GAPS AND THE EXPONENT OF CONVERGENCE OF AN INTEGER SEQUENCE

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Abstract. Professor Tibor Šalát, at one of his seminars at Comenius University, Bratislava, asked to study the influence of gaps of an integer sequence $A = \{a_1 < a_2 < \dots < a_n < \dots\}$ on its exponent of convergence. The exponent of convergence of A coincides with its upper exponential density. In this paper we consider an extension of Professor Šalát's question and we study the influence of the sequence of ratios $(\frac{a_m}{a_{m+1}})_{m=1}^{\infty}$ and of the sequence $(\frac{a_{m+1}-a_m}{a_m})_{m=1}^{\infty}$ on the upper and on the lower exponential densities of A.

1. Introduction

The concept of exponent of convergence is introduced in [10]. The authors of this treatise proved that for any real sequence $r = (r_n)_{n=1}^{\infty}$, $0 < r_1 \le r_2 \le \dots \le r_n \le \dots$, with $\lim\limits_{n\to\infty} r_n = +\infty$, there exists $\tau \in [0, +\infty]$, such that the series $\sum\limits_{n=1}^{\infty} r_n^{-\sigma}$ is convergent whenever $\sigma > \tau$ and divergent whenever $\sigma < \tau$ ([10, Part I, Exercises 113,114]). The number τ is called the *exponent of convergence* of the sequence r and denoted by $\tau(r)$. The exponent of convergence of real non-decreasing sequences was also studied in [7, 8, 11]. It was proved by Pólya and Szegö [10, Part I, Exercises 113,114] that $\tau(r)$ can be calculated by the formula

$$\tau(r) = \limsup_{n\to\infty} \frac{\log n}{\log r_n}. \tag{1}$$

In particular, if r is an *integer sequence* $A = \{a_1 < a_2 < \dots < a_n < \dots\}$ (that is, A is an infinite subset of $\mathbb{N} = \{1, 2, \dots\}$), then A has an exponent of convergence $\tau(A) \in [0, 1]$.

This simple observation indicates that when dealing with sequences of positive integers, then the exponent of convergence could be related to the number-theoretic densities.

We recall the notion of exponential density [3, 9].

Definition 1.1. The *upper* and *lower exponential densities* of an infinite subset A of $\mathbb{N}$ are defined by

$$\overline{\varepsilon}(A) = \limsup_{k\to\infty} \frac{\log A(k)}{\log k},$$
$$\underline{\varepsilon}(A) = \liminf_{k\to\infty} \frac{\log A(k)}{\log k},$$

where $A(x)$ denotes $|A \cap [1, x]|$.

If $\overline{\varepsilon}(A) = \underline{\varepsilon}(A)$, then we say that A has the exponential density $\varepsilon(A) = \overline{\varepsilon}(A) = \underline{\varepsilon}(A)$.

One can easily see that, for an infinite subset $A = \{a_1 < a_2 < \ldots < a_n < \ldots\}$ of $\mathbb{N}$, we have $\overline{\varepsilon}(A) = \tau(A) = \limsup_{n\to\infty} \frac{\log n}{\log a_n}$, and $\underline{\varepsilon}(A) = \liminf_{n\to\infty} \frac{\log n}{\log a_n}$.

The purpose of this paper is the investigation of the influence of gaps

$$g_n = a_{n+1} - a_n$$

in the set $A = \{a_1 < a_2 < \cdots < a_n < \ldots\} \subseteq \mathbb{N}$ on its exponent of convergence. This study was suggested to us by the late Professor Tibor Šalát. We will be also concerned with a slightly more general question about the influence of gaps in A on both exponential densities.

2. Ratios of consecutive terms and exponent of convergence

We are interested in the influence of gaps

$$g_n = a_{n+1} - a_n$$

in the set A on the exponent of convergence. Since a gap of given length has less influence if it is situated far from the origin, we might expect that (at least to some extent) we can describe the behavior of the exponent of convergence in terms of the asymptotic behavior of the fractions

$$\frac{g_n}{a_{n+1}} \qquad \text{or} \qquad \frac{a_n}{a_{n+1}}.$$

Note that $\frac{a_n}{a_{n+1}} + \frac{g_n}{a_{n+1}} = 1$.

