On hereditary coreflective subcategories of **Top**

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Abstract. Let A be a topological space which is not finitely generated and $\mathrm{CH}(A)$ denote the coreflective hull of A in **Top**. We construct a generator of the coreflective subcategory $\mathrm{SCH}(A)$ consisting of all subspaces of spaces from $\mathrm{CH}(A)$ which is a prime space and has the same cardinality as A. We also show that if $\mathbf{A}$ and $\mathbf{B}$ are coreflective subcategories of **Top** such that the hereditary coreflective kernel of each of them is the subcategory **FG** of all finitely generated spaces, then the hereditary coreflective kernel of their join $\mathrm{CH}(\mathbf{A} \cup \mathbf{B})$ is again **FG**.

Keywords: coreflective subcategory, hereditary coreflective subcategory, hereditary coreflective hull, hereditary coreflective kernel, prime space

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Introduction

Let X be a topological space which is not finitely generated and $\mathrm{SCH}(X)$ be the hereditary coreflective hull of X in the category **Top** of topological spaces. The aim of this paper is to construct a prime space Y_X with the same cardinality as X such that $\mathrm{SCH}(X) = \mathrm{CH}(Y_X)$ where $\mathrm{CH}(Y_X)$ is the coreflective hull of Y_X. Obviously, if X is finitely generated, then $\mathrm{CH}(X) = \mathrm{SCH}(X)$. If X is not finitely generated, then, using the prime factors of X we can easily construct a prime space P_X such that $\mathrm{SCH}(X) = \mathrm{SCH}(P_X)$. Thus, it suffices to restrict our investigation to the case of prime spaces.

For the prime space $C(\omega_0)$ consisting of a convergent sequence and its limit point the problem was studied in [5], where a countable generator for the category $\mathrm{SCH}(C(\omega_0))$ of subsequential spaces was produced.

Our procedure of constructing a generator Y_A of the category $\mathrm{SCH}(A)$ (where A is a prime space that is not finitely generated) consists of two main steps. In the first step, using similar methods as in [5], we produce a set of special prime spaces which generates $\mathrm{SCH}(A)$. Then, in the second step, we construct the generator Y_A of $\mathrm{SCH}(A)$ with the required properties.

This construction was inspired by the space S_ω from [2] and in the case $A = C(\omega_0)$ it gives a countable generator for the category of subsequential spaces different from that one presented in [5].

Finally, as an application of some above mentioned results we prove that if $\mathbf{A}$ and $\mathbf{B}$ are coreflective subcategories of **Top** such that the hereditary coreflective kernel of $\mathbf{A}$ as well as the hereditary coreflective kernel of $\mathbf{B}$ is the category **FG** of finitely generated spaces, then **FG** is also the hereditary coreflective kernel of their join $\mathrm{CH}(\mathbf{A} \cup \mathbf{B})$. As a consequence of this result and some results of [9] we obtain that the collection of all those coreflective subcategories of **Top** the hereditary coreflective kernel of which is **FG** and the hereditary coreflective hull of which is **Top** is closed under the formation of non-empty finite joins (in the lattice of all coreflective subcategories of **Top**) and arbitrary non-empty intersections.

1. Preliminaries

We recall some known facts about coreflective subcategories of the category **Top** of topological spaces (see [6]). All subcategories are supposed to be full and isomorphism-closed. The topological sum is denoted by $\sqcup$.

Let $\mathbf{A}$ be a subcategory of **Top**. $\mathbf{A}$ is *coreflective* if and only if it is closed under the formation of topological sums and quotient spaces. If $\mathbf{A}$ is a subcategory of **Top** or a class of topological spaces, then the *coreflective hull* of $\mathbf{A}$ is the smallest coreflective subcategory of **Top** which contains $\mathbf{A}$ and we denote it by $\mathrm{CH}(\mathbf{A})$. $\mathrm{CH}(\mathbf{A})$ consists of all quotients of topological sums of spaces that belong to $\mathbf{A}$. If $\mathbf{B} = \mathrm{CH}(\mathbf{A})$, then we say that $\mathbf{A}$ *generates* $\mathbf{B}$ and the members of $\mathbf{A}$ are called *generators* of $\mathbf{B}$. If $\mathbf{B} = \mathrm{CH}(\{X\})$, then $\mathbf{B}$ is called *simple generated* and X is said to be a *generator* of $\mathbf{B}$. We use the notation $\mathbf{B} = \mathrm{CH}(X)$ in this case.

Let $\mathbf{A}$ be a subcategory of **Top** and let $\mathrm{S}\mathbf{A}$ denote the subcategory of **Top** consisting of all subspaces of spaces from $\mathbf{A}$. Then the following result is known (see [8, Remark 2.4.4(5)] or [3, Proposition 3.1]).

PROPOSITION 1.1. *If* $\mathbf{A}$ *is a coreflective subcategory of* **Top***, then* $\mathrm{S}\mathbf{A}$ *is also a coreflective subcategory of* **Top**. *(*$\mathrm{S}\mathbf{A}$ *is the hereditary coreflective hull of* $\mathbf{A}$*.)*

By **FG** we denote the category of all finitely generated spaces. It is well known (see e.g. [6]) that if X is not finitely generated, then $\mathbf{FG} \subseteq \mathrm{CH}(X)$.

We say that a subcategory $\mathbf{A}$ of **Top** is *hereditary* if with each topological space X it contains also all its subspaces. It is well known that the category of all finitely generated spaces and all its subcategories that are coreflective in **Top** are hereditary.

Some known hereditary coreflective subcategories of **Top** are $\mathbf{Gen}(\alpha)$ and $\mathbf{Top}(\alpha)$, where α is an infinite cardinal. $\mathbf{Gen}(\alpha)$ is the subcategory of all spaces having tightness not exceeding α. $\mathbf{Top}(\alpha)$ is the category of all topological spaces such that the intersection of every family of open sets, which has cardinality less than α, is an open set.

Let A be a topological space. We say that A is a *prime space* if it has precisely one accumulation point. The following assertion is obvious.

LEMMA 1.2. *Let X be a prime space with an accumulation point a and let Y be a subspace of X containing the point a, then the map $f\colon X \to Y$, defined by $f(x) = x$ for $x \in Y$ and $f(x) = a$ for $x \in X \setminus Y$, is a quotient map.*

Given a topological space X and a point $a \in X$, denote by X_a the space constructed by making each point, other than a, isolated with a retaining its original neighborhoods. (I.e. a subset $U \subseteq X$ is open in X_a if and only if $a \notin U$ or there exists an open subset V of X such that $a \in V \subseteq U$.) The topological space X_a is called *prime factor of X at the point a*. It is clear that any prime factor is either a prime space or a discrete space.

PROPOSITION 1.3 ([3, Proposition 3.5]). *If $\mathbf{A}$ is a hereditary coreflective subcategory of $\mathbf{Top}$ with $\mathbf{FG} \subseteq \mathbf{A}$, then for each $X \in \mathbf{A}$ and each $a \in X$ the prime factor X_a of X at a belongs to $\mathbf{A}$.*

Let A be a prime space with an accumulation point a. A subspace B of A is said to be a *prime subspace* of A if B is a prime space (i.e. $a \in B$ and $\overline{B \setminus \{a\}} \ni a$).

LEMMA 1.4. *Let $(A_i; i \in I)$ be a family of prime spaces and let $a_i \in A_i$ be an accumulation point of A_i for $i \in I$. A topological space X belongs to $\mathrm{CH}(\{A_i; i \in I\})$ if and only if for every non-closed subset M of X there exists $i \in I$, a prime subspace B of A_i and a continuous map $f \colon B \to X$ such that $f[B \setminus \{a_i\}] \subseteq M$ and $f(a_i) \notin M$.*

Proof. Let $\mathbf{B} \subseteq \mathbf{Top}$ be the class of all topological spaces satisfying the given condition. First we show that $\mathbf{B}$ is a coreflective subcategory of $\mathbf{Top}$. It is evident that $\mathbf{B}$ is closed under the formation of topological sums. Now let $X \in \mathbf{B}$ and $q \colon X \to Y$ be a quotient map. Let M be a non-closed subset of Y. Then $q^{-1}[M]$ is a non-closed subset of X, $X \in \mathbf{B}$, so that there exists $i \in I$, a prime subspace B of A_i and a continuous map $g \colon B \to X$ such that $g[B \setminus \{a_i\}] \subseteq q^{-1}[M]$ and $g(a_i) \notin q^{-1}[M]$. Then for $f = q \circ g \colon B \to X$ we get $f[B \setminus \{a_i\}] \subseteq M$ and $f(a_i) \notin M$. Hence, $Y \in \mathbf{B}$ and $\mathbf{B}$ is a coreflective subcategory of $\mathbf{Top}$.

