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Hereditary, additive and divisible classes in epireflective subcategories of Top

Abstract Hereditary coreflective subcategories of an epireflective subcategory **A** of **Top** such that $I_2 \notin \mathbf{A}$ (here I_2 is the 2-point indiscrete space) were studied in [4]. It was shown that a coreflective subcategory **B** of **A** is hereditary (closed under the formation of subspaces) if and only if it is closed under the formation of prime factors. The main problem studied in this paper is the question whether this claim remains true if we study the (more general) subcategories of **A** which are closed under topological sums and quotients in **A** instead of the coreflective subcategories of **A**.

We show that this is true if $\mathbf{A} \subseteq \mathbf{Haus}$ or under some reasonable conditions on **B**. E.g., this holds if **B** contains either a prime space, or a space which is not locally connected, or a totally disconnected space or a non-discrete Hausdorff space.

We touch also other questions related to such subclasses of **A**. We introduce a method extending the results from the case of non-bireflective subcategories (which was studied in [4]) to arbitrary epireflective subcategories of **Top**. We also prove some new facts about the lattice of coreflective subcategories of **Top** and **ZD**.

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1 Introduction

Motivated by [11, Problem 7] J. Činčura studied in [3] hereditary coreflective subcategories of the category **Top** of all topological spaces and continuous maps. He proved a nice characterization of hereditary coreflective subcategories using prime factors of topological spaces. However, it would be interesting to study the hereditary coreflective subcategories also in other categories of topological spaces, as **Haus** or, more generally, any epireflective subcategory of **Top**. In this case the situation becomes more complicated than in **Top**. For instance, in **Top** we obtain the hereditary coreflective hull of a coreflective subcategory **C** simply by taking the subcategory S**C** consisting of all subspaces of spaces from **C**. This is not true in the case of coreflective subcategories of **Haus** anymore, as the example of T_2-subsequential spaces (see [7] or [4]) shows. The hereditary coreflective hull of Hausdorff sequential spaces in **Haus** are precisely the Hausdorff subsequential spaces. But not every Hausdorff subsequential space is a subspace of a Hausdorff sequential space. So the description of the hereditary coreflective hull mentioned above does not work in **Haus**.

Although we see that this new situation leads to some complications, in [4] it is proved that the same characterization holds if we study the same problem in an epireflective subcategory **A** of **Top**, which is not bireflective. Namely, it is shown that a coreflective subcategory of **A** is hereditary if and only if it is closed under the formation of prime factors. This paper is an attempt to study a similar situation and to add a few new results in this area of research.

We study here mainly the subcategories which are additive and divisible (i.e., closed under sums and quotient spaces) in **A**. We call them briefly AD-classes. The AD-classes include coreflective subcategories as a special case. If **A** is a quotient-reflective subcategory of **Top** (in particular if **A** = **Top**), then there is no difference between these two notions. We show that in many cases an AD-class **B** in **A** is hereditary if and only if it is closed under the formation of prime factors. E.g., this holds if **A** $\subseteq$ **Haus** or **B** contains at least one prime space.

We also present a method how to extend our results to bireflective subcategories of **Top**. (Maybe it is more precise to say that the restriction to non-bireflective subcategories is in fact not so restrictive.) For this purpose we use the correspondence between bireflective subcategories of **Top** and epireflective subcategories of **Top** consisting only of T_0-spaces. This correspondence was introduced in [19] (see also [21]).

2 Preliminaries

Topological terminology follows [6] with a few exceptions. We do not assume the T_1 separation axiom for zero-dimensional spaces. A neighborhood of x is any set V such that there exists an open subset U with $x \in U \subseteq V$. (So the neighborhoods in the sense of [6] are open neighborhoods in our terminology.) Compact spaces are not necessarily Hausdorff. For the notions and results from category theory we refer to [1], in particular for reflective

and coreflective subcategories of the category **Top** of topological spaces and continuous maps to [9].

All subcategories are assumed to be full and isomorphism-closed. To avoid some trivial cases we assume that every subcategory of **Top** contains at least one space with at least two points.

By $X \prec Y$ we mean that the spaces X and Y have the same underlying set and X has a finer topology than Y. By *initial map* we mean an initial morphism in the category **Top**. I.e., $f\colon X \to Y$ is said to be initial if X has the initial topology w.r.t. f.

Any ordinal is the set of its predecessors ordered by $\in$. Cardinal numbers are the initial ordinals. The class of all cardinals will be denoted by Cn.

2.1 Epireflective and coreflective subcategories

Perhaps the most important notions from category theory, which we will use in this paper, are those of reflective and coreflective subcategory. We review here some basic facts, more can be found in [1], [9] or [12].

A subcategory **A** of a category **B** is *reflective* if for any **B**-object there exists an **A**-reflection. The **A**-reflection of $X \in \mathbf{B}$ is an object $RX \in \mathbf{A}$ together with a morphism $r\colon X \to RX$ (called the **A**-*reflection arrow*) which has the following universal property: For any morphism $f\colon X \to A$ with $A \in \mathbf{A}$ there exists a unique morphism $\overline{f}\colon RX \to A$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{\ r\ } & RX \\
& \llap{$\scriptstyle f$}\searrow & \big\downarrow\rlap{$\scriptstyle \overline{f}$} \\
& & A
\end{array}
$$

The **A**-reflection is determined uniquely up to homeomorphism.

The functor $R\colon \mathbf{B} \to \mathbf{A}$ which assigns to each **B**-object its **A**-reflection (and acts on morphisms in the natural way) is called a *reflector*. This functor is coadjoint to the embedding functor $\mathbf{A} \hookrightarrow \mathbf{B}$.

We say that **A** is *epireflective* (*bireflective*) in **B** if all **A**-reflection arrows are epimorphisms (bimorphisms) in **B**. If $\mathbf{B} = \mathbf{Top}$ and all **A**-reflections are quotient maps, we speak about a *quotient-reflective* subcategory.

A subcategory **A** of **Top** is epireflective in **Top** if and only if it is closed under the formation of topological products and subspaces.

By EH(**A**) we denote the *epireflective hull* of a subcategory **A**. A topological space X belongs to EH(**A**) if and only if it is a subspace of a product of spaces from **A**. An equivalent condition is that there exist an initial monosource with domain X and codomain in **A**. (See e.g. [19, Theorem 2] or [1, Theorem 16.8].) A similar characterization holds for *bireflective hulls*. A topological space X belongs to the bireflective hull BH(**A**) of **A** if and only if there is an initial source from X to **A** ([15, Corollary 2] or [1, Theorem 16.8]).

By I_2 we will denote the two-point indiscrete space. An epireflective subcategory **A** of **Top** is bireflective if and only if $I_2 \in \mathbf{A}$. Therefore BH(**A**) = EH(**A** $\cup \{I_2\}$).

Mostly we will work in an epireflective subcategory $\mathbf{A}$ of $\mathbf{Top}$ which does not contain I_2. (We will show in Section 5 how to get rid of this assumption.) The same assumption on $\mathbf{A}$ was used in [4]. It is motivated by the fact that only these epireflective subcategories of $\mathbf{Top}$ are closed under the formation of prime factors. Under this assumption $\mathbf{A}$ is closed under topological sums, too. (Recall that we made an agreement that each subcategory contains a space with at least 2 points. Hence, $\mathbf{A}$ contains all discrete spaces whenever $I_2 \notin \mathbf{A}$.)

The largest such subcategory of $\mathbf{Top}$ is the category $\mathbf{Top}_0$ of T_0-spaces. The largest subcategory with these properties such that moreover $\mathbf{A} \subsetneq \mathbf{Top}_0$ is the category $\mathbf{Top}_1$ of T_1-spaces.

The smallest such subcategory of $\mathbf{Top}$ is the subcategory $\mathbf{ZD_0}$ of zero-dimensional T_0-spaces. (Note that for a zero-dimensional spaces the conditions T_0 and T_2 are equivalent.) The subcategory $\mathbf{ZD_0}$ is the epireflective hull of the 2-point discrete space D_2.

An epireflective subcategory of $\mathbf{Top}$ is quotient-reflective if and only if it is closed under the formation of spaces with finer topologies. In a quotient-reflective subcategory of $\mathbf{Top}$ regular (extremal) epimorphisms are exactly the quotient maps.

Let $\mathbf{A}$ be an epireflective subcategory of $\mathbf{Top}$ and $\mathbf{A} \neq \mathbf{Ind}$ (the subcategory of all indiscrete spaces). A subcategory $\mathbf{B} \subseteq \mathbf{A}$ is *coreflective* in $\mathbf{A}$ if and only if it is closed under topological sums and $\mathbf{A}$-extremal quotients. In particular $\mathbf{B}$ is coreflective in $\mathbf{Top}$ if it is closed under sums and quotients. For each subcategory $\mathbf{B}$ of $\mathbf{A}$ there exists the smallest coreflective subcategory of $\mathbf{A}$ containing $\mathbf{B}$. It is called the *coreflective hull* of $\mathbf{B}$ in $\mathbf{A}$ and denoted by $\mathrm{CH}_{\mathbf{A}}(\mathbf{B})$. If $\mathbf{B} = \{B\}$ consists of a single space we use the notation $\mathrm{CH}_{\mathbf{A}}(B)$. If $\mathbf{A}$ is an epireflective subcategory of $\mathbf{Top}$ and $\mathbf{A} \neq \mathbf{Ind}$, then the members of $\mathrm{CH}_{\mathbf{A}}(\mathbf{B})$ are exactly the $\mathbf{A}$-extremal quotients of topological sums of spaces from $\mathbf{B}$. If $\mathbf{A} = \mathbf{Top}$, then the notation $\mathrm{CH}(\mathbf{B})$ (resp. $\mathrm{CH}(B)$) is used and $\mathrm{CH}(\mathbf{B})$ is called the coreflective hull of $\mathbf{B}$. $\mathrm{CH}(\mathbf{B})$ is formed by quotients of topological sums of spaces from $\mathbf{B}$.

The class $\mathbf{FG}$ of all finitely generated spaces is the coreflective hull of all finite spaces in $\mathbf{Top}$. A space is *finitely generated* if and only if any intersection of its open sets is again open. The subcategory $\mathbf{FG}$ is the smallest coreflective subcategory of $\mathbf{Top}$ containing a space, which is not a sum of indiscrete spaces, and it is the coreflective hull of the Sierpiński space S. The *Sierpiński space S* is the two-point space in which only one point is isolated.

A subcategory $\mathbf{B}$ of $\mathbf{Top}$ is said to be *hereditary*, if it is closed under subspaces, and *additive*, if it is closed under topological sums. We say that $\mathbf{B}$ is *divisible in $\mathbf{A}$* if for every quotient map $q\colon X \to Y$ with $X \in \mathbf{B}$ and $Y \in \mathbf{A}$ we have $Y \in \mathbf{B}$.

A class $\mathbf{B}$ which is additive and divisible in $\mathbf{A}$ will be called briefly an *AD-class* in $\mathbf{A}$. If $\mathbf{B}$ is moreover hereditary, we say that it is an *HAD-class* in $\mathbf{A}$.

We define the *AD-hull (HAD-hull)* of $\mathbf{B} \subseteq \mathbf{A}$ as the smallest (hereditary) AD-class in $\mathbf{A}$ containing $\mathbf{B}$. It will be denoted by $\mathrm{AD}_{\mathbf{A}}(\mathbf{B})$, resp. $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})$. It is clear that $\mathrm{AD}_{\mathbf{A}}(\mathbf{B})$ consists precisely of all spaces from $\mathbf{A}$, which are quotient spaces of topological sums of spaces from $\mathbf{B}$.

If $\mathbf{A} = \mathbf{Top}$ or $\mathbf{A}$ is quotient-reflective in $\mathbf{Top}$, then the notion of AD-class (HAD-class) coincides with the notion of (hereditary) coreflective subcategory.

Whenever $\mathbf{C}$ is coreflective in $\mathbf{Top}$, the subcategory $S\mathbf{C}$ consisting of all subspaces of spaces from $\mathbf{C}$ is known to be coreflective as well (see e.g. [16, Remark 2.4.4(5)] or [3, Proposition 3.1]). Clearly the category $S(CH(\mathbf{B}))$ is the *hereditary coreflective hull* of $\mathbf{B}$. It will be denoted also by $HCH(\mathbf{B})$. The hereditary coreflective hull of a single space A in $\mathbf{Top}$ is denoted by $HCH(A)$.

For the future reference we state some obvious relations between AD-hulls in $\mathbf{A}$ and coreflective hulls in $\mathbf{Top}$ in the following lemma.