Definition 2.1. We define the upper and the lower *limit ratios* of A by

$$\overline{\varrho}(A) = \limsup_{n\to\infty} \frac{a_n}{a_{n+1}},$$
$$\underline{\varrho}(A) = \liminf_{n\to\infty} \frac{a_n}{a_{n+1}}.$$

We remark that several related concepts have been studied in various contexts. The gap density $\lambda(A) = \limsup \frac{a_{n+1}}{a_n}$ was introduced in [5] to study properties of the density set $D(A) = \{(\underline{d}(B), \overline{d}(B)); B \subseteq A\}$ of A, and further studied in [4]. Clearly $\underline{\varrho}(A) = \frac{1}{\lambda(A)}$ (using the convention $\frac{1}{\infty} = 0$).

The sets A with $\overline{\varrho}(A) = 0$ (called thin sets) play a role in the study of measures which can be regarded as certain extensions of asymptotic density [1]. The sets with $\overline{\varrho}(A) < 1$ (called almost thin sets) are studied in connection with some ultrafilters on $\mathbb{N}$ [2].

First we show that the exponent of convergence of any set $A \subseteq \mathbb{N}$ with $\overline{\varrho}(A) < 1$ is equal to zero. We will need the following well-known result (see e.g. [6, Problem 2.3.11], [12]):

Theorem 2.2 (Stolz Theorem)**.** *Let $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ be sequences of real numbers such that $(y_n)_{n=1}^\infty$ is strictly increasing, unbounded and*

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = g,$$

$g \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = g.$$

Proposition 2.3. *If $\overline{\varrho}(A) < 1$, then $\tau(A) = 0$.*

Proof. We will apply Theorem 2.2 to the sequences $x_n = \log n$ and $y_n = \log a_n$. Clearly, y_n is strictly increasing and unbounded. Note that $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = \lim\limits_{n\to\infty} \log \frac{n+1}{n} = 0$. Therefore

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{\log \frac{n+1}{n}}{\log \frac{a_{n+1}}{a_n}} = 0, \tag{2}$$

whenever $\log \frac{a_{n+1}}{a_n}$ is bounded from zero. Thus the assumption $\overline{\varrho}(A) < 1$ is sufficient to infer this.

From (2) we get

$$\lim_{n\to\infty} \frac{\log n}{\log a_n} = 0$$

by Stolz theorem. Thus $\tau(A) = 0$. □

It remains only to analyse the case $\overline{\varrho}(A) = 1$. The following examples show that in this case nothing can be said about $\tau(A)$ in general.

Example 2.4. Let $a \in]0, 1]$, and let $A = \{\lfloor n^{\frac{1}{a}} \rfloor; n \in \mathbb{N}\}$. Then $\varrho(A) = 1$ and $\varepsilon(A) = a$.

Example 2.5. Let $A = \{2^n; n \in \mathbb{N}\} \cup \{2^{2^N} + 1; N \in \mathbb{N}\}$. Then $\overline{\varrho}(A) = 1$ and $\varepsilon(A) = 0$.

Example 2.6. Let $A = \{a_n = \lfloor u_n \rfloor; n \geq 1, u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n\}$. Then $\varrho(A) = 1$ and $\varepsilon(A) = 0$.

Proof. First observe that u_n tends to infinity as $n \to \infty$, since

$$\lim_{n\to\infty} \log u_n = \lim_{n\to\infty} n \log \left(1 + \frac{1}{\sqrt{n}}\right) = +\infty.$$

This yields that $a_n \sim u_n$ and $\log a_n \sim \log u_n$.

We have

$$\lim_{n\to\infty} \frac{\log n}{\log a_n} = \lim_{n\to\infty} \frac{\log n}{\log u_n} = \lim_{n\to\infty} \frac{\log n}{n \log \left(1 + \frac{1}{\sqrt{n}}\right)} = 0$$

and so $\tau(A) = \varepsilon(A) = 0$.