Since evidently $A_i \in \mathbf{B}$ for each $i \in I$, we have $\mathrm{CH}(\{A_i; i \in I\}) \subseteq \mathbf{B}$. To prove the reverse inclusion we construct a quotient map from a sum of subspaces of A_i to arbitrary space $X \in \mathbf{B}$. (Every subspace of A_i belongs to $\mathrm{CH}(A_i)$ by Lemma 1.2.)

Let $X \in \mathbf{B}$. Let $f_j \colon B_j \to X$, $j \in J$, be the family of all continuous maps such that B_j is a prime subspace of some A_i, $i \in I$. Let $D(X)$ be the discrete space on the set X and $id_X \colon D(X) \to X$ be the identity map. It is easy to check that the map $f \colon D(X) \sqcup (\coprod_{j \in J} B_j) \to X$ given by the maps id_X and $f_j, j \in J$, is a quotient map. □

Cardinals are initial ordinals where each ordinal is the (well-ordered) set of its predecessors. We denote the class of all ordinals by ON. If α is a cardinal, then by α^+ we denote the cardinal which is a successor of α. A net in a topological space defined on an ordinal α we call an *α-sequence.*

From now on we assume that A is a prime space with an accumulation point a which is not finitely generated and the tightness of the space A is $t(A) = \alpha$.

2. Closure operator describing CH(A)

The notion of sequential closure was used in [5] when studying sequential and subsequential spaces. Now we introduce a corresponding closure operator for the subcategory CH(A).

Let X be an arbitrary space and $M \subseteq X$. The set $M_1 = \{x \in X$: there exists a prime subspace B of A and a continuous map $f\colon B \to X$ such that $f[B \setminus \{a\}] \subseteq M$ and $f(a) = x\}$ is called the *A-closure of M*. Using transfinite induction we can define the set M_β (the β-th A-closure of M) for each ordinal β as follows. $M_0 = M$, $M_{\beta+1} = (M_\beta)_1$ for each ordinal β and $M_\gamma = \bigcup_{\beta<\gamma} M_\beta$ for each limit ordinal $\gamma > 0$. Put $\widetilde{M} = \bigcup_{\beta\in \mathrm{ON}} M_\beta$.

Evidently $(\widetilde{M})_1 = \widetilde{M}$, $\widetilde{M} \subseteq \overline{M}$. It is also clear that $M_\beta \subseteq M_\gamma$ holds for $\beta < \gamma$. If $A \subseteq B \subseteq X$, then $A_\beta \subseteq B_\beta$ for each ordinal β and $\widetilde{A} \subseteq \widetilde{B}$. If $M_\beta = M_{\beta+1}$ for some ordinal β, then $\widetilde{M} = M_\beta$.

The following proposition characterizes the spaces belonging to CH(A) using the closure operator $M \mapsto \widetilde{M}$. It is a special case of [8, Theorem 3.1.7] which includes more general cases of closure operators.

PROPOSITION 2.1. *A topological space X belongs to* CH(A) *if and only if $\overline{M} = \widetilde{M}$ for every subset $M \subseteq X$.*

Proof. Let $X \in$ CH(A) and $M \subseteq X$. Then $(\widetilde{M})_1 \setminus \widetilde{M} = \emptyset$, so that by Lemma 1.4 $\widetilde{M}$ is closed and $\widetilde{M} = \overline{M}$.

Conversely, if $\overline{M} = \widetilde{M}$ for each $M \subseteq X$ and M is non-closed, then $M_1 \setminus M \neq \emptyset$ and there exists a prime subspace B of A and a continuous map $f\colon B \to X$ such that $f[B \setminus \{a\}] \subseteq M$ and $f(a) \notin M$. Hence, according to Lemma 1.4, we conclude that $X \in$ CH(A). □

PROPOSITION 2.2. *Let A be a prime space with an accumulation point a, $X \in$ CH(A) and $\alpha = t(A)$. Then for every subset $M \subseteq X$ it holds $M_{\alpha^+} = \overline{M}$.*

Proof. If suffices to prove that $(M_{\alpha^+})_1 = M_{\alpha^+}$. Let $c \in (M_{\alpha^+})_1$. Then there exists a prime subspace B of A and a continuous map $f\colon B \to X$ with $f(a) = c$ and $f[B \setminus \{a\}] \subset M_{\alpha^+}$. Since $t(A) = \alpha$ and $a \in \overline{B \setminus \{a\}}$, there exists $C \subseteq B \setminus \{a\}$ with $\operatorname{card} C \leq \alpha$ such that $a \in \overline{C}$. The subspace $B_1 = C \cup \{a\}$ of A is a prime subspace, $f|_{B_1}\colon B_1 \to X$ is continuous and $f|_{B_1}[C] \subseteq M_{\alpha^+}$.

For each $x \in C$ choose $\beta_x < \alpha^+$ such that $x \in M_{\beta_x}$ (α^+ is a limit ordinal). Since $\operatorname{card} C \leq \alpha < \alpha^+$ and α^+ is a regular cardinal we obtain that $\gamma = \sup\{\beta_x, x \in C\} < \alpha^+$. Then $C \subseteq M_\gamma$ and, obviously, $f|_{B_1}(a) = f(a) = c \in M_{\gamma+1} \subseteq M_{\alpha^+}$. Thus, $(M_{\alpha^+})_1 \subseteq M_{\alpha^+}$. □

3. A-sum

The notion of A-sum is a special case of the brush defined in [8] and a generalization of the sequential sum introduced in [2]. The sequential sum was used in [5] for constructing the set of "canonical" prime spaces which generates the category of subsequential spaces. The notion of the A-sum will be used in a similar way to produce the set of special prime spaces that generates SCH(A).

Definition 3.1. Let A be a prime space with an accumulation point $a \in A$. Let us denote $B := A \setminus \{a\}$. Let for each $b \in B$ X_b be a topological space and $x_b \in X_b$. Then the *A-sum* $\sum\limits_A \langle X_b, x_b \rangle$ is the topological space on the set $F = A \cup (\bigcup\limits_{b \in B} \{b\} \times (X_b \setminus \{x_b\}))$ such that the map $\varphi\colon A \sqcup (\coprod\limits_{b \in B} X_b) \to F$ given by $\varphi(x) = x$ for $x \in A$, $\varphi(x) = (b, x)$ for $x \in X_b \setminus \{x_b\}$ and $\varphi(x_b) = b$ for every $b \in B$ is a quotient map. (We assume A and all $\{b\} \times X_b$ to be disjoint.) The map φ will be called the *defining map* of the A-sum.

Often it will be clear from the context what we mean under A and we will abbreviate the notation of the A-sum to $\sum \langle X_b, x_b \rangle$ or $\sum X_b$. The A-sum is obtained simply by identifying every $x_b \in X_b$ with the point $b \in A$. It is easy to see that the subspace $\varphi[X_b]$ is homeomorphic to X_b and A is also a subspace of the A-sum $\sum \langle X_b, x_b \rangle$.

The A-sum is defined using topological sum and quotient map, thus if $\mathbf{A}$ is a coreflective subcategory of **Top** and $\mathbf{A}$ contains A and all X_b's, then the A-sum $\sum X_b$ belongs to $\mathbf{A}$.

The following lemma follows easily from the definition of the A-sum.

LEMMA 3.2. *A subset $U \subseteq \sum\limits_A \langle X_b, x_b \rangle$ is open (closed) if and only if $U \cap A$ is open (closed) in A and $U \cap \varphi[X_b]$ is open (closed) in $\varphi[X_b]$ for every $b \in B$.*

Let for every $b \in B$ X_b and Y_b be topological spaces, $x_b \in X_b$, $y_b \in Y_b$ and let $f_b\colon X_b \to Y_b$ be a function with $f(x_b) = y_b$. Then we can define a map $f =: \sum f_b \colon \sum\limits_A \langle X_b, x_b \rangle \to \sum\limits_A \langle Y_b, y_b \rangle$ by $y_b = f(x_b)$, $f(b, x) = (b, f_b(x))$ for $x \in X_b \setminus \{x_b\}$ and $f(x) = x$ for $x \in A$. Let us note that $f \circ \varphi_1|_{X_b} = \varphi_2|_{Y_b} \circ f_b$ where φ_1 and φ_2 are the defining maps of the A-sums $\sum X_b$ and $\sum Y_b$ respectively.