Lemma 2.1. *Let $\mathbf{A}$ be an epireflective subcategory of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$. Then $AD_\mathbf{A}(\mathbf{B}) = CH(\mathbf{B}) \cap \mathbf{A}$ and $S(AD_\mathbf{A}(\mathbf{B})) \subseteq HAD_\mathbf{A}(\mathbf{B}) \subseteq HCH(\mathbf{B}) \cap \mathbf{A}$.*

2.2 Prime spaces and prime factors

We say that a space P is a *prime space*, if it has precisely one accumulation point a. All prime spaces are T_0. It is easy to see that all prime T_2-spaces are zero-dimensional.

If the point a is not isolated in a subspace P' of a prime space P, i.e., if P' is itself a prime space, we say briefly that P' is a *prime subspace* of P.

Lemma 2.2. *Let P be a prime space with the accumulation point a and P' be a prime subspace of P. Then the map $f \colon P \to P'$, such that $f(x) = x$ if $x \in P'$ and $f(x) = a$ otherwise is a retraction.*

The fact that the discrete spaces form the smallest coreflective subcategory $\mathbf{Disc}$ of $\mathbf{Top}$ together with Lemma 2.2 imply that for any prime space P all its subspaces are contained in $CH(P)$. (They are moreover contained in $CH_\mathbf{A}(P)$ for any epireflective subcategory $\mathbf{A}$ of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$.)

For any space X and any point $a \in X$ we define the *prime factor* of X at a as the topological space on the same set in which all points different from a are isolated and the neighborhoods of a are the same as in the original topology. Clearly, X_a is a discrete or prime space.

Note that a prime space is T_2 if and only if it is T_1. Thus a prime factor of a T_1-space is either a prime T_2-space or a discrete space.

Each topological space X is a quotient of the sum of all its prime factors. The quotient map is obtained simply by mapping a point x in a summand X_a to the same point x of the space X.

3 Heredity and prime factors

Hereditary coreflective subcategories of $\mathbf{Top}$ were studied in [3]. In this paper we are interested in a slightly more general situation - we use an epireflective subcategory $\mathbf{A}$ (with $I_2 \notin \mathbf{A}$) instead of $\mathbf{Top}$. There are two natural generalizations of coreflective subcategories - we can study coreflective subcategories of $\mathbf{A}$ or AD-classes in $\mathbf{A}$. (Both of them were already studied in [4], AD-classes only in two special cases $\mathbf{A} = \mathbf{ZD_0}$ and $\mathbf{A} = \mathbf{Tych}$.)

Clearly, every coreflective subcategory **B** of **A** is an AD-class in **A**. (The opposite implication does not hold. The counterexample is the subcategory of k-spaces in **Tych**. It is the AD-hull of compact spaces in **Tych**, but the coreflective hull of compact spaces in **Tych** is the larger subcategory of k_R-spaces. For more details see Example 3.18 at the end of this section.)

It was shown in [3] that a coreflective subcategory of **Top** different from CH(**Ind**) (the coreflective hull of indiscrete spaces) is hereditary if and only if it is closed under the formation of prime factors. The same result was shown in [4] for coreflective subcategories of **A** (**A** being an epireflective subcategory of **Top** with $I_2 \notin \mathbf{A}$) and for AD-classes in **ZD₀** and **Tych**. We would like to generalize this result for AD-classes in more epireflective subcategories.

The main results of this section are Theorem 3.10 and its consequences. They say that an AD-class is hereditary if and only if it is closed under prime factors whenever this AD-class contains a prime space (or a space with "good" properties). Using this fact we can show in Theorem 3.14 another interesting result: If $\mathbf{A} \subseteq \mathbf{Haus}$, then an AD-class in **A** is hereditary if and only if it is closed under prime factors. So if we work only with Hausdorff spaces, the desired equivalence between heredity and closedness under prime factors is true. In section 6 we try to find some other cases when this statement holds.

3.1 When heredity implies closedness under prime factors?

It is easy to show that for an AD-class closedness under prime factors implies heredity. The proof follows the proof of [4, Theorem 2, Theorem 7].

Lemma 3.1. *Let* **B** *be additive and divisible in* **A***,* **A** *being an epireflective subcategory of* **Top** *with* $I_2 \notin \mathbf{A}$*. If* **B** *is closed under prime factors, then it is hereditary.*

Proof. Let $X \in \mathbf{B}$ and Y be a nonempty subspace of X. We want to show that $Y \in \mathbf{B}$. Let $a \in Y$. The prime factor Y_a is a subspace of the corresponding prime factor X_a of X. Since **B** is closed under prime factors, $X_a \in \mathbf{B}$ and, by Lemma 2.2, $Y_a \in \mathbf{B}$ as well. Since $Y \in \mathbf{A}$ and it is a quotient space of all its prime factors, $Y \in \mathbf{B}$.   □

This paper is mostly devoted to the effort to show that the opposite implication holds too (under some assumptions on **A** or **B**).

We first need to define the space $X \triangle_b Y$ which was used in a similar context in [4].

Definition 3.2. If X and Y are topological spaces, $b \in Y$ and $\{b\}$ is closed in Y, then we denote by $X \triangle_b Y$ the topological space on the set $X \times Y$ which has the final topology w.r.t the family of maps $\{f, g_a; a \in X\}$, where $f \colon X \to X \times Y$, $f(x) = (x, b)$ and $g_a \colon Y \to X \times Y$, $g_a(y) = (a, y)$.

In other words, $X \triangle_b Y$ is the quotient of $X \sqcup (\coprod_{a \in X} Y)$ with respect to the map obtained as the combination of the maps f and g_a, $a \in X$. Since the space $X \triangle_b Y$ is constructed from X and Y using only topological sums

and quotient maps, any coreflective subcategory of **Top** containing X and Y contains $X \triangle_b Y$, too.

A local base for the topology of $X \triangle_b Y$ at a point (a, y), $y \neq b$, is $\{\{a\} \times V; V$ is an open neighborhood of y in $Y\}$. A local base at (a, b) consists of all sets of the form $\bigcup_{x \in U} \{x\} \times V_x$ where U is an open neighborhood of a in X and each V_x is an open neighborhood of b in Y.

Figure 1 depicts the space $X \triangle_b Y$ by showing typical sets from the neighborhood basis.
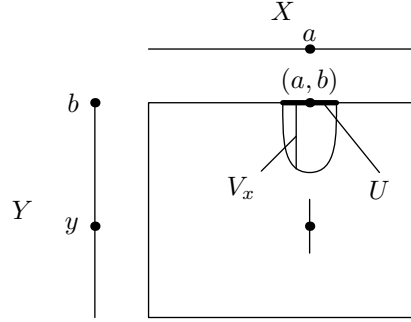
Fig. 1 The space $X \triangle_b Y$

Let $X^a_{(Y,b)}$ be the subspace of $X \triangle_b Y$ on the subset $\{(a, b)\} \cup (X \setminus \{a\}) \times (Y \setminus \{b\})$ and X_a be the prime factor of X at a. It was shown in [4] that, for any space Y in which the subset $\{b\}$ is closed but not open, the map $q \colon X^a_{(Y,b)} \to X_a$ given by $q(x, y) = x$ is quotient. This yields the following proposition:

Proposition 3.3. *Let* **B** *be an HAD-class in an epireflective subcategory* **A** *of* **Top** *with* $I_2 \notin \mathbf{A}$. *Let for any* $X \in \mathbf{B}$ *there exist* $Y \in \mathbf{B}$ *and a non-isolated point* $b \in Y$ *with* $\{b\}$ *being closed in* Y *such that* $X \triangle_b Y$ *belongs to* **A**. *Then* **B** *is closed under the formation of prime factors.*

Proof. Let $X \in \mathbf{B}$ and $a \in X$. We want to show that $X_a \in \mathbf{B}$. By the assumption there exists $Y \in \mathbf{A}$ and $b \in Y$ such that $\{b\}$ is closed but not open and $X \triangle_b Y \in \mathbf{A}$.

Since $X \triangle_b Y$ is constructed using quotients and sums, we get $X \triangle_b Y \in \mathbf{B}$, as well. Therefore also its subspace $X^a_{(Y,b)}$ belongs to **B** and X_a is a quotient of this space. $\square$

Definition 3.4. We say that a subcategory **A** of **Top** is *closed under* $\triangle$ if $X \triangle_b Y \in \mathbf{A}$ whenever $X, Y \in \mathbf{A}$ and $b \in Y$.

Proposition 3.5. *Let* **A** *be an epireflective subcategory of* **Top** *with* $I_2 \notin \mathbf{A}$. *If* **A** *is closed under* $\triangle$ *and* **B** *is hereditary, additive and divisible in* **A**, *then* **B** *is closed under prime factors.*

Proof. It suffices to choose any space $Y \in \mathbf{B}$ and $b \in Y$ such that $\{b\}$ is closed and not open. (We can w.l.o.g. assume that $\mathbf{B}$ contains a non-discrete space, since discrete spaces are closed under prime factors trivially. If $\mathbf{A} = \mathbf{Top}_0$, then the Sierpiński space S belongs to $\mathbf{B}$ and we can take for Y the space S. If $\mathbf{A} \neq \mathbf{Top}_0$, then $\mathbf{A} \subseteq \mathbf{Top}_1$ and in this case it suffices to take any non-discrete space for Y.) By Proposition 3.3 then $X_a \in \mathbf{B}$ whenever $a \in X \in \mathbf{B}$.
□

By [4, Proposition 1] every quotient-reflective subcategory of $\mathbf{Top}$ (in particular $\mathbf{Top}_0$, $\mathbf{Top}_1$, $\mathbf{Haus}$) is closed under the operation $\triangle$. It is also relatively easy to show that the subcategories $\mathbf{Reg}$, $\mathbf{Tych}$, $\mathbf{ZD_0}$ are closed under $\triangle$.

In Example 3.17 we will show that the epireflective subcategories of $\mathbf{Top}$ need not be closed under $\triangle$ in general. Therefore it could be interesting to show the above result under some less restrictive conditions on $\mathbf{A}$.

Proposition 3.3 suggests that it would be useful to have some conditions on spaces X, Y which imply $X \triangle_b Y \in \mathbf{A}$. Such a condition will be obtained in Theorem 3.10. We first introduce the operation $X \bigtriangledown_b Y$, which was defined in [4].

Definition 3.6. Let X, Y be topological spaces and $b \in Y$ with the set $\{b\}$ closed in Y. The space $X \bigtriangledown_b Y$ is the topological space on the set $X \times Y$ which has the initial topology w.r.t. the family $h_a \colon X \times Y \to X \times Y$, $h_a(x, y) = (x, b)$ for $x \neq a$ and $h_a(a, y) = (a, y)$.

A subbase for this topology is formed by the sets $h_a^{-1}(U \times V) = (\{a\} \times V) \cup (U \setminus \{a\}) \times Y$, where $a \in X$, V is an open neighborhood of b in Y and U is a neighborhood of a in X, and by the sets of the form $\{a\} \times V$, where V is an open set in Y not containing b.

The space is illustrated by Figure 2.
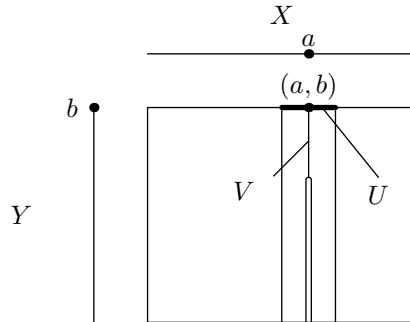
Fig. 2 The space $X \bigtriangledown_b Y$

Observe that $X \triangle_b Y \prec X \bigtriangledown_b Y$.

Since we have the initial monosource $(h_a \colon X \bigtriangledown_b Y \to X \times Y)$, it follows that $X \bigtriangledown_b Y \in \mathrm{EH}(X, Y)$. In other words, every epireflective subcategory of

Top containing X and Y must contain $X \bigtriangledown_b Y$ too. (For the basic properties of $X \bigtriangledown_b Y$ see [4, Proposition 1]).)

Now we introduce a (sufficient) condition on a space Y, under which $X \bigtriangleup_b Y \in \mathbf{A}$ for each $X \in \mathbf{A}$.

Definition 3.7. Let Y, Z be topological spaces and $b \in Y$ be a point such that the set $\{b\}$ is closed in Y. We say that $P(b, Y, Z)$ holds if there exists an open local base $\mathcal{B}$ at b in Y, a point $a \in Z$ and an open neighborhood U_0 of a in Z such that for any $V \in \mathcal{B}$ there exists a continuous map $f \colon Y \to Z$ with $f(b) = a$, $f^{-1}(U_0) = V$.

