I've put this text online on November 2, 2011. Since then I've occasionally corrected some typos.

The answer to [SZ, Problem 1] is affirmative, $\underline{\underline{d}}(A) = \underline{d_\infty}(A)$ holds for any $A \subseteq \mathbb{N}$.

Proof of $\underline{\underline{d}}(A) = \underline{d_\infty}(A)$

Pólya [P]:

$$\underline{\underline{d}}(A) = \lim_{\theta\to1^-} \liminf_{n\to\infty} \frac{A(n)-A(\theta n)}{n-\theta n}.$$

Suppose $\underline{\underline{d}}(A) = 0$. We want to show that $\underline{d_\infty}(A) = 0$.

We have

$$\lim_{\theta\to1^-} \liminf_{n\to\infty} \frac{A(n)-A(\theta n)}{n-\theta n} = 0.$$

Fix $\varepsilon > 0$. Then there exists $\theta_0 < 1$ such that

$$\liminf_{n\to\infty} \frac{A(n)-A(\theta n)}{n-\theta n} < \varepsilon$$

whenever $\theta_0 < \theta < 1$. This implies that

$$\frac{A(n)-A(\theta n)}{n-\theta n} < 2\varepsilon \qquad (1) \quad \{\texttt{EQTHETA}\}$$

for infinitely many n's.

If n fulfills (1) then

$$A_\alpha(n) \le A_\alpha(\theta n) + n^\alpha(A(n)-A(\theta n)) < A_\alpha(\theta n) + n^{\alpha+1}(1-\theta)2\varepsilon$$
$$\frac{A_\alpha(n)}{n^{\alpha+1}} \le \frac{A_\alpha(\theta n)}{(\theta n)^{\alpha+1}} \cdot \theta^{\alpha+1} + (1-\theta)2\varepsilon$$
$$\frac{A_\alpha(n)}{n^{\alpha+1}} \le \frac{\mathbb{N}_\alpha(\theta n)(\alpha+1)}{(\theta n)^{\alpha+1}} \cdot \frac{\theta^{\alpha+1}}{\alpha+1} + (1-\theta)2\varepsilon$$

Since the above inequality holds for infinitely many n's we get

$$\underline{d_\alpha}(A) = \liminf_{n\to\infty} \frac{A_\alpha(n)(\alpha+1)}{n^{\alpha+1}} \le \theta^{\alpha+1} + (1-\theta)2\varepsilon(\alpha+1) \qquad (2) \quad \{\texttt{INEQTHETA}\}$$

for any $\theta \in (\theta_0, 1)$ and any $\alpha > 0$.

Let us assume that, moreover,

$$(1-\theta)(\alpha+1) = \frac{1}{\sqrt{\varepsilon}}. \qquad (3) \quad \{\texttt{EQTHETAALPHA}\}$$

For a given $\theta < 1$ we get $\alpha + 1 = \frac{1}{\sqrt{\varepsilon}(1-\theta)}$. Hence we can find such α and $\alpha \to \infty$ as $\theta \to 1^-$ (for a fixed $\varepsilon > 0$).

Similarly, if we fix $\alpha > 0$ then $\theta = 1 - \frac{1}{(\alpha+1)\sqrt{\varepsilon}}$; i.e., there exists a corresponding θ and we have $\theta \to 1^-$ for $\alpha \to \infty$ (for a fixed $\varepsilon > 0$).

Using (3) we get from (2)

$$\underline{d_\alpha}(A) \le \left(1 - \frac{1}{(\alpha+1)\sqrt{\varepsilon}}\right)^{\alpha+1} + 2\sqrt{\varepsilon}. \qquad (4) \quad \{\texttt{EQALPEPS}\}$$

This inequality is valid for any α and $\theta \in (\theta_0, 1)$ that fulfill (3).

Now, if $\alpha \to \infty$ (note that this means $\theta \to 1^-$, hence we can always find $\theta \in (\theta_0, 1)$ such that (3) holds), we get

$$\underline{d_\infty}(A) \le e^{-\frac{1}{\sqrt{\varepsilon}}} + 2\sqrt{\varepsilon}. \qquad (5)$$ {EQINFEPS}

As the RHS tends to 0 for $\varepsilon \to 0^+$ and $\varepsilon > 0$ can be chosen arbitrarily, we finally get

$$\underline{d_\infty}(A) = 0.$$

If we combine the above with [SZ, Corollary 6], we have so far proved $\underline{\underline{d}}(A) = \underline{d}_\infty(A)$ for the case that some of these values is zero.

By [SZ, Proposition 2] we know that $\underline{\underline{d}}(A) = d(B)$ for some $B \subseteq A$, $B \in \mathcal{D}$. For this set we have $\underline{\underline{d}}(A \setminus B) = 0$. This implies

$$\underline{\underline{d}}(A) = \underline{\underline{d}}(B \cup A \setminus B) = d(B) + \underline{\underline{d}}(A \setminus B) = d(B) + \underline{d}_\infty(A \setminus B) = \underline{d}_\infty(A).$$

(We have used [SZ, Proposition 1] and some properties of α-densities which are recapitulated in [SZ, Theorem 1, Corollary 1, Corollary 2].)

References

[P] G. Pólya. Untersuchungen über Lücken und Singularitäten von Potenzreihen. *Math. Zeit.*, 29:549–640, 1929.

[SZ] M. Sleziak and M. Ziman. Range of density measures. *Acta Mathematica Universitatis Ostraviensis*, 17:33–50, 2009.