We will need the following simple lemma:

LEMMA 3.3. *Let $f\colon X \to Y$ be a quotient map, $A \subseteq Y$ and let f be one-to-one outside A. Then $f|_{f^{-1}[A]}\colon f^{-1}[A] \to A$ is a quotient map.*

LEMMA 3.4. *Let A be a prime space with an accumulation point a and $B = A \setminus \{a\}$. Let for every $b \in B$ $f_b\colon X_b \to Y_b$ be a map between topological spaces, $x_b \in X_b$, $y_b \in Y_b$ and $f(x_b) = y_b$.*

(i) If all f_b's are continuous, then $\sum f_b$ is continuous.

(ii) If all f_b's are quotient maps, then $\sum f_b$ is a quotient map.

(iii) If all f_b's are embeddings, then $\sum f_b$ is an embedding.

(iv) If all f_b's are homeomorphisms, then $\sum f_b$ is a homeomorphism.

(v) Let C be a prime subspace of A. Then $\sum\limits_{C}\langle X_b, x_b\rangle$ is a subspace of the space $\sum\limits_{A}\langle X_b, x_b\rangle$.

Proof. Put $f = \sum f_b$ and let φ_1, φ_2 be the defining maps of the A-sums $\sum\langle X_b, x_b\rangle$, $\sum\langle Y_b, y_b\rangle$ respectively. Let us denote $id_A \sqcup (\coprod_{b\in B} f_b)$ by h. In this situation the following diagram commutes.

$$\begin{tikzcd} A \sqcup (\coprod X_b) \arrow[r, "h"] \arrow[d, "\varphi_1"'] & A \sqcup (\coprod Y_b) \arrow[d, "\varphi_2"'] \\ \sum\langle X_b, x_b\rangle \arrow[r, "f"'] & \sum\langle Y_b, y_b\rangle \end{tikzcd}$$

The validity of (i) and (ii) follows easily from the fact that φ_1 and φ_2 are quotient maps.

(iii) Now, suppose that all f_b's are embeddings. W.l.o.g. we can assume that $X_b \subseteq Y_b$ and f_b is the inclusion of X_b into Y_b for every $b \in B$. Let X' be the subspace of the space $\sum Y_b$ on the set $\sum X_b$. We have the following situation:

$$\begin{tikzcd} A \sqcup (\coprod X_b) \arrow[r, hook, "h"] \arrow[d, "\varphi_1"'] & A \sqcup (\coprod Y_b) \arrow[d, "\varphi_2"'] \\ X' \arrow[r, hook, "f"'] & \sum Y_b \end{tikzcd}$$

We only need to prove that X' has the quotient topology with respect to φ_1, because this implies that $X' = \sum X_b$ and f is an embedding of $X' = \sum X_b$ to $\sum Y_b$. But φ_2 is one-to-one outside the set $A \sqcup (\coprod X_b)$ and Lemma 3.3 implies that φ_1 is a quotient map.

(iv) It is an easy consequence of (ii) and (iii). (v) It follows easily from the definition of the A-sum. □

COROLLARY 3.5. *Let A be a prime space with an accumulation point a and let C be a prime subspace of A. Let for every $b \in A \setminus \{a\}$ X_b be a topological space and $x_b \in X_b$. Let for every $b \in C$ Y_b be a subspace of X_b such that $x_b \in Y_b$. Then $\sum\limits_{C}\langle Y_b, x_b\rangle$ is a subspace of the space $\sum\limits_{A}\langle X_b, x_b\rangle$.*

Let us note, that if for every $b \in A \setminus \{a\}$ f_b is an embedding which maps isolated points of X_b to isolated points of Y_b, then the embedding $\sum f_b$ has the same property.

4. The sets TS_γ, TSS_γ

In this section we construct the set of special prime spaces that generates SCH(A) (where A is a prime space which is not finitely generated and $t(A) =$

α). We start with defining the set TS_γ of topological spaces for each ordinal $\gamma < \alpha^+$.

Let $TS_0 = \emptyset$ and TS_1 be the set of all prime subspaces of A.

If $\beta \geq 1$ is an ordinal, then $TS_{\beta+1}$ consists of all B-sums $\sum\limits_B \langle X_b, x_b\rangle$ where B is a prime subspace of A, each $X_b \in TS_\beta$ and $x_b = a$.

If $\gamma > 0$ is a limit ordinal, then $TS_\gamma = \bigcup_{\beta<\gamma} TS_\beta$.

Sometimes, if we want to emphasize which prime space A is used to construct this set, we use the notation $TS_\gamma(A)$.

Every space belonging to TS_γ contains B as a subspace and therefore it contains a. All spaces from TS_γ are constructed from A using B-sums, where $B \in \mathrm{CH}(A)$, thus $TS_\gamma \subseteq \mathrm{CH}(A)$ for each γ.

The following lemma is a generalization of [5, Lemma 6.2].

LEMMA 4.1. *Let X be a topological space and $M \subseteq X$. If $p \in M_\beta \setminus M_\gamma$ for any $\gamma < \beta$, then there exists a space $S \in TS_\beta$ and a continuous map $f\colon S \to X$, which maps all isolated points of S into M and maps only the point a to p.*

Proof. For $\beta = 1$ the claim follows from the definition of M_1.

From the definition of M_β it follows that β is a non-limit ordinal. According to Proposition 2.2 $\beta < \alpha^+$. Suppose the assertion is true for any subset K of X and for any $\beta' < \beta$.

For a non-limit $\beta > 1$ there exists a prime subspace B of A and a continuous map $f\colon B \to X$ such that $f(a) = p$ and $f[B \setminus \{a\}] \subseteq M_{\beta-1}$.

If $\beta - 1$ is non-limit, we can moreover assume that $f[B \setminus \{a\}] \subseteq M_{\beta-1} \setminus M_{\beta-2}$. (If necessary, we choose $B' = \{b \in B : f(b) \in M_{\beta-1} \setminus M_{\beta-2}\}$ and $f' = f|_{B'}$. B' is a prime subspace of A, otherwise we get $x \in M_{\beta-1}$.)

If $\beta - 1$ is a limit ordinal, then for each point $x \in M_{\beta-1}$ there exists the smallest ordinal $\gamma < \beta - 1$ such that $x \in M_\gamma$. Obviously, γ is a non-limit ordinal.

Thus for each $x \in f[B \setminus \{a\}]$ there exists a continuous map $f_x\colon S_x \to X$, where $S_x \in TS_{\beta-1}$, which sends all isolated points of S_x into M and a to x.

Then $\sum\limits_B \langle S_{f(b)}, a\rangle \in TS_\beta$ and we can define a map $g\colon \sum\limits_B \langle S_{f(b)}, a\rangle \to X$ such that $g|_B = f$ and $g|_{\{x\}\times(S_x\setminus\{a\})}(x, y) = f_x(y)$ for $y \in S_x \setminus \{a\}$. Clearly, g maps isolated points into M. It remains only to show that g is continuous.

The defining map $\varphi\colon B \sqcup (\coprod_{b\in B\setminus\{a\}} S_{f(b)}) \to \sum \langle S_{f(b)}, a\rangle$ is a quotient map. Thus, $g\colon \sum \langle S_{f(b)}, a\rangle \to X$ is continuous if and only if $g \circ \varphi$ is continuous. But $g \circ \varphi|_B = f$ and $g \circ \varphi|_{S_x} = f_x$ are continuous, thus g is continuous. $\square$

For any $S \in TS_\gamma$ we denote by $P(S)$ the subspace of the space S which consists of all isolated points of S and of the point a. Clearly, $P(S)$ is a prime space. We denote by TSS_γ the set of all spaces $P(S)$ where $S \in TS_\gamma$. The above lemma implies:

LEMMA 4.2. *If $p \in M_\beta$ and $p \notin M_\gamma$ for any $\gamma < \beta$, then there exists a space $T \in TSS_\beta$ and a continuous map $f\colon T \to X$, which maps all isolated points of the space T into M and such that $f(a) = p$.*

PROPOSITION 4.3. SCH(A) *is generated by the set* $\bigcup\limits_{\gamma<\alpha^+} TSS_\gamma$.

Proof. Let $X \in$ SCH(A). According to Lemma 1.4 it suffices to prove that for any subset $M \subseteq X$ and any $x \in \overline{M} \setminus M$ there exists $T \in \bigcup\limits_{\gamma<\alpha^+} TSS_\gamma$ and a continuous map $f\colon T \to X$ such that $f(a) = x$ and $f[T \setminus \{a\}] \subseteq M$.