We first present some simple examples of spaces with this property. Let $I = \langle 0, 1 \rangle$. We claim that $P(b, I, I)$ holds for any $b \in I$. We take $\mathcal{B} = \{\langle 0, \varepsilon \rangle\}$ if $b = 0$, symmetrically $\mathcal{B} = \{(1-\varepsilon, 1\rangle\}$ if $b = 1$, otherwise $\mathcal{B} = \{(b-\varepsilon, b+\varepsilon) \cap I\}$. (For a and U_0 we can take e.g. 0 and $\langle 0, \frac{1}{2})$, 1 and $(\frac{1}{2}, 1\rangle$, b and $(\frac{b}{2}, \frac{1+b}{2})$ respectively.)

Another example: Let Y be any zero-dimensional space, D_2 be the 2-point discrete space. Any point has a clopen local base, therefore $P(b, Y, D_2)$ holds for any $b \in Y$.

Example 3.8. Let X be any infinite space with the cofinite topology and $b \in X$. Then $P(b, X, X)$.

To show this, we take all neighborhoods of b for $\mathcal{B}$. Let $a = b$, $u \neq b$ and $U_0 = X \setminus \{u\}$. Then U_0 is an open neighborhood of a.

Any $V \in \mathcal{B}$ has the form $V = X \setminus F$, where F is a finite set and $b \notin F$. Then there exists a bijection $h \colon V \to U_0$ such that $h(b) = b$. We can define a map $f \colon X \to X$ by $f|_V = h$ and $f[F] = \{u\}$. Clearly, f is a continuous map.

Lemma 3.9. *Let $\mathbf{A}$ be an epireflective subcategory of* **Top**. *Let $X, Y, Z \in \mathbf{A}$, $b \in Y$ be a non-isolated point such that $\{b\}$ is closed and $P(b, Y, Z)$ holds. Then $X \bigtriangleup_b Y \in \mathbf{A}$.*

Proof. Using the base $\mathcal{B}$ from the definition of $P(b, Y, Z)$, we can obtain a local base at $(a, b) \in X \bigtriangleup_b Y$ consisting of sets of the form $\bigcup_{x \in U} V_x$, where U is an open neighborhood of a in X and $V_x \in \mathcal{B}$ for each $x \in U$. Let $\mathcal{S} = \{f_i : i \in I\}$ be the set of all maps $f_i \colon X \to \mathcal{B}$. For any basic neighborhood $\bigcup_{x \in U} V_x$ there is an $i \in I$ with $f_i(x) = V_x$ for every $x \in U$. Let us denote the topology of $X \bigtriangleup_b Y$ by $\mathcal{T}_\bigtriangleup$.

Now we define the following maps: $p \colon X \bigtriangleup_b Y \to X$ is the projection $p(x, y) = x$, $q \colon X \bigtriangleup_b Y \to X \bigtriangledown_b Y$ is the identity map $q(x, y) = (x, y)$, and for $i \in I$ we define a map $h_i \colon X \bigtriangleup_b Y \to Z$ as follows: For the open neighborhood $f_i(x) \in \mathcal{B}$ there exists a continuous map $g_{x,i} \colon Y \to Z$ with $g_{x,i}^{-1}(U_0) = f_i(x)$ and $g_{x,i}(b) = a$. We put $h_i(x, y) = g_{x,i}(y)$. Note that the maps p, q, h_i are continuous.

We claim that $\mathcal{T}_\bigtriangleup$ is the initial topology with respect to the family of maps $\{p, q, h_i; i \in I\}$. Let us denote this initial topology by $\mathcal{T}$. Let us recall that an open subbase for the initial topology is formed by sets $p^{-1}(U)$, $q^{-1}(U)$, $h_i^{-1}(U) = \bigcup_{x \in X} g_{x,i}^{-1}(U)$, where U is any open set in the codomain of the respective map.

$\boxed{\mathcal{T}_\triangle \prec \mathcal{T}}$ We first compare open neighborhoods of points (a, b) with $a \in X$. Any such neighborhood contains a basic neighborhood $\bigcup_{x \in U} V_x$ with $V_x \in \mathcal{B}$ for each $x \in U$. This set can be expressed as $p^{-1}(U) \cap h_i^{-1}(U_0)$ for the $i \in I$ satisfying $f_i(x) = V_x$ for every $x \in U$.

As for neighborhoods of (a, y), $y \neq b$, it suffices to note that they have the same neighborhood bases in $X \triangle_b Y$ and $X \triangledown_b Y$.

$\boxed{\mathcal{T} \prec \mathcal{T}_\triangle}$ It suffices to notice that the subbasic sets $p^{-1}(U)$, $q^{-1}(U)$, $h_i^{-1}(U)$ belong to $\mathcal{T}_\triangle$.

This family of maps forms a monosource, since it contains the identity map. So we have found an initial monosource from $X \triangle_b Y$ with a codomain in $\mathbf{A}$ and this implies $X \triangle_b Y \in \mathbf{A}$. $\square$

Theorem 3.10. *Let $\mathbf{A}$ be an epireflective subcategory of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$ and $\mathbf{B}$ be an HAD-class in $\mathbf{A}$. If $\mathbf{B}$ contains a space Y with $P(b, Y, Z)$ for some $Z \in \mathbf{A}$ and a non-isolated point $b \in Y$ such that the set $\{b\}$ is closed, then $\mathbf{B}$ is closed under prime factors.*

Proof. Follows easily from Proposition 3.3 and Lemma 3.9. $\square$

Corollary 3.11. *Let $\mathbf{A}$ be an epireflective subcategory of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$ and $\mathbf{B}$ be an HAD-class in $\mathbf{A}$. If $\mathbf{B}$ contains an infinite space with the cofinite topology or it contains a non-discrete zero-dimensional space (in particular a prime T_2-space), then $\mathbf{B}$ is closed under prime factors.*

Corollary 3.12. *Let $\mathbf{A}$ be an epireflective subcategory of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$ and $\mathbf{B}$ be an HAD-class in $\mathbf{A}$. If $\mathbf{B}$ contains a prime space, then $\mathbf{B}$ is closed under prime factors.*

Proof. If $\mathbf{A}$ contains a prime T_2-space, then the claim follows from Corollary 3.11.

Now assume that $\mathbf{A}$ contains a non-Hausdorff prime space P. To resolve this case we provide an argument which will be used several more times in this paper.

Since P is not T_2, there exists a point b which cannot be separated from the non-isolated point a of P. The subspace on the set $\{a, b\}$ is homeomorphic to the Sierpiński space S. Thus we get $S \in \mathbf{A}$, $\mathbf{A} = \mathbf{Top}_0$, and the result now follows from Proposition 3.5. (Note that $\mathbf{Top}_0$ is closed under $\triangle$.) $\square$

We have seen that the prime T_2-spaces are more convenient in this context, because they belong to $\mathbf{ZD_0}$ and thus they are automatically contained in any epireflective non-bireflective subcategory. Let us note that in most cases it suffices to consider the prime T_2-spaces only.

Indeed, the only epireflective subcategory $\mathbf{A}$ of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$, which contains non-T_2 prime spaces, is $\mathbf{Top}_0$. Moreover, even for $\mathbf{A} = \mathbf{Top}_0$, the non-$T_2$ prime spaces are needed only for AD-classes with $\mathbf{B} \subseteq \mathbf{FG}$.

Using the Corollary 3.11 we can show that if $\mathbf{A} \subseteq \mathbf{Haus}$ then every HAD-class $\mathbf{B} \neq \mathbf{Disc}$ in $\mathbf{A}$ contains a prime space and, consequently, it is closed under the formation of prime factors.

Proposition 3.13. *Let X be Hausdorff and not discrete. Then X contains a subspace Y, such that there exists a prime T_2-space P which is a quotient space of Y.*

Proof. Let a be any non-isolated point in X. We would like to get a subspace Y in which a is again non-isolated and which contains enough disjoint open subsets.

By transfinite induction we construct a system U_β, $\beta < \alpha$, of non-empty open subsets of X such that for each $\beta, \gamma < \alpha$ the following holds:

(1) If $\beta \neq \gamma$ then $U_\beta \cap U_\gamma = \emptyset$;

(2) $a \in V_\beta = X \setminus \overline{\bigcup_{\eta \leq \beta} U_\eta} = \mathrm{Int}(X \setminus \bigcup_{\eta \leq \beta} U_\eta)$;

(3) if $\gamma < \beta$ then $U_\beta \subseteq V_\gamma$;

(4) $a \in \overline{\bigcup_{\eta < \alpha} U_\eta}$.

$\beta = 0$: Since a is non-isolated, there exists $b \neq a$ in X. By Hausdorffness we have non-empty open sets U, V with $U \cap V = \emptyset$, $a \in V$, $b \in U$. We put $U_0 := U$. Since $a \in V \subseteq X \setminus U$ and V is open, the condition $a \in V_0 = \mathrm{Int}(X \setminus U_0)$ is fulfilled. The conditions (1) and (3) are vacuously true in this step of induction.

Now suppose that U_γ for $\gamma < \beta$ have already been defined. There are two possibilities. Either $a \in \overline{\bigcup_{\eta < \beta} U_\eta}$ and we can stop the process (putting $\alpha := \beta$) or $a \notin \overline{\bigcup_{\eta < \beta} U_\eta}$.

In the latter case the set $W := X \setminus \overline{\bigcup_{\eta < \beta} U_\eta}$ is an open neighborhood of a such that $W \cap (\bigcup_{\eta < \beta} U_\eta) = \emptyset$. Since a is not isolated, there exists $b \in W$, $b \neq a$. Again, by T_2-axiom, there exist open sets U, V such that $U \cap V = \emptyset$, $a \in V$, $b \in U$. We put $U_\beta := U \cap W$.

Since $U_\beta \subseteq W$ and $W \cap (\bigcup_{\eta < \beta} U_\eta) = \emptyset$, we do not violate (1).

The point a belongs to the open set $V \cap W$ and $(V \cap W) \cap U_\gamma = \emptyset$ for every $\gamma \leq \beta$, thus we get $a \in \mathrm{Int}(X \setminus \bigcup_{\eta \leq \beta} U_\eta)$, so (2) is fulfilled as well.

For $\gamma < \beta$ we have $W \subseteq V_\gamma = X \setminus \overline{\bigcup_{\eta \leq \gamma} U_\eta}$, thus $U_\beta \subseteq V_\gamma$ and (3) holds.

The condition (4) does make sense only at the end of induction, when we have finished the process and said, what α is. Note, that this procedure must stop at some ordinal α, otherwise we would obtain a proper class of open subsets of X.

Now we put $Y := \{a\} \cup (\bigcup_{\beta < \alpha} U_\beta)$. The prime space P will be obtained as the space on the set $\alpha \cup \{\alpha\}$ which is quotient with respect to $q \colon Y \to P$ defined by $q(a) = \alpha$ and $q[U_\beta] = \{\beta\}$ for any $\beta < \alpha$. By (1) and (2) the map q is well-defined.

Since $q^{-1}(\{\beta\}) = U_\beta$, each $\beta < \alpha$ is isolated. By (4) and $q^{-1}(\alpha) = \{a\}$, the point α is not isolated. Hence P is a prime space.

Since the set $q^{-1}(\{\gamma \in \alpha \cup \{\alpha\}; \gamma > \beta\}) = V_\beta \cap Y$ is open in Y for each $\beta < \alpha$, the prime space P is T_2 (every isolated point can be separated from the accumulation point). □

From Proposition 3.13 and Corollary 3.11 we get

Theorem 3.14. *If* **A** *is an epireflective subcategory of* **Top** *such that* **A** $\subseteq$ **Haus** *and* **B** *is an HAD-class in* **A***, then* **B** *is closed under prime factors.*

Corollary 3.15. *Let* **A** *be an epireflective subcategory of* **Top** *such that* **A** $\subseteq$ **Haus***. For every HAD-class* **B** *in* **A** *there exists a class* **S** *of prime spaces such that* **B** $= \mathrm{AD}_\mathbf{A}(\mathbf{S})$.

3.2 Two related examples

In the rest of this section we present two examples which are connected with HAD-classes and hereditary coreflective subcategories. In the first one we will deal with closedness of epireflective subcategories under $\triangle$. The second one is an example of an AD-class in **Tych** which is not coreflective in **Tych**.

We have observed in Proposition 3.5 that if **A** is closed under $\triangle$ then every HAD-class in **A** is closed under prime factors. We also noticed that this condition is fulfilled for many familiar epireflective subcategories of **Top**. Now we provide an example showing that it does not hold in general.