We will show that $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$, which implies $\varrho(A) = 1$. Note that

$$\frac{u_n}{u_{n+1}} = \frac{t_n}{1 + \frac{1}{\sqrt{n+1}}},$$

where

$$t_n = \left(\frac{1 + \frac{1}{\sqrt{n}}}{1 + \frac{1}{\sqrt{n+1}}}\right)^n.$$

So it is sufficient to show that $\lim\limits_{n\to\infty} t_n = 1$ or equivalently $\lim\limits_{n\to\infty} \ln t_n = 0$.

Indeed

$$\begin{aligned}\ln t_n &= n\left(\ln\left(1+\frac{1}{\sqrt{n}}\right) - \ln\left(1+\frac{1}{\sqrt{n+1}}\right)\right) = \\ &= n\left(\frac{1}{\sqrt{n}} - \frac{1}{2n}(1+o(1)) - \frac{1}{\sqrt{n+1}} + \frac{1}{2(n+1)}(1+o(1))\right) = \\ &\qquad = \frac{n}{\sqrt{n}\sqrt{n+1}(\sqrt{n}+\sqrt{n+1})} + \frac{1}{2}\left(\frac{n}{n+1}-1\right) + o(1)\end{aligned}$$

tends to 0 as $n \to \infty$.

Thus it remains to prove that $u_{n+1} - u_n > 1$ for all sufficiently large n, and hence the elements (integers) a_n of the set A are pairwise distinct for n large enough. We shall prove that $\lim\limits_{n\to\infty}(u_{n+1} - u_n) = +\infty$ using the function

$$f(x) = \left(1+\frac{1}{\sqrt{x}}\right)^x = e^{x\ln\left(1+\frac{1}{\sqrt{x}}\right)}.$$

An easy computation gives

$$f'(x) = f(x)\left(\ln\left(1+\frac{1}{\sqrt{x}}\right) - \frac{1}{2(\sqrt{x}+1)}\right) \geq \frac{f(x)}{2(\sqrt{x}+1)}. \tag{3}$$

From the inequality $\left(1+\frac{1}{t}\right)^t \geq 2$, which is valid for $t \geq 1$, we get $f(x) \geq 2^{\sqrt{x}}$ and

$$f'(x) \geq \frac{2^{\sqrt{x}}}{2(\sqrt{x}+1)} \tag{4}$$

for $x \geq 1$. This implies that $\lim\limits_{x\to\infty} f'(x) = +\infty$ and

$$\lim_{n\to\infty}(u_{n+1}-u_n) = \lim_{n\to\infty}(f(n+1)-f(n)) = +\infty$$

by the mean value theorem. □

The above examples suggest the following question:

Problem 2.7. Given $a, b, c \in [0,1]$, $a \leq b$, does there exist a subset $A \subseteq \mathbb{N}$ such that $\underline{\varepsilon}(A) = a$, $\overline{\varepsilon}(A) = b$, $\underline{\varrho}(A) = c$ and $\overline{\varrho}(A) = 1$?

3. Rate of proximity of $\frac{a_n}{a_{n+1}}$ to 1 and the exponential densities

The following three examples provide a motivation for the questions studied in this section.

Example 3.1. Let $a_n = \alpha b^n(1+o(1))$, for some $\alpha > 0$ and $b > 1$ (geometric-like sequence). Then $\tau(A) = 0$ and

$$\lim_{n\to\infty}\frac{g_n}{a_{n+1}} = \lim_{n\to\infty} 1 - \frac{a_n}{a_{n+1}} = 1 - \frac{1}{b} \in]0,1[.$$

Example 3.2. If $a_n = k + ln$ is an arithmetic sequence then $\tau(A) = 1$ and

$$\frac{g_n}{a_{n+1}} = \frac{1}{n}(1+o(1)).$$

Example 3.3. If $a_n = k + ln + tn^2$, then $\tau(A) = \frac{1}{2}$ and

$$\frac{g_n}{a_{n+1}} = \frac{2}{n}(1 + o(1)).$$

More generally, for any real $d > 1$ and any integer sequence $(a_n)_{n=1}^{\infty}$ satisfying $a_n = tn^d(1 + o(1))$ we get $\tau(A) = \frac{1}{d}$ and

$$\frac{g_n}{a_{n+1}} = \frac{d}{n}(1 + o(1)).$$

Thus for a real number $d > 0$ and $(a_n)_{n=1}^{\infty}$ as above, we have $\tau(A) = \max\{1, \frac{1}{d}\}$.