Since $X \in$ SCH(A) there exists $Y \in$ CH(A) such that X is a subspace of Y. Denote by $\overline{M}^Y$ the closure of M in Y. Then $\overline{M} = \overline{M}^Y \cap X$ and $x \in \overline{M}^Y \setminus M$ in Y. By Proposition 2.2 $\overline{M}^Y = M_{\alpha^+} = \bigcup\limits_{\beta<\alpha^+} M_\beta$. Let β be the smallest ordinal with $x \in M_\beta$. Then $\beta > 0$ and for any $\gamma < \beta$ $x \notin M_\gamma$. By Lemma 4.1 there exists $S \in TS_\beta$ and a continuous map $f\colon S \to Y$ with $f(a) = x$ and $f(c) \in M$ for any isolated point of S. Then $P(S) \in TSS_\gamma$ and $f[P(S)] \subseteq X$. Hence, $f|_{P(S)}\colon P(S) \to X$ is a continuous map satisfying the required conditions. Consequently, $X \in$ CH$(\bigcup\limits_{\gamma<\alpha^+} TSS_\gamma)$. □

Remark 4.4. It can be easily seen that if we define the sets $T'S_\gamma$, $\gamma < \alpha^+$, similarly as the sets TS_γ but we use only the A-sums (and not all B-sums for prime subspaces B of A) and then we put $T'SS_\gamma = \{P(S) : S \in T'S_\gamma\}$ we obtain the set $\bigcup\limits_{\gamma<\alpha^+} T'SS_\gamma$ which also generates SCH(A). This follows from the fact that any space from $\bigcup\limits_{\gamma<\alpha^+} TSS_\gamma$ is a prime subspace of some space from $\bigcup\limits_{\gamma<\alpha^+} T'SS_\gamma$.

Similarly, if we put $T'SS'_\gamma = \{S_a : S \in T'S_\gamma\}$ (S_a is the prime factor of S at a), then the set $\bigcup\limits_{\gamma<\alpha^+} T'SS'_\gamma$ generates SCH(A) because $\bigcup\limits_{\gamma<\alpha^+} T'SS'_\gamma \subseteq$ SCH(A) and for every $S \in \bigcup\limits_{\gamma<\alpha^+} T'S_\gamma$ $P(S)$ is a subspace of S_a.

5. The spaces A_ω and $(A_\omega)_a$

The space A_ω is defined similarly as S_ω in [2] using the A-sum and the space A instead of the sequential sum and the space $C(\omega_0)$. We start with defining the space A_n for each $n \in \mathbb{N}$ putting $A_1 = A$ and $A_{n+1} = \sum\limits_A \langle A_n, a\rangle$. Clearly, A_1 is a subspace of A_2 and if A_{n-1} is a subspace of A_n, then, according to Lemma 3.4, $A_n = \sum\limits_A \langle A_{n-1}, a\rangle$ is a subspace of $A_{n+1} = \sum\limits_A \langle A_n, a\rangle$. Hence, A_n is a subspace of A_{n+1} for each $n \in \mathbb{N}$.

The Figure 1 represents the space A_3 for $A = C(\omega_0)$. (The space $C(\omega_0)$ is defined in Example 5.7.)

The space A_ω is a topological space defined on the set $\bigcup_{n\in\mathbb{N}} A_n$ such that a subset U of $\bigcup_{n\in\mathbb{N}} A_n$ is open in A_ω if and only if $U \cap A_n$ is open in A_n for every $n \in \mathbb{N}$. It is obvious that for every $n \in \mathbb{N}$ the space A_n is a subspace of A_ω and A_ω is a quotient space of the topological sum $\coprod_{n\in\mathbb{N}} A_n$. Consequently,

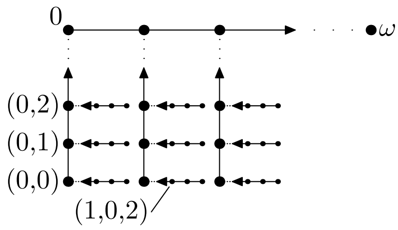

Figure 1. The space A_3 for $A = C(\omega_0)$

A_ω belongs to CH(A). Observe that A_ω can be considered as an inductive limit of its subspaces A_n, $n \in \mathbb{N}$.

Similarly as the space S_ω in [2] the space A_ω has the following important property.

PROPOSITION 5.1. $A_\omega = \sum\limits_A \langle A_\omega, a\rangle$

Proof. Put $X = \sum\limits_A \langle A_\omega, a\rangle$. For each $n \in A$ the space A_n is a subspace of A_ω and it follows that $A_{n+1} = \sum\limits_A \langle A_n, a\rangle$ is a subspace of X (Lemma 3.4). Obviously, $A = A_1$ is also a subspace of X and we obtain that for each $n \in \mathbb{N}$ A_n is a subspace of X. Clearly, $X = \bigcup\limits_{n\in\mathbb{N}} A_n$. To finish the proof it suffices to check that if U is a subset of X and $U \cap A_n$ is open in A_n for each $n \in \mathbb{N}$, then U is open in X.

Let us denote by A_n^b the subspace of X on the set $\{b\} \cup (\{b\} \times (A_n \setminus \{\alpha\}))$ and by A_ω^b the subspace of X on the set $\{b\} \cup (\{b\} \times (A_\omega \setminus \{\alpha\}))$. Clearly, A_n^b is homeomorphic to A_n and A_ω^b is homeomorphic to A_ω, A_n^b is a subspace of A_ω^b and a subset V of A_ω^b is open in A_ω^b if and only if $V \cap A_n^b$ is open in A_n^b for each $n \in \mathbb{N}$.

If $U \subseteq X$ and $U \cap A_n$ is open in A_n for all $n \in \mathbb{N}$, then $U \cap A$ is open in A and $U \cap A_{n+1}$ is open in $A_{n+1} = \sum\limits_A \langle A_n, a\rangle$ for all $n \in \mathbb{N}$. Then $U \cap A_n^b$ is open in A_n^b for each $n \in \mathbb{N}$ and $b \in A \setminus \{a\}$ and it follows that $U \cap A_\omega^b$ is open in A_ω^b for each $b \in A \setminus \{a\}$. Hence, U is open in X. $\square$

The following lemma is evident.

LEMMA 5.2. $\operatorname{card} A_\omega = \operatorname{card} A$

LEMMA 5.3. *For every ordinal γ, $1 \le \gamma < \alpha^+$ and every space $S \in TS_\gamma$ the space S is a subspace of A_ω. (Clearly, the point a of S coincides with the point a of A_ω.)*

Proof. If $\gamma = 1$, then $S = B$ is a prime subspace of $A = A_1$. Let γ be an ordinal, $1 < \gamma < \alpha^+$ and suppose that the assertion holds for every ordinal β, $1 \le \beta < \gamma$. If $S = \sum\limits_B X_b \in TS_\gamma$, then for each $b \in B \setminus \{a\}$ $X_b \in TS_{\beta_b}$ with $1 \le \beta_b < \gamma$. Hence, for each $b \in B \setminus \{a\}$, X_b is a subspace of A_ω and, according to Corollary 3.5, S is a subspace of $A_\omega = \sum\limits_A \langle A_\omega, a\rangle$. $\square$

THEOREM 5.4. *Let $(A_\omega)_a$ be the prime factor of the space A_ω at a. Then $(A_\omega)_a$ is a prime space,* $\mathrm{CH}((A_\omega)_a) = \mathrm{SCH}(A)$ *and* $\mathrm{card}(A_\omega)_a = \mathrm{card}\, A$.

Proof. Evidently, $(A_\omega)_a$ is a prime space and $\mathrm{card}(A_\omega)_a = \mathrm{card}\, A$. Since A_ω belongs to $\mathrm{CH}(A) \subseteq \mathrm{SCH}(A)$, according to Proposition 1.3 $(A_\omega)_a$ belongs to $\mathrm{SCH}(A)$. Hence, it suffices to check that $\bigcup\limits_{\gamma<\alpha^+} TSS_\gamma \subseteq \mathrm{CH}((A_\omega)_a)$.

Let $T \in \bigcup\limits_{\gamma<\alpha^+} TSS_\gamma$. Then there exists an ordinal γ, $1 \le \gamma < \alpha^+$, and $S \in TS_\gamma$ such that $T = P(S)$. By Lemma 5.3 S is a subspace of A_ω and, clearly, it follows that $T = P(S)$ is a subspace of $(A_\omega)_a$. Consequently, there exists a quotient map $(A_\omega)_a \to T$ and we obtain that T belongs to $\mathrm{CH}((A_\omega)_a)$. □

Finally, let X be an arbitrary topological space which is not finitely generated and $\{X_c, c \in Y\}$ be the set of all prime factors of X that are not discrete spaces. Denote by A_X the quotient space of the topological sum $\coprod\limits_{c\in Y} (\{c\} \times X_c)$ obtained by collapsing all points of the subset $\{(c,c), c \in Y\}$ of the space $\coprod\limits_{c\in Y} (\{c\} \times X_c)$ into one point a.