We first recall the notion of a strongly rigid space. A topological space X is called *strongly rigid* if any continuous map $f: X \to X$ is either constant or id_X. (See [17], it should be noted that such spaces are called rigid by some authors.) We will show in Example 3.17 that for a strongly rigid space which is not "too trivial" $X \triangle_b X \in \mathrm{EH}(X)$ does not hold.

Lemma 3.16. *Let* X *be a topological space and* $b \in X$*. If* X *is a strongly rigid space and* $X \triangle_b X \in \mathrm{EH}(X)$*, then for* $x \neq b$ *the set* $\{U \times V; U$ *is an open neighborhood of* x *and* V *is an open neighborhood of* $b\}$ *is a local base for the topology of* $X \triangle_b X$ *at* (x, b)*, i.e., this topology has the same local base at the point* (x, b) *as the product topology.*

Proof. Recall, that the underlying set of $X \triangle_b X$ is $X \times X$ (see Definition 3.2). Clearly, $X \triangle_b X \prec X \times X$. So it remains to show that any neighborhood of (x, b) in $X \triangle_b X$ contains a neighborhood of the form $U \times V$ with U and V as above. Since we assume that $X \triangle_b X \in \mathrm{EH}(X)$, the space $X \triangle_b X$ has the initial topology w.r.t. the family $C(X \triangle_b X, X)$; i.e., the subbase for the topology of this space consists of sets $f^{-1}(U)$ where $f \in C(X \triangle_b X, X)$ and U is open subset of X. (By $C(Y, Z)$ we mean the family of all continuous mapping between spaces Y and Z.)

The subspaces of $X \triangle_b X$ on the sets $\{a\} \times X$ for any $a \in X$ and $X \times \{b\}$ are homeomorphic to X. To be more precise, the homeomorphisms are given by $h_a(a, x) = x$ (between the subspace $\{a\} \times X$ and X) and $h(x, b) = x$ (between the subspace $X \times \{b\}$ and X). Thus for any $f \in C(X \triangle_b X, X)$ the restrictions to these subspaces are either constant or coincidental with h_a resp. h. We next investigate in detail all maps f in $C(X \triangle_b X, X)$.

First, assume that $f|_{X \times \{b\}}$ is not constant. Then $f(x, b) = x$ for any $x \in X$. Thus for $a \neq b$ we get $f(a, b) = a \neq b = h_a(a, b)$. Therefore the restriction of f to the subspace $\{a\} \times X$, $a \neq b$, is the constant map $f(a, x) = a$. For the subspace $\{b\} \times X$ we have two possibilities: h_b or a constant map. In this case we obtain two continuous maps: f_1 such that $f_1(x, y) = x$ and f_2 given by $f_2(x, y) = x$ for $x \neq b$ and $f_2(b, y) = y$.

The second possibility remains: $f(x, b) = a_0$ for any $x \in X$. If $a_0 \neq b$ then for any $a \in X$ we have $f(a, b) = a_0 \neq b = h_a(a, b)$ and f is a constant map. Thus the only interesting case is $a_0 = b$. In this case some restrictions are equal to h_a's and some are constant. I.e., every such map corresponds to a subset A of X in the following way: $f_A(x, y) = y$ if $x \in A$ and $f_A(x, y) = b$ otherwise.

We showed that the family $C(X \triangle_b X, X)$ consists precisely of all constant maps, the maps f_1, f_2 and the maps of the form f_A, $A \subseteq X$. The set $U \times V$ can be obtained as $f_X^{-1}(V) \cap f_1^{-1}(U)$. Moreover, every subbasic set $f^{-1}(U)$, where $f \in C(X \triangle_b X, X)$, contains a subset of the form $U \times V$. Thus such sets from a local base. $\square$

Example 3.17. If X is a strongly rigid space and $\bigcap_{x \in X} U_x$ is an intersection of open neighborhoods of a point b in X which fails to be a neighborhood, then the set $\bigcup_{x \in X} \{x\} \times U_x$ is open in $X \triangle_b X$, but it does not contains any subset from the local basis described in Lemma 3.16. Therefore in such case $X \triangle_b X \notin \mathrm{EH}(X)$.

This means that to obtain a counterexample, it suffices to have a strongly rigid space with a non-isolated point b such that at the same time $\{b\}$ is an intersection of a family U_i, $i \in I$, of open sets, with $\operatorname{card} I \leq \operatorname{card} X$. Any of the examples of strongly rigid T_2-spaces constructed in [5], [8] or [17] satisfies this condition.

We next include an example of an AD-class which is not coreflective. We will work in the epireflective subcategory $\mathbf{A} = \mathbf{Tych}$ of all completely regular (Tychonoff) spaces.

A topological space X is called k_R-*space* if it is completely regular and if every map $f \colon X \to \mathbb{R}$, whose restriction to every compact subset $K \subseteq X$ is continuous, is continuous on X. For more information about k_R-spaces see e.g. [13] or [18].

A topological space X is a k-*space* if a subset $U \subseteq X$ is open whenever $U \cap K$ is open for every compact subset $K \subseteq X$. The class of all k-spaces is the coreflective hull of compact spaces in $\mathbf{Top}$. In the next example we will denote the corresponding coreflector by C. It is known that X and CX have precisely the same compact subsets and the relative topology on every compact subset is the same.

Example 3.18. E. Michael constructed in [20, Lemma 3.8] a normal k_R-space X such that CX is not regular. This means that X is not a k-space. We will show that $RCX = X$, where R denotes the $\mathbf{Tych}$-reflection. This implies that X is in the coreflective hull of compact spaces in $\mathbf{Tych}$, since there exists a quotient map q from the sum of all compact subspaces of X to CX and consequently the map Rq with the codomain $RCX = X$ is a $\mathbf{Tych}$-extremal

epimorphism. (Since every reflector is coadjoint functor, it preserves regular epimorphisms.) But X is not in the AD-hull of compact spaces in **Tych**, since $CX \neq X$. Therefore the AD-hull of compact spaces in **Tych** (which consists precisely of Hausdorff k-spaces) is an example of an AD-class in **Tych** which is not a coreflective subcategory of **Tych**.

To show that X is the **Tych**-reflection of CX it suffices to show that the continuous maps from both spaces to $\mathbb{R}$ are the same. A map $f \colon CX \to \mathbb{R}$ is continuous if and only if all restrictions $f|_K$ with K compact are continuous. Since compact subsets of X and CX are the same and moreover the corresponding subspaces are homeomorphic, this implies that f is continuous as the map from X to $\mathbb{R}$. (Since X is a k_R-space and we have shown that the restrictions on compact subsets are continuous.)

4 The space A_ω and HAD-classes

In [23] the space A_ω was constructed for any prime space A and it was shown that the prime factor $(A_\omega)_a$ is a generator of the hereditary coreflective hull of the space A in **Top**. The goal of this section is to show some useful properties of the space A_ω. Namely we will prove that this space is zero-dimensional for any prime T_2-space A. This implies that, if A is a prime T_2-space, then A_ω is contained in every epireflective subcategory **A** with $I_2 \notin$ **A**. Using this property of A_ω we can show that if **B** is an HAD-class in **A** which contains a prime space then the coreflective hull CH(**B**) of **B** in **Top** is hereditary.

We first introduce some notions which are necessary to define the space A_ω. This space is very similar to the space S_ω defined in [2], the difference lies in using an arbitrary prime space A instead of the space $C(\omega)$ (see Definition 6.1) and A-sums instead of sequential sums. Let us note that the space S_ω is also a special case of the space $\mathcal{T}_{\mathcal{F}}$ defined in [24].

We first recall the definition of A-sums from [23]. Apart from the sequential sums of [2] this construction is also similar to the brush of [16].

Definition 4.1. Let A be a prime space with the accumulation point a. Let us denote $B := A \setminus \{a\}$. Suppose that for each $b \in B$ we are given a topological space X_b and a point $x_b \in X_b$. Then the A-sum $\sum_A \langle X_b, x_b \rangle$ is the topological space on the set $F = A \cup (\bigcup_{b \in B} \{b\} \times (X_b \setminus \{x_b\}))$ which is quotient with respect to the map $\varphi \colon A \sqcup (\coprod_{b \in B} X_b) \to F$, $\varphi(x) = x$ for $x \in A$, $\varphi(x) = (b, x)$ for $x \in X_b \setminus \{x_b\}$ and $\varphi(x_b) = b$ for every $b \in B$.

This means that the A-sum is defined simply by identifying each $x_b \in X_b$ with the corresponding point $b \in A$.

For the sake of convenience, we adopt some terminology from [16]. The map φ is called the *defining map* of the A-sum. The subspace of $\sum \langle X_b, x_b \rangle$ on the subset $\varphi[X_b] = \{b\} \cup (\{b\} \times \{X_b \setminus \{x_b\}\})$ is called the *bristle*.

The following easy lemma states that the bristles are homeomorphic to the spaces from which the A-sum is constructed.

Lemma 4.2. *The space X_b is homeomorphic to the subspace of the space $\sum \langle X_b, x_b \rangle$ on the subset $\varphi[X_b] = \{b\} \cup (\{b\} \times \{X_b \setminus \{x_b\}\})$. (The homeomorphism is given by the restriction of φ to the summand X_b.)*

Now we are ready to define the space A_ω using the A-sum. We first define inductively the spaces A_n for $n \in \mathbb{N}$. We put $A_1 = A$ and $A_{n+1} = \sum \langle A_n, a \rangle$. Note that A_n is a subspace of A_{n+1} for each n. Figure 3 depicts the space A_3 for $A = C(\omega)$.
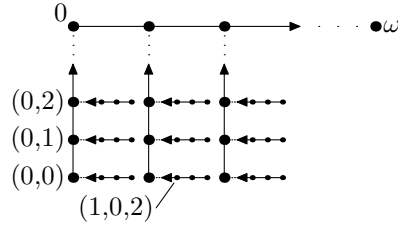
Fig. 3 The space A_3 for $A = C(\omega)$

Another possibility how to obtain A_{n+1} is to attach the space A (by its accumulation point) to each isolated point of A_n. Clearly, the underlying set is the same. It can be shown by induction that the topologies are the same too.

For $n = 1$ this is clear from the definition of A_2. If $n > 1$ then each stem of the space A_{n+1} is homeomorphic to A_n. By the induction hypothesis it can be obtained by gluing the space A to each isolated point of A_{n-1}. The isolated points in $A_n = \sum A_{n-1}$ are precisely the isolated points of the bristles. Hence by attaching the space A to isolated points we get A_n from each bristle and the resulting space will be $A_{n+1} = \sum A_n$.

Definition 4.3. Let A be a prime space with the accumulation point a. Then A_ω is the space on the set $\bigcup_{n \in \mathbb{N}} A_n$ where $U \subseteq \bigcup_{n \in \mathbb{N}} A_n$ is open if and only if $U \cap A_n$ is open for every $n \in \mathbb{N}$.

We see that A_ω is a quotient space of $\coprod_{n \in \mathbb{N}} A_n$, so $A_\omega \in \mathrm{CH}(A)$. (Let us note that A_ω is the inductive limit of spaces A_n and each A_n is embedded into A_ω.)

Observe that the underlying set of the space A_ω is $\{a\} \cup \bigcup_{n \in \mathbb{N}} (A \setminus \{a\})^n$. The space A_n is homeomorphic to the subspace on the subset $\{a\} \cup \bigcup_{k=1}^{n} (A \setminus \{a\})^k$ and the bristles of A_n are the subspaces on sets $\{b\} \cup \{(b, x_2, x_3, \ldots, x_n); x_i \in A \setminus \{a\}\}$.

We introduce some terminology analogous to [2]. We say that a point $x \in A_\omega$ is a point of *k-th level* if it belongs to $(A \setminus \{a\})^k$. Point a is the only point of 0-th level.

It was proved in [23, Proposition 5.1] that A_ω homeomorphic to the A-sum of several copies of itself: $A_\omega \cong \sum_A \langle A_\omega, a \rangle$.

From this property and Lemma 4.2 we can see that the bristles are homeomorphic to A_ω.

Lemma 4.4. *For $b \in A \setminus \{a\}$ denote by X the subspace of A_ω on the set $X :=$ $\{b\} \cup (\{b\} \times \bigcup_{n \in \mathbb{N}} (A \setminus \{a\})^n)$. Then X is homeomorphic to A_ω. Moreover, if A is T_2, then X is a clopen subset of A_ω.*

Using this lemma repeatedly we can find for any point of A_ω a (clopen, if A is T_2) neighborhood homeomorphic to A_ω (by induction on the level of points). Namely, if $x = (x_1, \ldots, x_k)$ then $U_x = \{(x_1, \ldots, x_k)\} \cup \{(x_1, \ldots, x_k)\} \times (\bigcup_{n \in \mathbb{N}} (A \setminus \{a\})^n)$ is a neighborhood of x homeomorphic to A_ω.

