for every $A \subset \mathbb{N}$ and every density measure μ, then $\pi \in \mathcal{G}$.*

6. Finitely additive measures and ℓ_∞^*

There is a very natural correspondence between finitely additive (signed) measures on $\mathbb{N}$ and the space ℓ_∞^*.

For a moment we will consider all finitely additive set functions on $\mathbb{N}$, i.e., functions $\mu\colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ such that

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever $A \cap B = \emptyset$. (We do not require positivity. We have also omitted the condition $\mu(\mathbb{N}) = 1$.)

Let us call such function a *finitely additive signed measure* on $\mathbb{N}$.

As usual, ℓ_∞ denotes the space of all bounded real sequences (with the norm $\|x\|_\infty = \sup |x_n|$) and ℓ_∞^* is the dual space, i.e., it contains all linear bounded functionals on ℓ_∞.

It is clear that if $f\colon \ell_\infty \to \mathbb{R}$ is a bounded linear functional, then

$$\mu(A) = f(\chi_A)$$

is a finitely additive signed measure.

On the other hand, for every finitely additive measure there exists precisely one $f \in \ell_\infty^*$ which produces this measure.

The process of obtaining a functional from a given measure is similar to definition of Riemann integral. It uses the fact that any bounded sequence can be uniformly approximated by step sequences. (By a *step sequence* we mean a sequence of the form $\sum_{i=1}^n c_i \chi_{A_i}$ for some $c_1, \dots, c_n \in \mathbb{R}$ and $A_1, \dots, A_n \subseteq \mathbb{N}$, i.e. a finite linear combination of characteristic sequences.)

More details about this construction can be found, for example, in [C, Theorem 16.7], [Mo, p.50, Example 1.19], [vD, Section 3].

It is relatively easy to see that the positive measures correspond to positive functionals, and positive normed measures correspond to functionals such that $\|f\| = 1$.

The advantage of this approach is that now we can view finitely additive measures (and in particular density measures) as a subset of the Banach space ℓ_∞^*, which means that we can use tools from functional analysis.

However, it would be useful to know whether we can somehow characterize the functionals corresponding to finitely additive measures.

If for a bounded sequence (x_n) exists the limit

$$C(x) = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n}$$

then $C(x)$ is called Cesàro mean of the sequence x.

Theorem 6.1. *Let μ be a measure and $f \in \ell_\infty^*$ be the corresponding functional. The measure μ is a density measure if and only if f extends Cesàro mean.*

We already know that density measures are characterized by the property that they are $\mathcal{G}$-invariant. It is relatively easy to show that if μ is $\mathcal{G}$-invariant, then the corresponding functional f also is $\mathcal{G}$-invariant. It was shown [BO, Theorem 2] a positive functional with $\|f\| = 1$ extends Cesàro mean if and only if it is $\mathcal{G}$-invariant.

6.1. **Existence of density measures using Banach-Alaoglu theorem.** As a very simple application of the correspondence between finitely additive measures and ℓ_∞^* we will show a different proof that density measures exist. Clearly, it suffices to show existence of a positive normed functional which extends Cesàro mean.

Let us define for $x = (x_n) \in \ell_\infty$

$$C_n(x) = \frac{x_1 + \cdots + x_n}{n}.$$

Each C_n is a positive linear functional belonging to ℓ_∞^* such that $\|C_n\| = 1$.

We know from Banach-Alaoglu theorem that unit ball of ℓ_∞^* is compact when endowed with the weak*-topology (i.e., the topology of pointwise convergence.)

So the set $\{C_n; n \in \mathbb{N}\}$ has a cluster point in ℓ_∞^*, i.e., there exists a subnet $(C_{n_d})_{d\in D}$ which is convergent to some $f \in \ell_\infty$.

We have that f is positive and $\|f\| = 1$, since the same is true for each C_n. And we also have

$$f(x) = \lim_{d\in D} C_{n_d}(x) = \lim_{n\to\infty} C_n(x) = C(x)$$

for every sequence which has a Cesàro mean.