The space A_X is a prime space which is not finitely generated, a is the accumulation point of A_X, $\mathrm{card}\, A_X = \mathrm{card}\, X$ and the following statement holds:

THEOREM 5.5. $\mathrm{SCH}(X) = \mathrm{CH}(((A_X)_\omega)_a)$, *$((A_X)_\omega)_a$ is a prime space and* $\mathrm{card}((A_X)_\omega)_a = \mathrm{card}\, X$.

Proof. Evidently, $A_X \in \mathrm{SCH}(X)$, $X \in \mathrm{CH}(A_X)$ and therefore $\mathrm{SCH}(X) = \mathrm{SCH}(A_X)$. According to Theorem 5.4 $\mathrm{SCH}(A_X) = \mathrm{CH}(((A_X)_\omega)_a)$, $((A_X)_\omega)_a$ is a prime space and $\mathrm{card}((A_X)_\omega)_a = \mathrm{card}\, A_X = \mathrm{card}\, X$. □

Recall, that a topological space X belongs to $\mathbf{Top}(\omega_1)$ if and only if every countable intersection of open subsets of X is open in X and $\mathbf{Top}(\omega_1)$ is a hereditary coreflective subcategory of **Top**. If the space A belongs to $\mathbf{Top}(\omega_1)$, then $\mathrm{SCH}(A) \subseteq \mathbf{Top}(\omega_1)$ and we can find smaller (and simpler) set of generators of $\mathrm{SCH}(A)$ than the set $\bigcup_{\gamma<\alpha^+} TSS_\gamma(A)$ constructed in Proposition 4.3.

PROPOSITION 5.6. *If $A \in \mathbf{Top}(\omega_1)$, then* $\mathrm{SCH}(A) = \mathrm{CH}(\{P(A_n); 0 < n < \omega_0\})$.

Proof. It suffices to show that $(A_\omega)_a \in \mathrm{CH}(\{P(A_n); n < \omega_0\})$. Clearly, each $P(A_n)$ is a subspace of $(A_\omega)_a$. Denote by $i_n \colon P(A_n) \hookrightarrow (A_\omega)_a$ the corresponding embedding and by $f \colon \coprod\limits_{n\in\mathbb{N}} P(A_n) \to (A_\omega)_a$ the continuous map given by the maps i_n, $n \in \mathbb{N}$. It is easy to see that this map is surjective. We claim that f is also a quotient map.

It suffices to show that if $a \in U \subseteq (A_\omega)_a$ and $U \cap P(A_n)$ is open in $P(A_n)$ for each $n < \omega_0$, then U is open in $(A_\omega)_a$. Since $P(A_n)$ is a subspace of A_ω, there exists an open subset W_n of A_ω such that $W_n \cap P(A_n) = U \cap P(A_n)$. Put $W = \bigcap_{n<\omega_0} W_n$. The set W is open in A_ω since A_ω belongs to $\mathbf{Top}(\omega_1)$.

We have $W \cap P(A_n) \subseteq U \cap P(A_n)$ and $\bigcup_{n<\omega_0} P(A_n) = A_\omega$, therefore $W \subseteq U$. Obviously, $a \in W$. Hence, the set U is open in $(A_\omega)_a$. □

Next we present some special cases of our construction.

Example 5.7. Sequential spaces. Recall that subspaces of sequential spaces are called subsequential. The category **Seq** of sequential spaces is the coreflective hull of the space $C(\omega_0)$. The space $C(\omega_0)$ is the topological space on the set $\omega_0 + 1 = \omega_0 \cup \{\omega_0\}$ such that all points of ω_0 are isolated and a set containing ω_0 is open if and only if its complement is finite. (Equivalently, the topology of $C(\omega_0)$ is the order topology given by the usual well-ordering of $\omega_0 + 1$.) The space $C(\omega_0)_\omega$ is homeomorphic to S_ω defined in [2]. Our results imply that the prime factor of the space $C(\omega_0)_\omega$ at ω_0 is a generator of the category of subsequential spaces. Another countable generator of this category was constructed before in [5].

Example 5.8. The coreflective hull of the space $C(\alpha)$. Let α be a regular cardinal and $C(\alpha)$ be the topological space on the set $\alpha + 1 = \alpha \cup \{\alpha\}$ such that all points of α are isolated and a set containing α is open if and only if its complement has cardinality less than α. It is well known that X belongs to $\mathrm{CH}(C(\alpha))$ if and only if a subset $V \subseteq X$ is closed in X whenever for each α-sequence of points from V the set V contains also all limits of this α-sequence. The subcategories $\mathrm{SCH}(C(\alpha))$ are minimal elements of the collection of all hereditary coreflective subcategories of **Top** above **FG**. We use the subcategories $\mathrm{SCH}(C(\alpha))$ in the next section. Our construction yields the generator $(C(\alpha)_\omega)_\alpha$ of $\mathrm{SCH}(C(\alpha))$ which has cardinality α.

6. Subcategories of Top having FG as their hereditary coreflective kernel

Recall that a *hereditary coreflective kernel* of a subcategory **A** of **Top** is the largest hereditary coreflective subcategory of **Top** contained in **A**. We denote it by HCK(**A**). In this section we prove that if **A** and **B** are coreflective subcategories of **Top** such that HCK(**A**) = HCK(**B**) = **FG**, then also HCK(CH(**A** ∪ **B**)) = **FG**. The analogous result does not hold for infinite countable joins of coreflective subcategories. This problem is closely related to the subcategories $\mathrm{SCH}(C(\alpha))$ because (see [3, Theorem 4.8]) **FG** is the hereditary coreflective kernel of a coreflective subcategory **A** of **Top** if and only if **FG** ⊆ **A** and for any regular cardinal α the category $\mathrm{SCH}(C(\alpha))$ is not contained in **A**.

In [7, Problem 7] H. Herrlich and M. Hušek suggest to study coreflective subcategories of **Top** such that their hereditary coreflective hull is the whole category **Top** (i.e. S**A** = **Top**) and their hereditary coreflective kernel is the subcategory **FG**. In the paper [9] it is shown that there exists the smallest such subcategory of **Top** and the collection of all such subcategories of **Top** is closed under the formation of arbitrary non-empty intersections. In this section we prove that this collection is also closed under the formation of non-empty finite joins without being closed under the formation of infinite countable joins in the lattice of all coreflective subcategories of **Top**.

Throughout this section we will apply the results obtained in preceding sections to prime spaces $C(\alpha)$, α being a regular cardinal, defined in Example 5.8. Note that α is an accumulation point of $C(\alpha)$ and $t(C(\alpha)) = \alpha$ for any regular cardinal α. Since any prime subspace of $C(\alpha)$ is homeomorphic to $C(\alpha)$ it suffices to use only $C(\alpha)$-sums in the definition of TS_γ. For instance, if n is a natural number, then TS_n as well as TSS_n contain precisely one space.

In order to prove the main result of this section, we first prove that if $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{A} \cup \mathbf{B})$ for some coreflective subcategories $\mathbf{A}$, $\mathbf{B}$ of **Top**, then one of these subcategories contains $\mathrm{SCH}(C(\alpha))$. We show it separately for the case $\alpha = \omega_0$ and $\alpha \geq \omega_1$.

We start with the case $\alpha = \omega_0$ where we can use some results presented in the paper [5]. As the sets TSS_γ, $\gamma < \omega_1$, introduced in [5] do not coincide with the sets $TSS_\gamma(C(\omega_0))$ defined in Section 4 we denote the sets used in [5] by $TSS'_\gamma(C(\omega_0))$.

The next lemma follows from [5, Theorem 7.1], resp. [5, Corollary 7.2].

LEMMA 6.1. *The category* $\mathbf{SSeq} = \mathrm{SCH}(C(\omega_0))$ *of subsequential spaces is the coreflective hull of the set* $\bigcup_{\gamma<\omega_1} TSS'_\gamma(C(\omega_0))$.

As a consequence of [5, Theorem 7.1] and [5, Theorem 6.4] we obtain:

LEMMA 6.2. *If* $\beta < \gamma < \omega_1$, *then* $TSS'_\beta(C(\omega_0)) \subseteq \mathrm{CH}(TSS'_\gamma(C(\omega_0)))$.

The following result concludes the part of this section concerning the subcategory $\mathrm{SCH}(C(\omega_0))$.