The following corollary is a straightforward consequence of Proposition 2.3.

Corollary 3.4. *If* $\liminf\limits_{n\to\infty} \frac{g_n}{a_{n+1}} > 0$ *then* $\tau(A) = \varepsilon(A) = 0$.

This can be improved as follows.

Proposition 3.5. *If* $\liminf\limits_{n\to\infty} \frac{g_n}{a_n} > 0$ *then* $\tau(A) = \varepsilon(A) = 0$.

Proof. The hypothesis $\liminf\limits_{n\to\infty} \frac{g_n}{a_n} > 0$ guarantees that there exists a $\delta > 0$ such that $\frac{g_n}{a_n} \geq \delta$ for each n. This implies

$$\frac{a_{n+1}}{a_n} = 1 + \frac{g_n}{a_n} \geq 1 + \delta.$$

Hence

$$\log a_n \geq \log c + n \log(1 + \delta), \text{ for some constant } c.$$

From this we get

$$0 \leq \lim_{n\to\infty} \frac{\log n}{\log a_n} \leq \lim_{n\to\infty} \frac{\log n}{\log c + n\log(1 + \delta)} = 0.$$

□

Examples 3.1, 3.2 and 3.3 show that there is a relation between the exponent of convergence and the limit behavior of $\frac{g_n/a_{n+1}}{1/n}$. This is generalized in Proposition 3.7.

We will need a slightly more general form of Stolz Theorem. For the sake of completeness we include the proof of this result.

Lemma 3.6. *Let* $(x_n)_{n=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty}$ *be sequences of positive real numbers such that* $(y_n)_{n=1}^{\infty}$ *is strictly increasing and unbounded. Then*

$$\liminf_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \leq \liminf_{n\to\infty} \frac{x_n}{y_n} \leq \limsup_{n\to\infty} \frac{x_n}{y_n} \leq \limsup_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}.$$

Proof. Put $l = \liminf\limits_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$. Given $\varepsilon > 0$, there exists n_0 such that $\frac{x_{n+1}-x_n}{y_{n+1}-y_n} \geq l - \varepsilon$ for all $n > n_0$. Using this fact we get, for all $n > n_0$,

$$x_n - x_{n_0} \geq (y_n - y_{n_0})(l - \varepsilon).$$

Thus

$$\liminf_{n\to\infty} \frac{x_n}{y_n} \geq l - \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we get

$$\liminf_{n\to\infty} \frac{x_n}{y_n} \geq l.$$

The proof of the second part of this lemma is analogous. □

Using the well-known equation $\sum_{k\le n}\frac1k=\ln n+\gamma+o\left(\frac1n\right)$ we get the alternative formulae for exponential densities

$$\overline{\varepsilon}(A)=\limsup_{n\to\infty}\frac{\sum_{k\le n}\frac1k}{\sum_{k\le a_n}\frac1k}, \tag{5}$$

$$\overline{\varepsilon}(A)=\liminf_{n\to\infty}\frac{\sum_{k\le n}\frac1k}{\sum_{k\le a_n}\frac1k}. \tag{6}$$

Proposition 3.7. *Let $A=\{a_1<a_2<\ldots<a_n<\ldots\}$ and*

$$\alpha(A)=\liminf_{n\to\infty} n\frac{g_n}{a_{n+1}},$$

$$\beta(A)=\limsup_{n\to\infty} n\frac{g_n}{a_n}.$$

Then

$$\frac{1}{\beta(A)}\le\underline{\varepsilon}(A)\le\overline{\varepsilon}(A)\le\frac{1}{\alpha(A)}. \tag{7}$$

(We use the convention $\frac{1}{\infty}=0$ and $\frac10=\infty$.)

In particular, if $\alpha(A)=\beta(A)$, then $\varepsilon(A)=\frac{1}{\alpha(A)}=\frac{1}{\beta(A)}$.