We now proceed to defining a clopen local base at $a \in A_\omega$.

We will show that $\mathcal{B} = \{U \subseteq A_\omega; a \in U; U$ is open in A_ω and (1) holds$\}$ is a base for A_ω at the point a.

$$(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) \in U \Rightarrow (x_1, x_2, \ldots, x_n) \in U \qquad (1)$$

Lemma 4.5. *$\mathcal{B}$ is a base for A_ω at the point a.*

Proof. Let V be an open neighborhood of a. We want to find $U \in \mathcal{B}$ such that $U \subseteq V$. Let us put

$$U_1 := V \cap A_1$$
$$U_2 := V \cap A_2 \cap [U_1 \cup (U_1 \times (A \setminus \{a\}))]$$
$$U_{n+1} := V \cap A_{n+1} \cap [U_n \cup (U_n \times (A \setminus \{a\}))]$$

and $U := \bigcup_{n \in \mathbb{N}} U_n$.

Observe that $a \in U \subseteq V$ and, for each $n \in \mathbb{N}$, $U_n \subseteq U_{n+1}$, U_n is open in A_n and $U \cap A_n = U_n$. Hence U is open in A_ω.

If $(x_1, \ldots, x_{n+1}) \in U_{n+1}$ then $(x_1, \ldots, x_{n+1}) \in U_n \times (A \setminus \{a\})$ and $(x_1, \ldots, x_n) \in U_n$. By induction we get that (1) holds for U. $\qquad \square$

Lemma 4.6. *All sets in $\mathcal{B}$ are clopen.*

Proof. Let U be any set from $\mathcal{B}$. If $x = (x_1, \ldots, x_k) \notin U$, then no point of the form $(x_1, \ldots, x_k, y_{k+1}, \ldots, y_{k+l})$ belongs to U, i.e., $U_x \cap U = \emptyset$ holds for the neighborhood $U_x = \{(x_1, \ldots, x_k)\} \cup \{(x_1, \ldots, x_k)\} \times (\bigcup_{n \in \mathbb{N}} (A \setminus \{a\})^n)$ of x. Hence $x \in \text{Int}(A_\omega \setminus U)$, $A_\omega \setminus U$ is open, U is closed. $\qquad \square$

If A is T_2 then by Lemma 4.4 we obtain from the clopen base $\mathcal{B}$ at a a clopen base at each point of A_ω. Thus we get finally

Proposition 4.7. *The space A_ω is zero-dimensional and T_2 for any prime T_2-space A.*

As A_ω is a zero-dimensional T_2-space, it is contained in any epireflective subcategory of **Top** with $I_2 \notin \mathbf{A}$. In the following proposition we summarize some properties of A_ω which were proved in [23].

Proposition 4.8. *If A is a prime space, then $\text{HCH}(A) = \text{CH}((A_\omega)_a) = \text{HCH}(A_\omega)$.*

If A is infinite then $A_\omega \in \text{CH}(A)$ and $\text{card}(A_\omega)_a = \text{card } A$.

Using Lemma 2.1 we obtain from Proposition 4.8 the following corollary.

Corollary 4.9. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$. *If* $A \in \mathbf{A}$ *is a prime space, then* $\mathrm{HAD}_{\mathbf{A}}(A) = \mathrm{AD}_{\mathbf{A}}((A_\omega)_a) = \mathrm{HCH}(A) \cap \mathbf{A}$. *Moreover, if* A *is infinite, then* $\mathrm{card}(A_\omega)_a = \mathrm{card}\, A$.

Proof. We first observe that $\mathbf{A}$ contains A_ω. If A is T_2 then this is true by Proposition 4.7. If A is not T_2, then $\mathbf{A} = \mathbf{Top}_0$ and A_ω is clearly a T_0-space.

By Lemma 2.1 and Proposition 4.8 $\mathrm{HAD}_{\mathbf{A}}(A) \subseteq \mathrm{HCH}(A) \cap \mathbf{A} = \mathrm{CH}((A_\omega)_a) \cap \mathbf{A} = \mathrm{AD}_{\mathbf{A}}((A_\omega)_a)$ holds.

On the other hand, $A_\omega \in \mathbf{A}$ implies $A_\omega \in \mathrm{HAD}_{\mathbf{A}}(A)$. The HAD-class $\mathrm{HAD}_{\mathbf{A}}(A)$ contains the prime space A. So by Corollary 3.12 it is closed under prime factors and $(A_\omega)_a \in \mathrm{HAD}_{\mathbf{A}}(A)$, which proves the opposite inclusion. □

In the rest of this section we show that coreflective hull $\mathrm{CH}(\mathbf{B})$ in $\mathbf{Top}$ of an HAD-class $\mathbf{B}$ in $\mathbf{A}$ is hereditary whenever $\mathbf{B}$ contains at least one prime space.

Lemma 4.10. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *such that* $I_2 \notin \mathbf{A}$. *If* $\mathbf{B} = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D})$, *where* $\mathbf{D} \subseteq \mathbf{A}$ *is a set of spaces and* $\mathbf{B}$ *contains at least one prime space, then there exists a prime space* $B \in \mathbf{A}$ *such that* $\mathbf{B} = \mathrm{HAD}_{\mathbf{A}}(B) = \mathrm{AD}_{\mathbf{A}}(B)$. *Moreover,* $\mathrm{CH}(B) = \mathrm{HCH}(B)$ *is hereditary.*

Proof. Let us denote by $\mathbf{D}'$ the set of all non-discrete prime factors of spaces from $\mathbf{D}$. By joining the accumulation points of all prime spaces in $\mathbf{D}'$ into one point we get a prime space A. We consider 2 cases. If $\mathbf{A} = \mathbf{Top}_0$ then clearly $A \in \mathbf{A}$. If $\mathbf{A} \subseteq \mathbf{Top}_1$ then all spaces in $\mathbf{D}'$ are T_2 and A is T_2 as well. Therefore in both cases $A \in \mathbf{A}$ and $\mathrm{HAD}_{\mathbf{A}}(A) = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D}')$.

Any space from $\mathbf{D}$ can be obtained as a quotient of the sum of its prime factors and consequently $\mathrm{AD}_{\mathbf{A}}(\mathbf{D}') = \mathrm{CH}(\mathbf{D}') \cap \mathbf{A}$ contains the whole $\mathbf{D}$ and $\mathbf{B} = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D}) = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D}') = \mathrm{HAD}_{\mathbf{A}}(A)$.

Using Corollary 4.9 we obtain that the claim of the lemma holds for $B = (A_\omega)_a$. □

With the help of Lemma 4.10 we can prove, using very similar methods as in [4, Proposition 4], the following theorem.

Theorem 4.11. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *such that* $I_2 \notin \mathbf{A}$. *If* $\mathbf{B}$ *is an HAD-class in* $\mathbf{A}$ *and* $\mathbf{B}$ *contains at least one prime space, then the coreflective hull* $\mathrm{CH}(\mathbf{B})$ *of* $\mathbf{B}$ *in* $\mathbf{Top}$ *is hereditary.*

Proof. We first represent $\mathbf{B}$ as a union of an ascending chain of HAD-classes $\mathbf{B}_\alpha$ in $\mathbf{A}$, such that each of them is generated by a single space (as an AD-class).

Let us denote by $\mathbf{B}_\alpha$ the HAD-hull of all spaces from $\mathbf{B}$ with cardinality at most α. Clearly, $\mathbf{B} = \bigcup_{\alpha \in \mathrm{Cn}} \mathbf{B}_\alpha$ and the system $\mathbf{B}_\alpha$, $\alpha \in \mathrm{Cn}$, is nondecreasing.

Since $\mathbf{B}$ contains a prime space, there exists the smallest α_0 such that $\mathbf{B}_{\alpha_0}$ contains a prime space. Then $\mathbf{B} = \bigcup_{\alpha \geq \alpha_0} \mathbf{B}_\alpha$ and for each $\alpha \geq \alpha_0$ the class

$\mathbf{B}_\alpha$ is a HAD-hull of a set of spaces and it contains a prime space. So we can use Lemma 4.10 and we get that for any cardinal number $\alpha \geq \alpha_0$ there exists a prime space $B_\alpha \in \mathbf{A}$ such that $\mathbf{B}_\alpha = \mathrm{HAD}_{\mathbf{A}}(B_\alpha) = \mathrm{AD}_{\mathbf{A}}(B_\alpha) \subseteq \mathrm{CH}(B_\alpha)$.

It is easy to see that $\mathrm{CH}(\mathbf{B})$ consists of quotients of spaces from $\mathbf{B}$. Thus if $Y \in \mathrm{CH}(\mathbf{B})$, then Y is quotient of some space $X \in \mathbf{B}$ and there exists $\alpha \geq \alpha_0$ such that $X \in \mathbf{B}_\alpha$. Consequently we get $Y \in \mathrm{CH}(B_\alpha)$.

Any subspace of Y belongs to $\mathrm{HCH}(B_\alpha) = \mathrm{CH}(B_\alpha) \subseteq \mathrm{CH}(\mathbf{B})$. Thus $\mathrm{CH}(\mathbf{B})$ is closed under the formation of subspaces.   □

Using the above theorem we can prove the result corresponding to [4, Corollary 1].

Corollary 4.12. *Let* $\mathbf{A}$ *be an epireflective subcategory of* **Top** *such that* $I_2 \notin \mathbf{A}$. *Let* $\mathbf{B} \subseteq \mathbf{A}$ *and* $\mathbf{B}$ *contain at least one prime space. Then* $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B}) = \mathrm{HCH}(\mathbf{B}) \cap \mathbf{A} = \mathrm{S}(\mathrm{CH}(\mathbf{B})) \cap \mathbf{A}$.

Proof. By Lemma 2.1 we have $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B}) \subseteq \mathrm{HCH}(\mathbf{B}) \cap \mathbf{A}$.

To obtain the opposite inclusion, we use Theorem 4.11 for the HAD-class $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})$. We get $\mathrm{CH}(\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})) = \mathrm{HCH}(\mathrm{HAD}_{\mathbf{A}}(\mathbf{B}))$. This implies $\mathrm{HAD}_{\mathbf{A}}(\mathbf{B}) \supseteq \mathrm{CH}(\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})) \cap \mathbf{A} = \mathrm{HCH}(\mathrm{HAD}_{\mathbf{A}}(\mathbf{B})) \cap \mathbf{A} \supseteq \mathrm{HCH}(\mathbf{B}) \cap \mathbf{A}$.
                                                                                 □

In particular, Theorem 3.14 implies that Theorem 4.11 and Corollary 4.12 are valid for any HAD-class in $\mathbf{A} \subseteq \mathbf{Haus}$.

Corollary 4.13. *For any epireflective subcategory* $\mathbf{A}$ *of* **Top** *such that* $\mathbf{A} \subseteq \mathbf{Haus}$ *the assignment given by* $\mathbf{C} \mapsto \mathbf{C} \cap \mathbf{A}$ *yields a bijection between the hereditary coreflective subcategories of* **Top** *with* $\mathbf{C} \supseteq \mathbf{FG}$ *and HAD-classes in* $\mathbf{A}$.

Proof. If $\mathbf{C}$ is a hereditary coreflective subcategory of **Top** then the class $\mathbf{C} \cap \mathbf{A}$ is an intersection of two hereditary classes, thus it is hereditary as well. It is clearly an AD-class in $\mathbf{A}$.

Let us denote by F the assignment defined in the claim. We will show that G given by $G(\mathbf{Disc}) = \mathbf{FG}$ and $G(\mathbf{B}) = \mathrm{CH}(\mathbf{B})$ for $\mathbf{B} \neq \mathbf{Disc}$ is inverse to F.

First, observe that if $\mathbf{B} \neq \mathbf{Disc}$ is an HAD-class in $\mathbf{A}$ then by Proposition 3.13 it contains a prime space and from Theorem 4.11 we get that $\mathrm{CH}(\mathbf{B})$ is hereditary.

Let $\mathbf{C} \supsetneq \mathbf{FG}$ be a hereditary coreflective subcategory of **Top**. Then $G(F(\mathbf{C})) = \mathrm{CH}(\mathbf{C} \cap \mathbf{A})$. Since $\mathbf{C} \cap \mathbf{A}$ contains all prime T_2-spaces from $\mathbf{C}$, we get $\mathrm{CH}(\mathbf{C} \cap \mathbf{A}) = \mathbf{C}$ (see [4, Lemma 1]). The equality $G(F(\mathbf{FG})) = \mathbf{FG}$ is also clear.