PROPOSITION 6.3. *If* $\mathrm{SCH}(C(\omega_0)) \subseteq \mathrm{CH}(\bigcup_{i\in I} \mathbf{A}_i)$, $\mathbf{A}_i$ *is a coreflective subcategory of* **Top** *for every* $i \in I$ *and* $\mathrm{card}\, I \leq \omega_0$, *then there exists* $i_0 \in I$ *such that* $\mathrm{SCH}(C(\omega_0)) \subseteq \mathbf{A}_{i_0}$.

Proof. Put $\beta_i = \sup\{\beta : TSS'_\beta(C(\omega_0)) \subseteq \mathbf{A}_i\}$ for $i \in I$. Since $\sup \beta_i = \omega_1$ (Lemma 6.1) and ω_1 is a regular cardinal, there exists $i_0 \in I$ such that $\beta_{i_0} = \omega_1$. By Lemma 6.1 and Lemma 6.2 we get that the coreflective subcategory $\mathbf{A}_{i_0}$ contains the subcategory $\mathbf{SSeq} = \mathrm{SCH}(C(\omega_0))$. □

Next we want to prove a result analogous to Proposition 6.3 for the space $C(\alpha)$, where $\alpha \geq \omega_1$ is a regular cardinal. In the case $\alpha \geq \omega_1$ the desired result holds only for non-empty finite joins of coreflective subcategories of **Top**.

Recall that $C(\alpha)_1 = C(\alpha)$ and $C(\alpha)_{n+1} = \sum_{C(\alpha)} \langle C(\alpha)_n, \alpha\rangle$. According to Corollary 3.5 we obtain that $P(C(\alpha)_{n+1}) = P(\sum_{C(\alpha)} \langle P(C(\alpha)_n), \alpha\rangle)$ and it is easy to see that $\alpha^{n+1} \cup \{\alpha\}$ is the underlying set of the space $P(C(\alpha)_{n+1})$

and the subspace of $\sum P(C(\alpha)_n)$ on the set $\{\eta\}\cup(\{\eta\}\times\alpha^n)$ is homeomorphic to $P(C(\alpha)_n)$ for each $\eta<\alpha$. To simplify the notation we will write $C(\alpha)_n^-$ instead of $P(C(\alpha)_n)$.

The following result is a special case of Proposition 5.6.

PROPOSITION 6.4. *If $\alpha\geq\omega_1$ is a regular cardinal, then* $\mathrm{SCH}(C(\alpha)) = \mathrm{CH}(\{C(\alpha)_n^-; 0<n<\omega_0\})$.

LEMMA 6.5. *Let $\alpha\geq\omega_1$ be a regular cardinal. If M is a subset of $C(\alpha)_n$ such that $\alpha\in\overline{M}$ and M contains only isolated points of $C(\alpha)_n$, then there exists a subset $M'\subseteq M$ such that the subspace of the space $C(\alpha)_n$ on the set $\overline{M'}$ is homeomorphic to $C(\alpha)_n$.*

Proof. The case $n=1$ is clear. Let the assertion be true for m. Denote the subspace of $C(\alpha)_{m+1}=\sum\limits_{C(\alpha)} C(\alpha)_m$ on the set $\{\eta\}\cup(\{\eta\}\times(C(\alpha)_m\setminus\{\alpha\}))$, where $\eta<\alpha$, by $C(\alpha)_m^\eta$.

Put $B=\overline{M}\cap C(\alpha)$. Then B is a prime subspace of $C(\alpha)$, for each $\eta\in B\setminus\{\alpha\}$ all points of the set $M_\eta=M\cap C(\alpha)_m^\eta$ are isolated in the space $C(\alpha)_m^\eta$ and $\eta\in\overline{M_\eta}$ in $C(\alpha)_m^\eta$ (observe that $\overline{M_\eta}$ in $C(\alpha)_m^\eta$ coincides with $\overline{M_\eta}$ in $C(\alpha)_{m+1}$ because $C(\alpha)_m^\eta$ is closed in $C(\alpha)_{m+1}$). Since $C(\alpha)_m^\eta$ is homeomorphic to $C(\alpha)_m$ by the induction assumption we obtain that there exists a subset $M'_\eta\subseteq M_\eta$ such that $\eta\in\overline{M'_\eta}$ and the subspace $\overline{M'_\eta}$ of $C(\alpha)_m^\eta$ is homeomorphic to some space $C(\alpha)_m$.

Let $B'=B\setminus\{\alpha\}$ and $M'=\bigcup\limits_{n\in B'} M'_\eta$. Clearly, $M'\subseteq M$, $\overline{M'}=\bigcup\limits_{\eta\in B'}\overline{M'_\eta}\cup\{\alpha\}$ in S and $\overline{M'_\eta}$ is homeomorphic to $C(\alpha)_m$ for each $\eta\in B'$.

The subspace B of $C(\alpha)$ is homeomorphic to $C(\alpha)$ and it is easy to check that $\overline{M'}$ is homeomorphic to $\sum\limits_{C(\alpha)} C(\alpha)_m = C(\alpha)_{m+1}$. □

COROLLARY 6.6. *Let $\alpha\geq\omega_1$ be a regular cardinal, $0<n<\omega_0$. Then every prime subspace T of $C(\alpha)_n^-$ is homeomorphic to $C(\alpha)_n^-$.*

Proof. Put $M=T\setminus\{\alpha\}$. Clearly, $\alpha\in\overline{M}$. According to Lemma 6.5 there exists a subset M' of M such that the subspace $M'\cup\{\alpha\}$ of $C(\alpha)_n^-$ is homeomorphic to $C(\alpha)_n^-$. It follows from the proof of Lemma 6.5 that $M'\setminus M$ is a discrete clopen subspace of $C(\alpha)_n^-$ with cardinality at most α. Hence, $T=M\cup\{\alpha\}$ is homeomorphic to $C(\alpha)_n^-$ as well. □

PROPOSITION 6.7. *Let $\alpha\geq\omega_1$ be a regular cardinal and $0<n<\omega_0$. If $C(\alpha)_n^-\in\mathrm{CH}(\bigcup_{i\in I}\mathbf{A}_i)$, where all $\mathbf{A}_i$'s are coreflective subcategories of* **Top**, *then there exists $i_0\in I$ such that $C(\alpha)_n^-\in\mathbf{A}_{i_0}$.*

Proof. The space $C(\alpha)_n^-$ is a prime space with an accumulation point α. If $C(\alpha)_n^-\in\mathrm{CH}(\bigcup_{i\in I}\mathbf{A}_i)$, then there exists a quotient map $f\colon\coprod\limits_{i\in I}B_i\to C(\alpha)_n^-$, where B_i belongs to $\mathbf{A}_i$ for each $i\in I$. Put $f_i=f|_{B_i}$ and let A_i be the space on the set $f_i[B_i]$ endowed with the quotient topology with respect to f_i for each $i\in I$.

The topology of every space A_i is finer than the topology of the corresponding subspace of $C(\alpha)_n^-$ and it follows that A_i is either discrete or prime space. Clearly, a set $U \subseteq C(\alpha)_n^-$ is open in $C(\alpha)_n^-$ if and only if $U \cap A_i$ is open in A_i for each $i \in I$ and $A_i \in \mathbf{A}_i$. Obviously, there exists $i_0 \in I$ such that α is an accumulation point of A_{i_0} (otherwise α would be isolated in $C(\alpha)_n^-$).

We show that $C(\alpha)_n^- \in \mathrm{CH}(A_{i_0})$. Let M be a non-closed subset of $C(\alpha)_n^-$. It suffices to find a continuous map $f\colon A_{i_0} \to C(\alpha)_n^-$ such that $f[A_{i_0} \setminus \{\alpha\}] \subseteq M$ and $f(\alpha) = \alpha$. According to Corollary 6.6 the subspace on the set $M \cup \{\alpha\}$ is homeomorphic to $C(\alpha)_n^-$. Let us denote the homeomorphism from $C(\alpha)_n^-$ to $M \cup \{\alpha\}$ by g. Moreover, there is a continuous map $i\colon A_{i_0} \to C(\alpha)_n^-$ defined by $i(x) = x$ for each $x \in A_{i_0}$. The desired continuous map is $f = g \circ i$. □

If X and Y are prime spaces, then a continuous map $f\colon X \to Y$ is called a *prime map* if it maps only the accumulation point of X to the accumulation point of Y.