Proof. Applying Lemma 3.6 to (5) we get

$$\limsup\frac{\sum_{k\le n}\frac1k}{\sum_{k\le a_n}\frac1k}\le\limsup\frac{\frac{1}{n+1}}{\sum_{a_n<k\le a_{n+1}}\frac1k}\le\limsup\frac{\frac{1}{n+1}}{\frac{a_{n+1}-a_n}{a_{n+1}}}=\limsup\frac{1}{n+1}\cdot\frac{1}{\frac{g_n}{a_{n+1}}}.$$

By a similar argument we get

$$\liminf_{n\to\infty}\frac{\sum_{k\le n}\frac1k}{\sum_{k\le a_n}\frac1k}\ge\liminf_{n\to\infty}\frac{\frac{1}{n+1}}{\sum_{a_n<k\le a_{n+1}}\frac1k}\ge\liminf_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{a_{n+1}-a_n}{a_n}}=\liminf_{n\to\infty}\frac{1}{n+1}\frac{1}{\frac{g_n}{a_n}}.$$

□

The above lemma yields

Theorem 3.8. *Let α, β be real numbers with $1\le\alpha\le\beta$.*

(i) *If there exists a real sequences $(e_n)_{n=1}^\infty$ tending to zero such that, for each $n\ge1$,*

$$(1+e_n)\frac{\alpha}{n}\le\frac{g_n}{a_{n+1}},$$

then

$$\overline{\varepsilon}(A)\le\frac1\alpha.$$

(ii) *If there exists a real sequences $(f_n)_{n=1}^\infty$ tending to zero such that, for each $n\ge1$,*

$$\frac{g_n}{a_n}\le\frac{\beta}{n}(1+f_n),$$

then

$$\frac1\beta\le\underline{\varepsilon}(A).$$

Example 3.9. Proposition 3.7 can be used to compute the exponential density for some sets from the above examples. By a straightforward computation we get the following values.

$A = \{a_n; n \in \mathbb{N}\}$	$\alpha(A)$	$\beta(A)$
$A = \{\lfloor n^{\frac{1}{a}} \rfloor; n \in \mathbb{N}\}$	$\frac{1}{a}$	$\frac{1}{a}$
$A = \{2^n; n \in \mathbb{N}\} \cup \{2^{2^N} + 1; N \in \mathbb{N}\}$	0	$+\infty$
$A = \{n^2; n \in \mathbb{N}, n \text{ is not square}\}$	2	4
$a_n = \alpha b^n(1 + o(1))$	$+\infty$	$+\infty$
$a_n = k + ln$	1	1
$a_n = k + ln + tn^2$	2	2
$a_n = tn^d(1 + o(1))$	d	d

We have shown in Example 2.5 that $\varepsilon(A) = 0$ for the set in the second row from this table. It can be shown easily that $\varepsilon(A) = \frac{1}{2}$ for the set in the fourth row. These two examples show that the inequalities in (7) can be strict.

In all other rows we have $\alpha(A) = \beta(A)$, and the value of $\varepsilon(A)$ can be computed using Proposition 3.7.

The computation for the set in Example 2.6 is slightly more complicated. We use the function $f(x) = \left(1 + \frac{1}{\sqrt{x}}\right)^x$ again. Notice that (3) implies that this function is increasing.

Using the mean value theorem and (3) we get

$$n\frac{f(n+1) - f(n)}{f(n)} \geq n\frac{\min\{f'(c); c \in [n, n+1]\}}{f(n)} \geq$$

$$\geq n\frac{\min\{\frac{f(c)}{2(\sqrt{c}+1)}; c \in [n, n+1]\}}{f(n)} \geq \frac{n}{f(n)} \cdot \frac{f(n)}{2(\sqrt{n+1}+1)} \frac{n}{2(\sqrt{n+1}+1)}$$

and the last expression tends to $+\infty$ as $n \to \infty$.

We found out that $\alpha(A) = \beta(A) = +\infty$. Thus $\varepsilon(A) = 0$ by Proposition 3.7.

Remark 3.10. If the set A satisfies the hypotheses of Theorem 3.8 with $\alpha = \beta$, then $\varepsilon(A) = \tau(A) = \frac{1}{\alpha}$. This type of sets A generalizes the sets considered in Example 3.2 (sequences increasing arithmetically; $\alpha = \beta = 1$) and in Example 3.3 (sequences increasing polynomially; $\alpha = \beta > 1$).