On the other hand, if $\mathbf{B}$ is an HAD-class in $\mathbf{A}$ and $\mathbf{B} \neq \mathbf{Disc}$, then $F(G(\mathbf{B})) = \mathrm{CH}(\mathbf{B}) \cap \mathbf{A} = \mathrm{AD}_{\mathbf{A}}(\mathbf{B}) = \mathbf{B}$.   □

Remark 4.14. Obviously, if $\mathbf{A}$ is an epireflective subcategory of **Top** with $\mathbf{A} \subseteq \mathbf{Haus}$ (e.g. $\mathbf{A} = \mathbf{Tych}, \mathbf{Reg_2}, \mathbf{ZD_0}$), then $\mathbf{C} \mapsto \mathbf{C} \cap \mathbf{A}$ yields a bijection between the hereditary coreflective subcategories (i.e., HAD-classes) in **Haus** and HAD-classes in $\mathbf{A}$.

For any epireflective subcategory of **Top** with $I_2 \notin \mathbf{A}$ the above assignment is a bijection between hereditary coreflective subcategories of **Top** such that $\mathbf{C} \supsetneq \mathbf{FG}$ and the HAD-classes in **A** containing at least one prime space.

5 Extension of the results to bireflective subcategories

Until now we have only dealt with the epireflective subcategories **A** of **Top** such that $I_2 \notin \mathbf{A}$, i.e., with **A** not bireflective. In this section we would like to find a method how to extend our results also to bireflective subcategories of **Top**. For this, we can use the one-to-one correspondence between bireflective and non-bireflective epireflective subcategories of **Top** given by the assignments $\mathbf{A} \mapsto \mathrm{BH}(\mathbf{A})$ and $\mathbf{B} \mapsto \mathbf{B} \cap \mathbf{Top}_0 = \{R_0B; B \in \mathbf{B}\}$ (see [19], [21]). For sake of simplicity we will ignore the trivial case $\mathbf{A} = \mathbf{Ind}$. (The bireflective subcategory **Ind** corresponds in this assignment to the subcategory containing only one-point spaces and the empty space.)

Recall that the category $\mathbf{Top}_0$ of all T_0-spaces is a quotient-reflective subcategory of **Top**. We will denote the T_0-reflector by R_0.

The T_0-reflection of a space X is the quotient space given by the following equivalence relation: $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$ (see e.g. [9, Beispiel 8.3(2)]). The T_0-reflection arrow is the quotient map corresponding to this equivalence relation. It is moreover an initial map and a retraction, i.e., the T_0-reflection R_0X is homeomorphic to a subspace of X obtained by choosing one point from each equivalence class.

Using the results of the foregoing section we obtain a simple characterization of hereditary AD-classes in a bireflective subcategory **A** in Theorem 5.4.

The following lemma says that in the collection of all AD-classes in **A** with $I_2 \in \mathbf{A}$ all AD-classes except **Disc** contain I_2.

Lemma 5.1. *Let* **C** *be an AD-class in an epireflective subcategory* **A** *with* $I_2 \in \mathbf{A}$ *and* $\mathbf{A} \neq \mathbf{Ind}$. *If* **C** *contains a non-discrete space then* $I_2 \in \mathbf{C}$.

Proof. Let $C \in \mathbf{C}$ be a non-discrete space and $c \in C$ be non-isolated. Let us define $f, g \colon C \to I_2$ by $f(c) = 0$, $f[C \setminus \{c\}] = \{1\}$ and $g(c) = 1$, $g[C \setminus \{c\}] = \{0\}$. We see at once that the map $h \colon C \sqcup C \to I_2$, obtained as the combination of f and g, is a quotient map. $\square$

Lemma 5.2. *Let* **C** *be an AD-class in an epireflective subcategory* **A**, $\mathbf{A} \neq \mathbf{Ind}$, $I_2 \in \mathbf{C}$ *and* $A \in \mathbf{A}$. *Then* $A \in \mathbf{C}$ *if and only if* $R_0A \in \mathbf{C}$.

Proof. Let $A \in \mathbf{C}$. Since R_0A is a subspace of A, we have $R_0A \in \mathbf{A}$. The T_0-reflection arrow $A \to R_0A$ is a quotient map, therefore $R_0A \in \mathbf{C}$.

Now let $R_0A \in \mathbf{C}$. Since the T_0-reflection arrow $r_A \colon A \to R_0A$ is an initial map and bireflective subcategories of **Top** are known to be closed under initial sources, we get $A \in \mathrm{BH}(R_0A) = \mathrm{EH}(\{R_0A, I_2\}) \subseteq \mathbf{A}$.

Since the equivalence classes of $\sim$ are indiscrete subspaces of A, the space A can be obtained as a quotient of the topological sum of R_0A and indiscrete spaces corresponding to these equivalence classes. (We identify each point of R_0A with some point from the indiscrete space representing its equivalence class.) Hence $A \in \mathrm{CH}(R_0A, I_2)$. Consequently, $A \in \mathbf{C}$. $\square$

Lemma 5.3. *If* **C** *is an AD-class in an epireflective subcategory* **A** $\neq$ **Ind**, *then* **C** $\cap$ **Top**$_0$ $= R_0$**C** $= \{R_0C; C \in$ **C**$\}$.

Proof. W.l.o.g. let **C** contains a non-discrete space.

If $X \in$ **C** $\cap$ **Top**$_0$, then $X = R_0X \in R_0$**C**. Hence **C** $\cap$ **Top**$_0 \subseteq R_0$**C**.

On the other hand, let $X = R_0C$ for $C \in$ **C**. By Lemma 5.2 $X \in$ **C**, hence $X \in$ **C** $\cap$ **Top**$_0$. So the opposite inclusion is true as well. $\square$

Theorem 5.4. *Let* **C** *be an AD-class in an epireflective subcategory* **A** $\neq$ **Ind** *with* $I_2 \in$ **A**. *Then* **C** *is hereditary if and only if* R_0**C** $=$ **C** $\cap$ **Top**$_0$ *is hereditary.*

Proof. We can assume $I_2 \in$ **C**, since otherwise **C** $\subseteq$ **Disc** and the claim is trivial.

If **C** is hereditary, then **C** $\cap$ **Top**$_0$ is hereditary as an intersection of two hereditary classes.

Now assume that R_0**C** is hereditary. Let $B \in$ **C** and $e \colon A \hookrightarrow B$ be an embedding. Then the map $R_0e \colon R_0A \hookrightarrow R_0B$ is an embedding as well. (Recall that R_0B is the subspace of B obtained by choosing one point from each equivalence class and note that the equivalence relation $\sim_A$ is the restriction of the relation $\sim_B$.) Therefore $R_0A \in$ **C** $\cap$ **Top**$_0$. Then Lemma 5.2 implies $A \in$ **C**. $\square$

We have shown that to answer the question whether an AD-class in **A** is hereditary it suffices to study the corresponding AD-class in **A** $\cap$ **Top**$_0$. Since for a bireflective subcategory **A** $\neq$ **Top** we have $I_2 \notin$ **A** $\cap$ **Top**$_0$, this is precisely the situation examined in the preceding parts of this paper.

Lemma 5.5. *If* $f \colon X \to Y$ *is a surjective initial map,* $b \in Y$ *and* $f^{-1}(b) = \{a\}$, *then* $X_a \in$ CH(Y_b).

Proof. Let $g \colon Y_b \to X_a$ be any map such that $f(g(x)) = x$ for any $x \in X_a$ (in particular, $g(b) = a$). We first show that g is continuous.

If $a \in U$ and U is open in X_a, then there exists an open set $U' \subseteq Y$ with $a \in f^{-1}(U') \subseteq U$. Then $g^{-1}(U) \supseteq g^{-1}(f^{-1}(U')) = U' \ni b$, hence $g^{-1}(U)$ is open in Y_b.

We have continuous maps f, g such that $f \circ g = id_{X_a}$. So f is a retraction, thus it is a quotient map and $X_a \in$ CH(Y_b). $\square$

Corollary 5.6. *Let* X *be a topological space,* $a \in X$ *be a point such that* X_a *is* T_2 *and* $r \colon X \to RX_0$ *be the* T_0*-reflection of* X. *Then* $X_a \in$ CH$((R_0X)_b)$, *where* $b = r(a)$.

Proof. Since $\overline{\{a\}} = \{a\}$, the equivalence class of the point a consists of this single point. Therefore $r^{-1}(b) = \{a\}$ holds for the **Top**$_0$-reflection r of X. The claim follows now from Lemma 5.5. $\square$

Note that, if **A** is none of the categories **Top**, **Top**$_0$, then all prime factors belonging to **A** are Hausdorff. So we see from the above corollary that, if **B** is an AD-class in an epireflective subcategory **A** $\neq$ **Top**, **Top**$_0$ and R_0**B**

is closed under the formation of prime factors, then $\mathbf{B}$ is closed under the formation of prime factors too.

So our results for arbitrary epireflective subcategories can be subsumed as follows:

Proposition 5.7. *Let* $\mathbf{A} \neq \mathbf{Ind}$ *be an epireflective subcategory of* $\mathbf{Top}$ *and* $\mathbf{B}$ *be an AD-class in* $\mathbf{A}$. *If* $\mathbf{A} \cap \mathbf{Top}_0 \subseteq \mathbf{Haus}$ *or the subcategories* $\mathbf{A} \cap \mathbf{Top}_0$ *and* $\mathbf{B} \cap \mathbf{Top}_0$ *fulfill the assumptions of Theorem 3.10 or those of Corollary 6.8, then* $\mathbf{B}$ *is hereditary if and only if it is closed under the formation of prime factors which belong to* $\mathbf{A}$.

In particular we get that $\mathbf{B}$ is closed under the formation of prime factors which are Hausdorff.

6 AD-classes and HAD-classes containing a prime space

We have shown in Corollary 3.11 that if an HAD-class contains a prime space then it is closed under the formation of prime factors. In connection with this result it seems useful to give some conditions on an HAD-class $\mathbf{B}$ which imply that $\mathbf{B}$ contains at least one prime space.

Unfortunately we were able neither to find a counterexample to the claim that every HAD-class (in an epireflective subcategory $\mathbf{A}$ of $\mathbf{Top}$ with $I_2 \notin \mathbf{A}$) contains a prime space nor to prove this in general.

We have already shown that if an HAD-class $\mathbf{B}$ contains a Hausdorff non-discrete space then it contains a prime space (Proposition 3.13). In this section we provide further sufficient conditions. The main results we obtain are the following: If $\mathbf{B}$ contains a space which is not locally connected, then it contains a prime space (Corollary 6.8). The same holds for non-discrete totally disconnected spaces.

We also show that an AD-class contains a prime space if and only if it contains the space $C(\alpha)$ for some regular cardinal α. At the end of this section we provide some consequences of our results for the lattices of all coreflective subcategories of $\mathbf{Top}$ and of some epireflective subcategories of $\mathbf{Top}$.

6.1 AD-classes containing $C(\alpha)$

Definition 6.1. For any infinite cardinal α we denote by $C(\alpha)$ the space on the set $\alpha \cup \{\alpha\}$ such that each $\beta \in \alpha$ is isolated and the sets $B_\beta = \{\xi \in \alpha \cup \{\alpha\}; \xi \geq \beta\}$ for $\beta < \alpha$ form a local base at α.

The most important case is the case when α is regular, since for any $\alpha \in$ Cn there exists a regular cardinal β with $\mathrm{CH}(C(\alpha)) = \mathrm{CH}(C(\beta))$. (Namely, β is the cofinality of α.)

If α is regular we have a simpler description of the topology of $C(\alpha)$: A subset V of $C(\alpha)$ is open if either $\alpha \notin V$ or $\mathrm{card}(C(\alpha) \setminus V) < \alpha$. Note that this implies that every injective map $f \colon C(\alpha) \to C(\alpha)$ such that $f(\alpha) = \alpha$ is continuous.

In this part we show that an AD-class contains a prime space if and only if it contains some space $C(\alpha)$. We first state two lemmas needed in the proof.

Lemma 6.2. *Let α be an infinite regular cardinal. If $Y \prec C(\alpha)$ is a prime space (with the accumulation point α), then $C(\alpha) \in \mathrm{CH}(Y)$.*

Proof. Let f_i, $i \in I$, be the family of all injective mappings $f_i\colon \alpha \cup \{\alpha\} \to \alpha \cup \{\alpha\}$ such that $f_i(\alpha) = \alpha$. Let us denote by X the topological space on $\alpha \cup \{\alpha\}$ with the final topology with respect to the family $f_i\colon Y \to X$. We claim that $X = C(\alpha)$.