LEMMA 6.8. *Let $\alpha \geq \omega_1$ be a regular cardinal and $0 < m < n < \omega_0$. There exists a quotient prime map $g\colon C(\alpha)_n^- \to C(\alpha)_m^-$.*

Proof. Obviously it suffices to prove the lemma for $n = m + 1$. In this case $C(\alpha)_{m+1}^- = P(\sum C(\alpha)_m^-)$ is a topological space on the set $\{\alpha\} \cup \alpha^{m+1}$ and $C(\alpha)_m^-$ is a topological space on the set $\{\alpha\} \cup \alpha^m$. We define a map $g\colon C(\alpha)_{m+1}^- \to C(\alpha)_m^-$ by $g(\alpha) = \alpha$ and $g((\eta, x)) = x$ for all $(\eta, x) \in C(\alpha)_{m+1}^- \setminus \{\alpha\}$. It is easy to check that the map g is continuous and quotient. □

COROLLARY 6.9. *If $\alpha \geq \omega_1$ is a regular cardinal and $0 < m < n < \omega_0$, then $C(\alpha)_m^- \in \mathrm{CH}(C(\alpha)_n^-)$.*

PROPOSITION 6.10. *If α is a regular cardinal and $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{A} \cup \mathbf{B})$, then $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{A})$ or $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{B})$.*

Proof. Since the case $\alpha = \omega_0$ follows immediately from Proposition 6.3 we can assume that $\alpha \geq \omega_1$.

By Proposition 6.7 for each n, $0 < n < \omega_0$, the space $C(\alpha)_n^-$ belongs either to $\mathbf{A}$ or to $\mathbf{B}$. By Lemma 6.8 we have a quotient map $f\colon C(\alpha)_n^- \to C(\alpha)_m^-$ for each $n > m$. Hence, one of these two coreflective categories contains all spaces $C(\alpha)_n^-$ and, consequently, it contains $\mathrm{SCH}(C(\alpha))$. □

Now we can state the main result of this section.

THEOREM 6.11. *If $\mathbf{A}$, $\mathbf{B}$ are coreflective subcategories of the category* **Top** *and $\mathrm{HCK}(\mathbf{A}) = \mathrm{HCK}(\mathbf{B}) = \mathbf{FG}$, then $\mathrm{HCK}(\mathrm{CH}(\mathbf{A} \cup \mathbf{B})) = \mathbf{FG}$.*

Proof. Suppose the contrary. Then according to [3, Theorem 4.8] there exists a regular cardinal α with $\mathrm{SCH}(C(\alpha)) \subseteq \mathrm{CH}(\mathbf{A} \cup \mathbf{B})$. Proposition 6.10 implies that $\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{A}$ or $\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{B}$, contradicting the assumption that the hereditary coreflective kernel of both these categories is **FG**. □

Let $\mathcal{C}$ be the conglomerate of all coreflective subcategories of **Top**. It is known that $\mathcal{C}$ partially ordered by inclusion is a complete lattice. The above theorem shows that the collection of all coreflective subcategories **A** of **Top** such that $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$ is closed under the formation of non-empty finite joins in the lattice $\mathcal{C}$. We next show that this family is not closed under the formation of infinite countable joins. Namely, for all categories $\mathrm{CH}(C(\alpha)^-_n)$ we have $\mathrm{HCK}(\mathrm{CH}(C(\alpha)^-_n)) = \mathbf{FG}$ and their join is the category $\mathrm{SCH}(C(\alpha))$ which does not have this property. The proof is divided into three auxiliary lemmas.

LEMMA 6.12. *Let $\alpha \geq \omega_1$ be a regular cardinal and $2 \leq n < \omega_0$. If there exists a prime map $f\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$, then there exists a prime map $f'\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$ such that $f'[\{\xi\} \times \alpha^{n-1}] \cap (\bigcup_{\eta<\xi}\{\eta\} \times \alpha^n) = \emptyset$ for each $\xi < \alpha$.*

Proof. Let $f\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$ be a prime map. Denote by B_ξ the subspace of $\sum C(\alpha)^-_{n-1}$ on the set $\{\xi\} \cup (\{\xi\} \times \alpha^{n-1})$ where $\xi < \alpha$. The subspace B_ξ is homeomorphic to $C(\alpha)^-_{n-1}$.

For each $\xi < \alpha$ the set $f^{-1}[\{\alpha\} \cup (\bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n)]$ is open in $C(\alpha)^-_n$, therefore there exists an ordinal $\gamma < \alpha$ such that for each $\gamma' > \gamma$ the set $\{\gamma'\} \cup (f^{-1}[\bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n] \cap B_{\gamma'})$ is open in $B_{\gamma'}$. Hence, we can define an increasing sequence $(\gamma_\xi)_{\xi<\alpha}$ such that $C_\xi := \{\gamma_\xi\} \cup (f^{-1}[\bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n] \cap B_{\gamma_\xi})$ is open in B_{γ_ξ}. Clearly, $f[C_\xi \setminus \{\gamma_\xi\}] \subseteq \bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n$.

According to Corollary 6.6 the subspace of B_{γ_ξ} on the set C_ξ is homeomorphic to $C(\alpha)^-_{n-1}$. Hence, for each $\xi < \alpha$ we can define an embedding $h_\xi\colon C(\alpha)^-_{n-1} \hookrightarrow \sum C(\alpha)^-_{n-1}$ such that $h_\xi[C(\alpha)^-_{n-1}] = C_\xi$. It is easy to see that the map $h\colon \sum C(\alpha)^-_{n-1} \to \sum C(\alpha)^-_{n-1}$ given by $h(\xi) = \gamma_\xi$ for each $\xi < \alpha$, $h(\alpha) = \alpha$ and $h(\xi, x) = h_\xi(x)$ for each $\xi < \alpha$ and $x \in \alpha^{n-1}$ is also an embedding. Put $A_\xi = \{\xi\} \times \alpha^{n-1}$ ($A_\xi \subseteq B_\xi$). Then $h[A_\xi] \subseteq h_\xi[C(\alpha)^-_{n-1}] = C_\xi$ and $f[h[A_\xi]] \subseteq f[C_\xi \setminus \{\gamma_\xi\}] \subseteq \bigcup_{\eta\geq\xi}\{\eta\} \times \alpha^n$. Consequently $f \circ h[A_\xi] \cap (\bigcup_{\eta<\xi}\{\eta\} \times \alpha^n) = \emptyset$ and the prime map $f' = f \circ (h|_{C(\alpha)^-_n})\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$ is a prime map satisfying the required condition. $\square$

LEMMA 6.13. *Let $\alpha \geq \omega_1$ be a regular cardinal and $0 < n < \omega_0$. Then there exists no prime map from $C(\alpha)^-_n$ to $C(\alpha)^-_{n+1}$.*

Proof. First let $n = 1$. For each $\gamma < \alpha$ the set $\{\gamma\} \times \alpha$ is closed in $C(\alpha)^-_2$. Consequently $f^{-1}[\{\gamma\} \times \alpha]$ is closed in $C(\alpha)$, hence it contains less than α points and there exists a set $U_\gamma \subseteq \alpha$ with $\mathrm{card}(\alpha \setminus U_\gamma) < \alpha$ such that $(\{\gamma\} \times U_\gamma) \cap f[C(\alpha)] = \emptyset$. Thus, $W = \{\alpha\} \cup \left(\bigcup_{\gamma<\alpha}\{\gamma\} \times U_\gamma\right)$ is an open neighborhood of α in $C(\alpha)^-_2$ such that $f^{-1}[W] = \{\alpha\}$ and this contradicts the continuity of f.

Let $n > 1$ and the lemma hold for $n - 1$. Suppose that there exists a prime map $f\colon C(\alpha)^-_n \to C(\alpha)^-_{n+1}$. By Lemma 6.12 we can assume w.l.o.g. that $f[\{\xi\} \times \alpha^{n-1}] \cap (\bigcup_{\eta<\xi}\{\eta\} \times \alpha^n) = \emptyset$ for each $\xi < \alpha$.

Recall the definition of the quotient prime map $g\colon C(\alpha)^-_n \to C(\alpha)^-_{n-1}$ from Lemma 6.8. The map g is defined by $g(\alpha) = \alpha$ and $g(\eta, x) = x$ for $\eta < \alpha$, $x \in \alpha^{n-1}$.

Put $A_\xi = \{\xi\} \times \alpha^{n-1}$. Let us denote the subspace of $\sum C(\alpha)^-_{n-1}$ on the set $\{\xi\} \cup A_\xi = \{\xi\} \cup (\{\xi\} \times \alpha^{n-1})$ by B_ξ for each $\xi < \alpha$. Clearly B_ξ is homeomorphic to $C(\alpha)^-_{n-1}$. We define a map $f_\xi\colon B_\xi \to C(\alpha)^-_{n+1}$ by $f_\xi(\xi) = \alpha$ and $f_\xi(\xi, x) = f(\xi, x)$ for each $x \in \alpha^{n-1}$.