Now we state two refinements of Theorem 3.8, Theorems 3.12 and 3.13.

Lemma 3.11. *For any $x \in [0, \frac{1}{2}]$ the inequality*

$$-\ln(1 - x) \leq x + x^2$$

holds.

Proof. By studying $f(x) = x + x^2 - [-\ln(1 - x)]$. □

Theorem 3.12. *Let β be a real number, $\beta \geq 1$. If there exists a real sequence $(f_n)_{n=1}^{\infty}$ tending to zero such that, for each $n \geq 1$,*

$$\frac{g_n}{a_{n+1}} \leq \frac{\beta}{n}(1 + f_n),$$

then

$$\underline{\varepsilon}(A) \geq \frac{1}{\beta}.$$

Proof. By the definition of $\underline{\varepsilon}(A)$ it is sufficient to show that for each $0 < \varepsilon < \frac{1}{\beta}$ there exists $n_0 = n_0(\varepsilon)$ such that for each $n \geq n_0$
$$\frac{\ln n}{\ln a_n} \geq \frac{1}{\beta} - \varepsilon.$$
We will verify that
$$\ln a_{n+1} \leq \frac{\ln(n+1)}{\frac{1}{\beta} - \varepsilon}$$
for every n large enough.

The hypothesis gives
$$\frac{a_{n+1} - a_n}{a_{n+1}} \leq \frac{\beta}{n}(1 + f_n),$$
$$\frac{a_{n+1}}{a_n} \leq \frac{1}{1 - \frac{\beta}{n}(1 + f_n)}.$$
Let us choose n_1 such that $|f_n| < \frac{1}{2}$, for each $n \geq n_1$. Then we get
$$\ln a_{n+1} = \ln a_{n_1} + \sum_{i=n_1}^{n} \ln \frac{a_{i+1}}{a_i} \leq \ln a_{n_1} - \sum_{i=n_1}^{n} \ln(1 - \frac{\beta}{n}(1 + f_n)).$$
Let $n_2 \geq n_1$ be such that $|\frac{\beta}{n}(1 + f_n)| \leq \frac{1}{2}$ whenever $n \geq n_2$. Then we get
$$\ln a_{n+1} \leq c_1 + \sum_{i=n_2}^{n} \frac{\beta}{i}(1 + f_i) + \sum_{i=n_2}^{n} \frac{\beta^2}{i^2}(1 + f_i)^2.$$
Put $\delta = \frac{1}{2}\frac{\beta\varepsilon}{1-\beta\varepsilon} > 0$ and choose $n_3 \geq n_2$ such that $|f_i| \leq \delta$, whenever $i \geq n_3$.

Then, for all $n \geq n_3$,
$$\ln a_{n+1} \leq c_2 + \sum_{i=n_3}^{n} \frac{\beta(1+\delta)}{i} + \sum_{i=n_3}^{n} \frac{\beta^2(1+\delta)^2}{i^2} \leq$$
$$\leq c_3 + \beta(1+\delta)\sum_{i=1}^{n} \frac{1}{i} \leq c_3 + \beta(1+\delta) + \beta(1+\delta)\ln(n+1)$$
holds.

Obviously
$$\beta(1+\delta) < \frac{1}{\frac{1}{\beta} - \varepsilon}$$
and therefore, the right hand side of the above inequality is at most $\frac{\ln(n+1)}{\frac{1}{\beta}-\varepsilon}$ for every sufficiently large n. □

Theorem 3.13. *Let α be a real number, $\alpha \geq 1$. Suppose that for all n*
$$\frac{g_n}{a_n} \geq \frac{\alpha}{n}(1 + e_n),$$
where $\lim\limits_{n\to\infty} e_n = 0$. Then
$$\overline{\varepsilon}(A) \leq \frac{1}{\alpha}.$$

Note that Proposition 3.5 can be deduced from the above theorem.

Proof. If $\alpha = 1$ we are done. Suppose that $\alpha > 1$.