One of the maps f_i is the identity, hence $Y \prec X$ and α is non-isolated in X. Since $Y \prec C(\alpha)$ and all f_i's considered as maps from $C(\alpha)$ to $C(\alpha)$ are continuous, we get $X \prec C(\alpha)$.

To verify that $C(\alpha) \prec X$ we show that any set which is not closed in $C(\alpha)$ is not closed in X.

From $X \prec C(\alpha)$ follows that X is a prime space, therefore it suffices to compare the sets not containing its accumulation point α. So let V be a subset of α with cardinality α and $\alpha \notin V$. Then there exists an $i \in I$ such that f_i maps bijectively the set α to V. Since the subset α is not closed in Y (the point α is not isolated), we get that V is not closed in X. $\square$

Lemma 6.3. *Let α be any infinite cardinal. If $Y \prec C(\alpha)$ is a prime space (with the accumulation point α), then there exists a regular cardinal β with $C(\beta) \in \mathrm{CH}(Y)$.*

Proof. Let β be the cofinality of α. There exists a quotient map $q\colon C(\alpha) \to C(\beta)$ which maps only the point α to β. Let Y' be the quotient of Y with respect to the same map q. Then Y' is a prime space, since $q^{-1}(\beta) = \{\alpha\}$ and α is not isolated in Y. Moreover, $Y' \prec C(\beta)$ and β is a regular cardinal, thus $C(\beta) \in \mathrm{CH}(Y') \subseteq \mathrm{CH}(Y)$ by Lemma 6.2.

Proposition 6.4. *If an AD-class $\mathbf{B}$ in an epireflective subcategory $\mathbf{A} \neq \mathbf{Ind}$ contains a prime T_2-space then it contains $C(\alpha)$ for some regular cardinal number α.*

Proof. Let P be a prime space with the accumulation point a. Denote by α the smallest cardinality of a non-closed subset of $P \setminus \{a\}$. Let C be some such subset.

If V is any subset of C with cardinality smaller than α then it is closed (since α was chosen as the smallest cardinality of a non-closed set). Therefore $C \cup \{a\}$ is a prime subspace of P and it is finer than $C(\alpha)$. (In the case that α is regular it is even homeomorphic, but in either case complements of all basic neighborhoods B_β of α are closed.)

The claim follows now from Lemma 6.3. $\square$

Since every prime T_2-space is zero-dimensional, Proposition 6.4 could be also deduced from Proposition 6.11. But the proof presented here is more straightforward.

6.2 How to obtain a prime space

We now turn our attention to some conditions which are sufficient to enforce that an HAD-class contains a prime space.

Let us denote by **Con** the class of all connected spaces. By [9, Satz 21.2.6] its coreflective hull CH(**Con**) consists precisely of sums of connected spaces. An equivalent characterization is that $X \in$ CH(**Con**) if and only if each point of X has an open connected neighborhood.

Proposition 6.5. *If X is not a sum of connected spaces then there exists a quotient map $f : X \to P$, where P is a prime T_2-space and $P \prec C(\alpha)$.*

Proof. Since X does not belong to CH(**Con**), there exists $a \in X$ such that no open neighborhood of a is connected. This means that for any open neighborhood U of a there exist disjoint open proper subsets V, W of U such that $V \cup W = U$. By transfinite induction we construct a decreasing family U_α of open neighborhoods of a. We put $U_0 = X$. For any β the neighborhood U_β can be divided into two disjoint open non-empty sets. Denote by $U_{\beta+1}$ that one which contains a. Now suppose that β is a limit ordinal and U_γ is already defined for each $\gamma < \beta$. We put $U_\beta := \bigcap_{\gamma < \beta} U_\gamma$ if this set is open. If not, we stop the process and put $\alpha := \beta$. (We must stop at some ordinal β, otherwise there would be a proper class of open sets in X.)

Thus we get a system $(U_\beta)_{\beta < \alpha}$ of open neighborhoods of a with the following properties: $U_\beta \subsetneq U_\gamma$ whenever $\beta > \gamma$. For any limit ordinal $\beta < \alpha$ the equality $U_\beta = \bigcap_{\gamma < \beta} U_\gamma$ holds. The set $U_\beta \setminus U_{\beta+1}$ is open for any $\beta < \alpha$, but $\bigcap_{\beta < \alpha} U_\beta$ is not open.

Now we define $f : X \to \alpha \cup \{\alpha\}$ by

$$f(x) = \sup\{\beta \in \alpha : x \in U_\beta\}.$$

Recall (Definition 6.1) that a neighborhood base for $C(\alpha)$ at α consists of the sets $B_\beta = \{\xi \in \alpha \cup \{\alpha\}; \xi \geq \beta\}$ for $\beta < \alpha$. We have $f^{-1}(B_\beta) = U_\beta$, $f^{-1}(\beta) = U_\beta \setminus U_{\beta+1}$ for any $\beta < \alpha$ and $f^{-1}(\alpha) = \bigcap_{\beta < \alpha} U_\beta$. Thus the quotient space w.r.t. the map f is finer than $C(\alpha)$ and the point α is non-isolated in it. Hence it is a prime T_2-space. $\square$

Propositions 6.5 and 3.13 imply that, if there exist an epireflective subcategory **A** of **Top**, **A** $\neq$ **Ind**, and an HAD-class **B** in **A** not containing a prime space, then **B** $\subseteq$ CH(**Con**) and **B** contains no non-discrete T_2-space.

A topological space X is *totally disconnected* if all components of X are singletons ([6, Notes after section 6.2], [25, Definition 29.1]). The class of totally disconnected spaces forms a quotient-reflective subcategory **TD** of **Top**. If a totally disconnected space X is a sum of connected spaces, then X is clearly discrete.

Corollary 6.6. *If X is non-discrete and totally disconnected then there exists a quotient map from X to a prime T_2-space.*

All zero-dimensional spaces T_0-spaces are totally-disconnected, thus the above corollary applies to the class **ZD₀** as well. We will see in Proposition 6.11 that in the case of zero-dimensional spaces this result can be slightly improved, which leads to the description of atoms above **Disc** in the lattice of coreflective subcategories of the category **ZD₀**.

We say that a space X is *locally connected* if for any open neighborhood U of x there is an open neighborhood $V \subseteq U$ of x, which is connected (see [6, Problem 6.3.3] or [25, Definition 27.7]). The class of locally connected spaces is a coreflective subcategory of **Top**.

Lemma 6.7. *Let X be a topological space. If X is not locally connected then there exists an open subspace V of X such that V is not a sum of connected spaces.*

Proof. If X is not locally connected then there exist a point x and an open neighborhood V of x such that no open neighborhood U of x with $U \subseteq V$ is connected. So x has no open connected neighborhood in the subspace V and $V \notin \mathrm{CH}(\mathbf{Con})$. $\square$

Corollary 6.8. *Let $\mathbf{A}$ be an epireflective subcategory of **Top** with $I_2 \notin \mathbf{A}$. If $\mathbf{B}$ is an HAD-class in $\mathbf{A}$ and $\mathbf{B}$ contains at least one space which is not locally-connected, then $\mathbf{B}$ is closed under prime factors.*

6.3 Lattices of coreflective subcategories

The rest of this section is devoted to showing some new facts concerning the (large) lattice of all coreflective subcategories of **Top** and of **ZD**, which follow from the results above or can be shown using similar methods.

Definition 6.9. Let α be a regular cardinal. Then $B(\alpha)$ is the topological space on the set $\alpha \cup \{\alpha\}$ whose open sets are precisely the sets $B_\beta = \{\xi \in \alpha \cup \{\alpha\}; \xi \geq \beta\}$ for $\beta < \alpha$. We will denote the coreflective hull of $B(\alpha)$ in **Top** by $\mathbf{B}_\alpha$.

Many interesting facts about the lattice of all coreflective subcategories of **Top** can be found in [9, §22] and [10]. It is shown that the atoms of this lattice above **FG** are precisely the subcategories $\mathbf{B}_\alpha$. It is also shown that $\mathbf{B}_\alpha \subseteq \mathrm{CH}(C(\alpha))$ and $\mathrm{CH}(C(\alpha)) \cap \mathrm{CH}(C(\beta)) = \mathbf{FG}$ for any regular cardinals $\alpha \neq \beta$.

Next we show that the minimal elements of the lattice of all coreflective subcategories of **Top** such that $\mathbf{C} \not\subseteq \mathrm{CH}(\mathbf{Con})$ are precisely the subcategories $\mathrm{CH}(C(\alpha))$. (Note that the spaces $B(\alpha)$ are connected whereas $C(\alpha) \notin \mathrm{CH}(\mathbf{Con})$.)

Proposition 6.10. *If $\mathbf{C}$ is a subcategory of **Top** with $\mathbf{C} \not\subseteq \mathrm{CH}(\mathbf{Con})$, then there exists a regular cardinal α such that $\mathrm{CH}(C(\alpha)) \subseteq \mathbf{C}$.*

Proof. If we have $X \in \mathbf{C}$, where $\mathbf{C}$ is coreflective and $X \notin \mathrm{CH}(\mathbf{Con})$, then by Proposition 6.5 and Lemma 6.3 we get $C(\alpha) \in \mathbf{C}$ for some cardinal α. $\square$

The subcategory $\mathbf{B}_\alpha = \mathrm{CH}(B(\alpha))$ is the smallest coreflective subcategory of **Top** such that in each space $X \in \mathbf{B}_\alpha$ any intersection of less than α open sets is open, and there exists a space $Y \in \mathbf{B}_\alpha$ and a system of α open sets in Y with a non-open intersection.

We show that if we have a zero-dimensional space with similar properties then we can obtain a prime T_2-space from it. Thus the atoms in the lattice of coreflective subcategories of **ZD** above the class **FG** $\cap$ **ZD** have a similar description. The proof of the following proposition is similar to the proof of [22, Proposition 4.4].

Proposition 6.11. *Let X be a zero-dimensional space and α be the smallest cardinal number such that there exists a system U_β, $\beta < \alpha$, of open subsets of X with non-open intersection $\bigcap_{\beta < \alpha} U_\beta$, but every intersection of less than α open subsets of X is open. Then there exists a prime space $Y \prec C(\alpha)$ and a quotient map $q \colon X \to Y$.*

Proof. Denote by $\{U_\beta; \beta < \alpha\}$ the system of α open sets in X whose intersection is not open. We can assume w.l.o.g. that this system is strictly decreasing and all sets U_β are clopen. (From an arbitrary decreasing system of open sets we obtain a system of clopen sets by choosing any point $a \in \bigcap_{\gamma < \alpha} U_\gamma \setminus \mathrm{Int}(\bigcap_{\gamma < \alpha} U_\gamma)$ and choosing a basic neighborhood U'_β with $a \in U'_\beta \subseteq U_\beta$ for each $\beta < \alpha$.) If necessary, we can modify this system in such a way that $U_0 = X$ and $U_\beta = \bigcap_{\gamma < \beta} U_\gamma$ for any limit ordinal $\beta < \alpha$.

Define $f \colon X \to \alpha \cup \{\alpha\}$ by

$$f(x) = \sup\{\beta \in \alpha : x \in U_\beta\}.$$

Let Y be the quotient space with respect to f.

The equality $f^{-1}(B_\beta) = U_\beta$ holds for any $\beta < \alpha$. Since each U_β is clopen, we see that B_β and its complement are open in the quotient topology.

The set $f^{-1}(\alpha) = \bigcap_{\beta < \alpha} U_\beta$ is not open, therefore $\{\alpha\}$ is not open. Since the sets U_β are clopen, all sets $\{\beta\} = f^{-1}(U_\beta \setminus U_{\beta+1})$ are open in Y. Thus Y is indeed a prime space and $Y \prec C(\alpha)$. $\square$

Theorem 6.12. *Let $\mathbf{A} = \mathbf{ZD}$ or $\mathbf{A} = \mathbf{ZD_0}$. Let $\mathbf{C}$ be a coreflective subcategory (an AD-class) in $\mathbf{A}$ such that $\mathbf{C} \nsubseteq \mathbf{FG} \cap \mathbf{A}$ and α be the smallest cardinal such that there exists a space $X \in \mathbf{C}$ and a system U_β, $\beta < \alpha$, of open sets in X whose intersection $\bigcap_{\beta < \alpha} U_\beta$ is not open. Then there exists a regular cardinal α such that $\mathrm{CH_A}(C(\alpha)) \subseteq \mathbf{C}$ (resp. $\mathrm{AD_A}(C(\alpha)) \subseteq \mathbf{C}$).*

7 Further applications

In this section we study some other questions which are related to HAD-classes.