Thus f is a positive normed linear functional extending Cesáro mean.

7. Applications

7.1. **An interesting density measure.** We will closed this talk with an interesting example of a density measure which answers several questions posed by van Douwen [vD] and Šalát and Tijdeman [ŠT]. This example of density measure was defined in the paper of Blümlinger [B].

Example 7.1. Let $\mathcal{F}$ be any ultrafilter. By $2\mathcal{F}$ we denote the ultrafilter given by the base $\{2A; A \in \mathcal{F}\}$, i.e., $2\mathcal{F} = \{B \subseteq \mathbb{N}; B \supseteq 2A$ for some $A \in \mathcal{F}\}$. Let us define μ by

$$\mu(A) = 2\,(2\mathcal{F})\text{-}\lim \frac{A(n)}{n} - \mathcal{F}\text{-}\lim \frac{A(n)}{n}.$$

It can be shown that μ is a density measure.

Now let us consider the set $A = \bigcup_{i=1}^{\infty} \{2^{2^i}, 2^{2^i} + 1, \dots, 2.2^{2^i} - 1\}$. Note that $A(2.2^{2^i} - 1) \geq \frac{1}{2}$ and $A(2^{2^i} - 1) \leq \frac{1}{2^{i-3}}$ for any positive integer i. It can be shown that $\overline{d}(A) = \frac{1}{2}$ and $\mu(A) = 1$ for any ultrafilter containing the set $\{2^{2^i}; i \in \mathbb{N}\}$.

This answers the Van Douwen's question [vD, Question 7A.1] whether $\mu(A) \leq \overline{d}(A)$ for every density measure. The same density measure μ is a counterexample to the following claim of Lauwers [L, p.46]:

Every density measure can be expressed in the form

$$\mu_\varphi(A) = \int_{\beta\mathbb{N}^*} \mathcal{F}\text{-}\lim \frac{A(n)}{n}\, \mathrm{d}\varphi(\mathcal{F}), \qquad A \subseteq \mathbb{N} \tag{7.1}$$

for some probability Borel measure φ on the set of all free ultrafilters $\beta\mathbb{N}^*$.
It is easy to notice that if this claim were true the answer to van Douwen's question would be positive.

Šalát and Tijdeman have posed another question concerning the density measures [ŠT, p.201]. They ask whether every density measure has the following properties:
a) If $A(n) \leq B(n)$ for all $n \in \mathbb{N}$ then $\mu(A) \leq \mu(B)$ (where $A, B \subseteq \mathbb{N}$).
b) If $\lim\limits_{n\to\infty} \frac{A(n)}{B(tn)} = 1$ then $\mu(A) = t\mu(B)$ (where $A, B \subseteq \mathbb{N}$ and $t \in \mathbb{R}$).
(The authors of [ŠT] conjectured that there exist density measures that do not fulfill a) and b). We will see that this conjecture was right.)

Clearly, any density measure of the form (7.1) has both these properties.

The question a) is closely related to van Douwen's question. Clearly, if a set A fulfills $\overline{d}(A) < \mu(A)$ there is a set B having asymptotic density $d(B) \in (\overline{d}(A), \mu(A))$. Since $d(B) > \overline{d}(A)$, there exists n_0 such that $B(n) \geq A(n)$ for $n > n_0$. Since changing only finitely many elements influences neither asymptotic density nor density measure, any such pair of sets yields a counterexample to the property a).

It is easy to verify that for the set A from the preceding example (and the measure given by an ultrafilter containing $\{2^{2^i}; i \in \mathbb{N}\}$) we get $\mu(2A) = 0$ and $\mu(A) = 1$. This shows that property b) is not valid in general. (A different density measure μ and a set A with $\mu(2A) \neq \frac{1}{2}\mu(A)$ was given by Van Douwen [vD, Example 5.6, Case 2].)