By the definition of $\overline{\varepsilon}(A)$ it suffices to show that for each $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon)$ such that

$$\frac{\ln n}{\ln a_n} \leq \frac{1}{\alpha} + \varepsilon,$$

whenever $n \geq n_0$. That is, $\ln n \leq \left(\frac{1}{\alpha} + \varepsilon\right) \ln a_n$ or equivalently

$$\ln a_n \geq \frac{\ln n}{\frac{1}{\alpha} + \varepsilon}.$$

Fix $\varepsilon > 0$. We shall prove that for all n sufficiently large

$$\ln a_{n+1} \geq \frac{\ln(n+1)}{\frac{1}{\alpha} + \varepsilon}. \tag{8}$$

Let $\delta = \frac{1}{2}\frac{\alpha\varepsilon}{1+\alpha\varepsilon}$. Obviously $\delta < 1$. Choose n_1 such that $|e_n| \leq \delta$ for all $n \geq n_1$. The hypothesis of the theorem implies that for all n

$$\frac{a_{n+1}}{a_n} \geq 1 + \frac{\alpha}{n}(1 + e_n).$$

Then for $n \geq n_1$, we have

$$\ln a_{n+1} = \ln a_{n_1} + \sum_{i=n_1}^{n} \ln \frac{a_{i+1}}{a_i} \geq \ln a_{n_1} + \sum_{i=n_1}^{n} \ln\left(1 + \frac{\alpha}{i}(1 + e_i)\right),$$

and by the inequality

$$\ln(1 + x) \geq x - \frac{x^2}{2}$$

(valid for $x \geq 0$), we have

$$\ln a_{n+1} \geq \ln a_{n_1} + \sum_{i=n_1}^{n} \frac{\alpha}{i}(1 + e_i) - \frac{1}{2}\sum_{i=n_1}^{n} \frac{\alpha^2}{i^2}(1 + e_i)^2 \geq$$

$$\geq \ln a_{n_1} + \alpha(1 - \delta)\sum_{i=n_1}^{n} \frac{1}{i} - \frac{1}{2}(1 + \delta)^2\alpha^2 \sum_{i=1}^{\infty} \frac{1}{i^2}.$$

Finally

$$\ln a_{n+1} \geq c_4 + \alpha(1 - \delta)\sum_{i=1}^{n} \frac{1}{i},$$

where c_4 does not depend on n. The last inequality implies that

$$\ln a_{n+1} \geq c_4 + \alpha(1 - \delta)\ln(n + 1),$$

since $\sum_{i=1}^{n} \frac{1}{i} > \ln(n + 1)$.

Now to deduce that (8) is valid for all n large enough, it suffices to verify that $\alpha(1 - \delta) > \frac{1}{\frac{1}{\alpha}+\varepsilon}$, which is straightforward. □

Note that by Proposition 3.5 the study of exponential densities is non-trivial only for sets A such that $\liminf_{n\to\infty} \frac{g_n}{a_n} = 0$. In view of this, the results stated in Theorems 3.12 and 3.13 are far from being complete since only comparison of $\frac{g_n}{a_n}$ or $\frac{g_n}{a_{n+1}}$ to sequences of the type "constant times $\frac{1}{n}$" was considered. Nevertheless these cases, motivated by Examples 3.2 and 3.3, are the most important ones as the following observations show.

Observation 1. If $\frac{g_n}{a_n}$ is approximatively $\frac{1}{n^\alpha}$ with $\alpha > 1$, then a_{n+1} is approximatively $a_n(1+n^\alpha)$ and a_n is approximatively $c\prod_{i=1}^{n}\left(1+\frac{1}{i^\alpha}\right)$. The product $\prod_{i=1}^{\infty}\left(1+\frac{1}{i^\alpha}\right)$ being convergent, we get that the set $\{\lfloor c\prod_{i=1}^{n}\left(1+\frac{1}{i^\alpha}\right)\rfloor; n\in\mathbb{N}\}$ is finite.

Observation 2. If $\frac{g_n}{a_n}$ is approximatively $\frac{1}{n^\alpha}$ with $0<\alpha<1$, again a_n would be close to $u_n := c\prod_{i=1}^{n}\left(1+\frac{1}{i^\alpha}\right)$. Then $\ln u_n \sim \frac{n^{1-\alpha}}{1-\alpha}$, so that $\lim\limits_{n\to\infty}\frac{\ln u_n}{\ln n} = +\infty$. In other words, A has zero exponential density.

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