7.1 HAD-hulls and hereditary coreflective hulls

The aim of this part is to show that if the coreflective hull of $\mathbf{D}$ in $\mathbf{A}$ is hereditary, it is at the same time the HAD-hull of $\mathbf{D}$ in $\mathbf{A}$.

Recall that the coreflective hull $\mathrm{CH_A}(\mathbf{D})$ of $\mathbf{D}$ in $\mathbf{A}$ can be formed by taking all $\mathbf{A}$-extremal quotients of topological sums of spaces from $\mathbf{D}$.

Lemma 7.1. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{D} \subseteq \mathbf{A}$. *Then the prime* T_2-*spaces contained in* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ *and the prime* T_2-*spaces contained in* $\mathrm{CH}(\mathbf{D})$ *are the same. I.e.,* $\{P \in \mathrm{CH}_{\mathbf{A}}(\mathbf{D}), P$ *is a prime* T_2-*space* $\} = \{P \in \mathrm{CH}(\mathbf{D}), P$ *is a prime* T_2-*space* $\}$.

In the case $\mathbf{A} = \mathbf{Top}_0$ *we moreover get* $\{P \in \mathrm{CH}_{\mathbf{A}}(\mathbf{D}), P$ *is prime* $\} = \{P \in \mathrm{CH}(\mathbf{D}), P$ *is prime* $\}$.

Proof. Let P be a prime T_2-space belonging to $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$. There is an $\mathbf{A}$-extremal epimorphism $e \colon A \to P$, where A is a sum of spaces from $\mathbf{D}$, which can be factorized as $m \circ q$ with q a quotient map and m an injective continuous map.

$$A \xrightarrow{\ q\ } X \xrightarrow{\ m\ } P, \quad e = m \circ q$$

Since $m \colon X \to P$ is an injective map, X is either discrete or a prime T_2-space. Thus $X \in \mathbf{ZD_0} \subseteq \mathbf{A}$, and m is an $\mathbf{A}$-monomorphism. Since e is $\mathbf{A}$-extremal epimorphism, we obtain that m is an isomorphism and $P \in \mathrm{CH}(\mathbf{D})$.

Any prime space is T_0. Therefore the second part is clear from the equality $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{CH}(\mathbf{D}) \cap \mathbf{A}$, which holds for $\mathbf{A} = \mathbf{Top}_0$. $\qquad\square$

Theorem 7.2. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{D} \subseteq \mathbf{A}$. *If* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ *is hereditary then* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{CH}(\mathbf{D}) \cap \mathbf{A}$.

Proof. The inclusion $\mathrm{CH}(\mathbf{D}) \cap \mathbf{A} \subseteq \mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ holds for any $\mathbf{D} \subseteq \mathbf{A}$. We show the opposite inclusion.

If $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ is a hereditary coreflective subcategory of $\mathbf{A}$, then it is closed under the formation of prime factors (see [4, Theorem 1]). Let $Y \in \mathrm{CH}_{\mathbf{A}}(\mathbf{D})$. If $\mathbf{A} \subseteq \mathbf{Top}_1$ then any prime factor Y_a of Y is T_2. All of them belong to $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$. According to Lemma 7.1 prime T_2-spaces in $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ and $\mathrm{CH}(\mathbf{D})$ are the same. As Y is a quotient of spaces Y_a belonging to $\mathrm{CH}(\mathbf{D})$, we get $Y \in \mathrm{CH}(\mathbf{D})$.

In the case $\mathbf{A} = \mathbf{Top}_0$ the equality $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{CH}(\mathbf{D}) \cap \mathbf{A}$ holds for any $\mathbf{D} \subseteq \mathbf{A}$. $\qquad\square$

Corollary 7.3. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{D} \subseteq \mathbf{A}$. *If* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ *is hereditary then* $\mathrm{CH}_{\mathbf{A}}(\mathbf{D}) = \mathrm{AD}_{\mathbf{A}}(\mathbf{D}) = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D})$.

This corollary implies that the results we proved about HAD-hulls in $\mathbf{A}$ can be applied in the case of hereditary coreflective hulls in $\mathbf{A}$ as well.

E.g., if $\mathbf{D} \subseteq \mathbf{A}$ is a set of spaces and $\mathbf{B} = \mathrm{CH}_{\mathbf{A}}(\mathbf{D})$ is hereditary, then by Corollary 7.3 it fulfills the assumptions of Lemma 4.10 and we get existence of a space B with $\mathbf{B} = \mathrm{CH}_{\mathbf{A}}(B)$ in this case.

7.2 Coreflective hull of a map invariant hereditary class need not be
hereditary

Finally, we turn our attention to another question. Relatively little is known
about conditions on a class of spaces which ensure the heredity of the core-
flective hull (AD-hull) of this class. V. Kannan has a result saying that if
B is a hereditary family closed under the formation of spaces with finer
topologies then the coreflective hull CH(**B**) of **B** in **Top** is hereditary as well
([16, Remark 2.4.4(6)]). Our Theorem 4.11 yields a kind of such condition,
too. We next present a well-known example of classes **B** such that CH(**B**) is
hereditary.

Example 7.4. Let α be an infinite cardinal and $\mathbf{G}_\alpha$ be the class of all spaces
with cardinality at most α. These classes are hereditary, map invariant (i.e.,
closed under continuous images) and closed under the formation of prime
factors. The coreflective hull of $\mathbf{G}_\alpha$ is hereditary for each α. Spaces from
the coreflective hull of $\mathbf{G}_\alpha$ are called α-*generated* and the subcategory of all
α-generated spaces is denoted $\mathbf{Gen}(\alpha)$.

On the other hand, let **B** be a class of topological spaces which is map
invariant and closed under prime factors. It is easy to show that if **B** contains
an infinite space, then either **B** = **Top** or **B** = $\mathbf{G}_\alpha$ for some cardinal α. If **B**
consists of finite spaces only, then CH(**B**) is either **FG** or **Disc**.

It is natural to ask whether we can somehow weaken the above mentioned
properties of the classes $\mathbf{G}_\alpha$ in such a way, that for every class **B** with these
properties the coreflective hull CH(**B**) of **B** in **Top** is hereditary.

One possible weakening is replacing the condition that **B** is map invariant
by divisibility. We can construct easily an example showing that for such a
class CH(**B**) need not be hereditary in general.

Example 7.5. Let **B** consist of all quotients of the space $C(\omega)$ and of all
discrete (at most) countable spaces. This class is clearly divisible. Every
space in **B** is prime or discrete, hence **B** is closed under the formation of
prime factors. A subspace of a prime space P is either a discrete space or a
quotient of P, thus **B** is hereditary.

The coreflective hull CH(**B**) = CH($C(\omega)$) = **Seq** is not hereditary.

Another possible weakening is omitting the closedness under prime fac-
tors. We show in the rest of this section that there exists a class **B** which is
hereditary and map invariant but CH(**B**) is not hereditary.

We start with two easy examples.

Example 7.6. Let **B** be the class of all continuous images of the space $B(\omega)$.
The class **B** is hereditary and map invariant, but CH(**B**) = CH($B(\omega)$) is not
hereditary. (This follows from the fact that the prime factor $(B(\omega))_\omega$ is $C(\omega)$
and $C(\omega) \notin$ CH($B(\omega)$).)

It is known, that if **A** is a map invariant class of topological spaces then
CH(**A**) coincides with the class $\mathbf{A}_{gen}$ of **A**-generated spaces (see [9, §21] or
[10]). A topological space X is said to be **A**-*generated* if $U \subseteq X$ is closed

whenever $U \cap V$ is closed in V for every subspace V of X which belongs to $\mathbf{A}$. The subcategory $\mathbf{Gen}(\alpha)$ and the class of k-spaces used in Example 3.18 are examples of such categories.

We denote the cardinality of the topology of a space X by $o(X)$ (in accordance with [14]). For any cardinal α let us denote by $\mathbf{A}_\alpha$ the class of all topological spaces such that $o(X) < 2^\alpha$. This class is hereditary and map invariant. Its coreflective hull (i.e., the class of all $\mathbf{A}_\alpha$-generated spaces) will be denoted by $\mathbf{C}_\alpha$.

Note that $o(B(\omega)) = \omega$, thus $\mathrm{CH}(\mathbf{B}) \subseteq \mathbf{C}_\omega$ holds for the category $\mathbf{B}$ from Example 7.6.

Example 7.7. We show that $\mathbf{A}_{\omega_0}$ is not closed under the formation of prime factors and consequently it is not hereditary.

Let X be a countable topological space with the cofinite topology. Clearly, $o(X) = \omega_0$, thus $X \in \mathbf{A}_{\omega_0}$. But the prime factor X_a of X is homeomorphic to $C(\omega)$. Only the finite subspaces of $C(\omega)$ belong to $\mathbf{A}_{\omega_0}$. Thus the point ω is isolated in each subspace belonging to $\mathbf{A}_{\omega_0}$ and $C(\omega) \notin \mathbf{C}_{\omega_0}$.

Note that by Proposition 3.13 in every non-discrete Hausdorff space X we have infinitely many disjoint open subsets in the subspace Y constructed in the proof of this proposition. Therefore $o(X) \geq \mathfrak{c}$. This implies $\mathbf{A}_{\omega_0} \cap \mathbf{Haus} \subseteq \mathbf{Disc}$ and, consequently, $\mathbf{C}_{\omega_0} \cap \mathbf{Haus} = \mathbf{Disc}$.

Note that, since the space constructed in the above example is T_1, we also obtain that $\mathrm{CH}_{\mathbf{A}}(\mathbf{A}_\alpha \cap \mathbf{A})$ is not hereditary for $\mathbf{A} = \mathbf{Top}_{0,1}$.

It is quite natural to look for a Hausdorff example after we have constructed a T_1-space with the required properties. We have already seen that such an example cannot be found in the subcategory $\mathbf{C}_{\omega_0}$. We were able to construct a Hausdorff example only under the assumption $2^{\omega_1} = 2^\mathfrak{c}$ (which is valid under CH).

Example 7.8 ($2^{\omega_1} = 2^\mathfrak{c}$). Suppose $2^{\omega_1} = 2^\mathfrak{c}$. Let X be the topological space on the set $\mathbb{R}$ with the topology $\mathcal{T} = \{U \setminus A; U$ is open in $\mathbb{R}$ and card $A \leq \omega_0\}$. Clearly, $o(X) = o(\mathbb{R}) \cdot \mathrm{card}\{A \subseteq \mathbb{R}; A$ is countable$\} = \mathfrak{c} \cdot \mathfrak{c}^{\omega_0} = \mathfrak{c}$. Thus $X \in \mathbf{A}_\mathfrak{c}$.

We claim that, for any $a \in X$, the prime factor X_a does not belong to $\mathbf{C}_\mathfrak{c}$. Indeed, if $a \in V$ and V is a subspace of X_a such that $V \in \mathbf{A}_\mathfrak{c}$, then card $V = \omega_0$ (otherwise V contains a discrete subspace $V \setminus \{a\}$ of cardinality ω_1 and $o(V) = 2^{\omega_1} = 2^\mathfrak{c}$). At the same time $a \notin \overline{V \setminus \{a\}}$ (since $\{a\} \cup (\mathbb{R} \setminus V)$ is a neighborhood of a). We see that a is isolated in all subspaces of X_a belonging to $\mathbf{A}_\mathfrak{c}$, but a is not isolated in X_a, thus $X_a \notin \mathbf{C}_\mathfrak{c}$.

Example 7.9. After we have shown that $\mathbf{C}_\alpha$ is not hereditary for some α, we can be interested in finding a concrete example of a space from $\mathbf{C}_\alpha$ and its subspace which is not in $\mathbf{C}_\alpha$. Such an example can be found with the help of the operation $\triangle$.

Suppose that $X \in \mathbf{C}_\alpha$ is such a space that $X_a \notin \mathbf{C}_\alpha$. Let $Y := X \triangle_a X$. Clearly, $Y \in \mathbf{C}_\alpha$. Recall that $X^a_{(X,a)}$ is the subspace on the set $\{(a,a)\} \cup (X \setminus \{a\}) \times (X \setminus \{a\})$. Since $X_a \notin \mathbf{C}_\alpha$ and X_a is a quotient of $X^a_{(X,a)}$, we get that the subspace $X^a_{(X,a)} \notin \mathbf{C}_\alpha$ as well.

Note that, since the subcategories **Top**$_1$, **Haus** are closed under $\triangle$, if we start with the space X from Example 7.7 (or Example 7.8), the resulting space Y will be T_1 (resp. Hausdorff) as well.

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