References

[B] M. Blümlinger. Lévy group action and invariant measures on $\beta\mathbb{N}$. *Trans. Amer. Math. Soc.*, 348(12):5087–5111, 1996.

[BFPRN] A. Blass, R. Frankiewicz, G. Plebanek, and C. Ryll-Nardzewski. A note on extensions of asymptotic density. *Proc. Amer. Math. Soc.*, 129(11):3313–3320, 2001.

[BO] M. Blümlinger and N. Obata. Permutations preserving Cesáro mean, densities of natural numbers and uniform distribution of sequences. *Ann. Inst. Fourier*, 41:665–678, 1991.

[BRBR] K. P. S. Bhaskara Rao and M. Bhaskara Rao. *Theory of Charges – A Study of Finitely Additive Measures.* Pure and Applied Mathematics – A series of Monographs and Textbooks. Academic Press, London–New York, 1983.

[BŠ] B. Balcar and P. Štěpánek. *Teorie množin.* Academia, Praha, 1986.

[C] N. L. Carothers. *A Short Course on Banach Space Theory.* New York, 2004. London Mathematical Society Student Texts 64.

[CK] D. E. Campbell and J. S. Kelly. Asymptotic density and social choice trade-offs. *Math. Social Sci.*, 29:181–194, 1995.

[Fe] M. Fey. May's theorem with an infinite population. *Social Choice and Welfare*, 23:275–293, 2004.

[Fr] J. A. Fridy. On statistical convergence. *Analysis*, 5:301–313, 1985.

[GAGM] R. Giuliano Antonini, G. Grekos, and L. Mišík. On weighted densities. *Czechosl. Math. J.*, 57(3):947–962, 2007.

[HJ] K. Hrbacek and T. Jech. *Introduction to set theory.* Marcel Dekker, New York, 1999.

[HR] Paul Howard and Jean E. Rubin. *Consequences of the axiom of choice.* Mathematical Surveys and Monographs. 59. Providence, RI: American Mathematical Society (AMS), 1998.

[L] L. Lauwers. Intertemporal objective functions: strong Pareto versus anonymity. *Mathematical Social Sciences*, 35:37–55, 1998.

[LMS] Peter Letavaj, Ladislav Mišík, and Martin Sleziak. Extreme points of the set of density measures. in preparation.

[Ma] D. Maharam. Finitely additive measures on the integers. *Sankhya, Ser. A*, 38:44–59, 1976.

[Mo] T. J. Morrison. *Functional Analysis: An Introduction to Banach Space Theory*. Wiley, 2000.

[P] G. Pólya. Untersuchungen über Lücken und Singularitäten von Potenzreihen. *Math. Zeit.*, 29:549–640, 1929.

[PS] David Pincus and Robert M. Solovay. Definability of measures and ultrafilters. *J. Symbolic Logic*, 42(2):179–190, 1977.

[Š] T. Šalát. On statistically convergent sequences of real numbers. *Mathematica Slovaca*, 30:139–150, 1980.

[ŠT] T. Šalát and R. Tijdeman. Asymptotic densities of sets of positive integers. *Mathematica Slovaca*, 33:199–207, 1983.

[SZ1] M. Sleziak and M. Ziman. Lévy group and density measures. *J. Number Theory*, 128(12):3005–3012, 2008.

[SZ2] M. Sleziak and M. Ziman. Range of density measures. *Acta Mathematica Universitatis Ostraviensis*, 17:33–50, 2009.

[T] V. Toma. Densities and social choice trade-offs. *Tatra Mt. Math. Publ.*, 31:55–63, 2005.

[V] Martin Väth. The dual space of L_∞ is L_1. *Indagationes Mathematicae*, 9(4):619–625, 1998.

[vD] E. K. van Douwen. Finitely additive measures on $\mathbb{N}$. *Topology and its Applications*, 47:223–268, 1992.