The map $g \circ f_\xi\colon B_\xi \to C(\alpha)^-_n$ cannot be continuous, otherwise we get a prime map from a space homeomorphic to $C(\alpha)^-_{n-1}$ to the space $C(\alpha)^-_n$. Therefore there exists an open subset of $C(\alpha)^-_n$ such that inverse image of this set is not open in B_ξ. This set can be written in the form $U_\xi \cup \{\alpha\}$, where $\alpha \notin U_\xi$, and we get that the set

$$f_\xi^{-1}[g^{-1}[U_\xi \cup \{\alpha\}]] = f_\xi^{-1}[\{\alpha\} \cup (\bigcup_{\eta<\alpha} (\{\eta\} \times U_\xi))] = \{\xi\} \cup (B_\xi \cap f^{-1}[\bigcup_{\eta<\alpha} \{\eta\} \times U_\xi])$$

is not open in B_ξ.

Put $V_\xi = \bigcap_{\eta \le \xi} U_\eta$ for $\xi < \alpha$. The family V_ξ is non-increasing and it has the same properties as the family U_ξ. Each $V_\xi \cup \{\alpha\}$ is open in $C(\alpha)^-_n$, because $C(\alpha)^-_n$ belongs to $\mathbf{Top}(\alpha)$ $(\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{Top}(\alpha))$. The set $\{\xi\} \cup (B_\xi \cap f^{-1}[\bigcup_{\eta<\alpha}\{\eta\} \times V_\xi])$ is not open in B_ξ since B_ξ is a prime space with an accumulation point ξ (and $\{\xi\} \cup (B_\xi \cap f^{-1}[\bigcup_{\eta<\alpha}\{\eta\} \times V_\xi]) \subseteq \{\xi\} \cup (B_\xi \cap f^{-1}[\bigcup_{\eta<\alpha}\{\eta\} \times U_\xi]))$.

Finally let us put $W = \bigcup_{\xi<\alpha}\{\xi\} \times V_\xi$. The set $W \cup \{\alpha\}$ is open in $C(\alpha)^-_{n+1}$. We claim that $f^{-1}[\{\alpha\} \cup W]$ is not open in $C(\alpha)^-_n$. It suffices to show that $\{\xi\} \cup (f^{-1}[W] \cap B_\xi)$ is not open in B_ξ for each $\xi < \alpha$.

Clearly $B_\xi = A_\xi \cup \{\xi\}$ and we get $\{\xi\} \cup (f^{-1}[\{\alpha\} \cup W] \cap B_\xi) = \{\xi\} \cup (f^{-1}[\bigcup_{\eta<\alpha}\{\eta\} \times V_\eta] \cap A_\xi)$. We have $f[A_\xi] \cap (\bigcup_{\eta<\xi}\{\eta\} \times \alpha^{n-1}) = \emptyset$, hence $f^{-1}[\bigcup_{\eta<\alpha}\{\eta\} \times V_\eta] \cap A_\xi = f^{-1}[\bigcup_{\eta\ge\xi}\{\eta\} \times V_\eta] \cap A_\xi$ and we obtain

$$\{\xi\} \cup (f^{-1}[\{\alpha\} \cup W] \cap B_\xi) = \{\xi\} \cup (f^{-1}[\bigcup_{\eta\ge\xi}\{\eta\} \times V_\eta] \cap B_\xi) \subseteq$$

$$\subseteq \{\xi\} \cup (f^{-1}[\bigcup_{\eta\ge\xi}\{\eta\} \times V_\xi] \cap B_\xi) \subseteq \{\xi\} \cup (f^{-1}[\bigcup_{\eta<\alpha}\{\eta\} \times V_\xi] \cap B_\xi).$$

The latter set is not open in B_ξ therefore $\{\xi\} \cup (f^{-1}[\{\alpha\} \cup W] \cap B_\xi)$ is not open in B_ξ as well. □

LEMMA 6.14. *Let $\alpha \ge \omega_1$ be a regular cardinal and $0 < n < \omega_0$. Then* $\mathrm{HCK}(\mathrm{CH}(C(\alpha)^-_n)) = \mathbf{FG}$.

Proof. Recall (see [6]) that if $\gamma > \delta$, then $\mathbf{Top}(\gamma) \cap \mathbf{Gen}(\delta) = \mathbf{FG}$. For $\beta < \alpha$ we have $\mathrm{SCH}(C(\beta)) \subseteq \mathbf{Gen}(\beta)$ and $C(\alpha)^-_n \in \mathbf{Top}(\alpha)$, hence $\mathrm{SCH}(C(\beta)) \nsubseteq \mathrm{CH}(C(\alpha)^-_n)$. Similarly if $\beta > \alpha$, then $\mathrm{SCH}(C(\beta)) \subseteq \mathbf{Top}(\beta)$ and $C(\alpha)^-_n \in \mathbf{Gen}(\alpha)$. Thus, $\mathrm{SCH}(C(\beta)) \nsubseteq \mathrm{CH}(C(\alpha)^-_n)$.

By Lemma 6.13 and Lemma 1.4 $C(\alpha)^-_{n+1} \notin \mathrm{CH}(C(\alpha)^-_n)$ (every prime subspace of $C(\alpha)^-_n$ is homeomorphic to $C(\alpha)^-_n$) and $C(\alpha)^-_{n+1} \in \mathrm{SCH}(C(\alpha))$, therefore $\mathrm{SCH}(C(\alpha)) \nsubseteq \mathrm{CH}(C(\alpha)^-_{n+1})$ as well. □

It follows from Proposition 6.4 and Lemma 6.14 that the conglomerate of all coreflective subcategories $\mathbf{A}$ of $\mathbf{Top}$ such that $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$ is not closed under the formation of infinite countable joins in $\mathcal{C}$.

Denote by $\mathcal{L}$ the collection of all coreflective subcategories $\mathbf{A}$ of $\mathbf{Top}$ such that $\mathrm{S}\mathbf{A} = \mathbf{Top}$ and $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$. In the paper [9] it is shown that $\mathcal{L}$ has the smallest element $\mathbf{A}_0 = \mathrm{CH}(\{S^\alpha; \alpha \text{ is a cardinal}\})$, where S is the Sierpiński doubleton, and $\mathcal{L}$ is closed under the formation of arbitrary non-empty intersections. This together with Theorem 6.11 yields:

THEOREM 6.15. *The collection $\mathcal{L}$ is closed under the formation of non-empty intersections, non-empty finite joins in $\mathcal{C}$ and has the smallest element.*

PROPOSITION 6.16. *There is no maximal coreflective subcategory* $\mathbf{A}$ *of* $\mathbf{Top}$ *such that* $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$. *Consequently, the collection $\mathcal{L}$ has no maximal element.*

Proof. Suppose that $\mathbf{A}$ is maximal coreflective subcategory of $\mathbf{Top}$ with the property $\mathrm{HCK}(\mathbf{A}) = \mathbf{FG}$. Let $\alpha \geq \omega_1$ be a regular cardinal. According to Lemma 6.14 and Theorem 6.11 $\mathrm{HCK}(\mathrm{CH}(\mathbf{A} \cup \{C(\alpha)_n^-\})) = \mathbf{FG}$ for each n, $0 < n < \omega_0$. Thus, we get $C(\alpha)_n^- \in \mathbf{A}$ for each $n < \omega_0$ and by Proposition 6.4 $\mathrm{SCH}(C(\alpha)) \subseteq \mathbf{A}$, a contradiction.

The proof that $\mathcal{L}$ has no maximal elements is analogous. □

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References

1. J. Adámek, H. Herrlich, G. Strecker: *Abstract and Concrete Categories*, Willey-Interscience, New York, 1989.
2. A. V. Archangel'skiĭ, S. P. Franklin: Ordinal invariants for topological spaces, *Michigan Math. J.* **15** (1968), 313–320.
3. J. Činčura: Heredity and Coreflective Subcategories of the Category of Topological Spaces, *Applied Categorical Structures* **9** (2001), 131–138.
4. R. Engelking: *General Topology*, PWN, Warsaw, 1977.
5. S. P. Franklin, M. Rajagopalan: On subsequential spaces, *Topology and its Applications* **35** (1990), 1–19.
6. H. Herrlich: *Topologische Reflexionen und Coreflexionen*, Springer Verlag, Berlin, 1968.
7. H. Herrlich, M. Hušek: Some open categorical problems in Top, *Applied Categorical Structures* **1** (1993), 1–19.
8. V. Kannan: Ordinal invariants in topology, *Memoirs of AMS* **245** (1981).
9. M. Sleziak: Subspaces of pseudoradial spaces, to appear in *Mathematica Slovaca.*

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