Hereditary, additive and divisible classes in epireflective subcategories of **Top** *

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1 Introduction

In this talk I will often use the notions of coreflective or epireflective subcategories. Let me start by explaining the meaning these notions in the category **Top** of topological spaces.

Coreflective subcategories are simply the classes of spaces closed under the formation of topological sums and quotient spaces. *Epireflective subcategories* can be characterized as the classes of spaces which are closed under the dual operations – products and subspaces.

For example Hausdorff or completely regular spaces form epireflective subcategories of **Top**, sequential spaces or k-spaces are coreflective subcategories of **Top**.

Of course, there is also a categorical characterization of such subcategories in terms of adjoint functors. And this is probably one of the reasons why they are interesting – that we have two nice characterization of such classes. But in this talk we will only need to know the characterizations using closedness under basic topological operations.

We will also study classes of spaces which are closed under the formation of subspaces – *hereditary.*

We will start with the following question: What can be said about the coreflective subcategories of **Top**, which are at the same time hereditary? This question was posed in [HH] and it was shown in [Č1] that

Theorem 1.1 ([Č1, Theorem 3.7]). *Let* $\mathbf{C}$ *be a coreflective subcategory of* **Top** *with* $\mathbf{FG} \subseteq \mathbf{C}$. *The subcategory* $\mathbf{C}$ *is hereditary if and only if for each* $X \in \mathbf{C}$ *all prime factors of* X *belong to* $\mathbf{C}$ *(i.e.,* $\mathbf{C}$ *is closed under the formation of prime factors).*

Here the *prime factor* X_a of X at a is the topological space on the same set in which all points different from a are isolated and the neighborhoods of a are the same as in the original topology.

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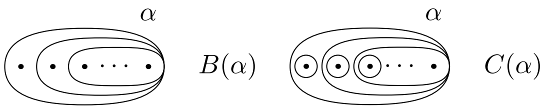
The following figure illustrates the notion of prime factor.



The reason why the condition $\mathbf{FG} \subseteq \mathbf{C}$ appears in this theorem is that the Sierpiński space S is used in the construction of X_a from X and S using sums, quotient map and subspace. This condition says, in other words, that the only exception to this theorem is the coreflective subcategory $\mathbf{Ind}_{\mathrm{sum}}$ consisting of all sums of indiscrete spaces.

Figure 1 illustrates the construction used in the proof of Theorem 1.1. We will later say more about a generalization of this construction.

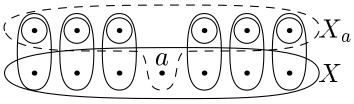


Figure 1: The construction used in the proof of Theorem 1.1

Analogous problem might be interesting also in other base categories than **Top**. In the paper [Č2] the same problem was studied in epireflective subcategories of **Top**.

Theorem 1.2 ([Č2, Theorem 1]). *If* $\mathbf{A}$ *is an epireflective subcategory of* **Top** *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{C}$ *is a coreflective subcategory of* $\mathbf{A}$*, then* $\mathbf{C}$ *is hereditary if and only if* $\mathbf{C}$ *is closed under the formation of prime factors.*

Here I_2 denotes the 2-point indiscrete space.

The coreflective subcategories in an epireflective subcategory $\mathbf{A}$ of **Top** have a similar characterization as coreflective subcategories in **Top**: They are precisely the classes closed under topological sums and $\mathbf{A}$-extremal quotients. We will study a slightly different (and more general) situation – the subclasses of $\mathbf{A}$ which are closed under topological sums and the usual topological quotients.

We will say that a subclass $\mathbf{C}$ of an epireflective subcategory $\mathbf{A}$ is *divisible in* $\mathbf{A}$ if it closed under the quotients with codomain in $\mathbf{A}$. A subclass $\mathbf{C}$ of $\mathbf{A}$ is *additive* if it is closed under topological sums.

Instead of additive and divisible class we will say shortly *AD-class.* Hereditary AD-classes will be called HAD-classes. (So we can see that the coreflective subcategories are precisely the AD-classes in **Top**.)

It is easy to show that if an AD-class in an epireflective subcategory $\mathbf{A}$ of **Top** is closed under the formation of prime factors then it is hereditary. The question we would like to answer is, whether this condition characterizes all AD-classes which are hereditary, i.e., whether every HAD-class is closed under the formation of prime factors.

We will find some sufficient conditions on an AD-class $\mathbf{B}$ and an epireflective subcategory $\mathbf{A}$ of **Top** under which $\mathbf{B}$ is closed under prime factors.

2 Are HAD-classes closed under prime factors?

We will use a construction denoted by $X \bigtriangleup_b Y$, which was defined in [Č2].

Definition 2.1. If X and Y are topological spaces, $b \in Y$ and $\{b\}$ is closed but not open in Y, then we denote by $X \bigtriangleup_b Y$ the topological space on the set $X \times Y$ which has the final topology w.r.t the family of maps $\{f, g_a; a \in X\}$, where $f\colon X \to X \times Y$, $f(x) = (x, b)$ and $g_a\colon Y \to X \times Y$, $g_a(y) = (a, y)$.

In other words, the space $X \bigtriangleup_b Y$ is obtained by attaching the space Y to each point of the space X.

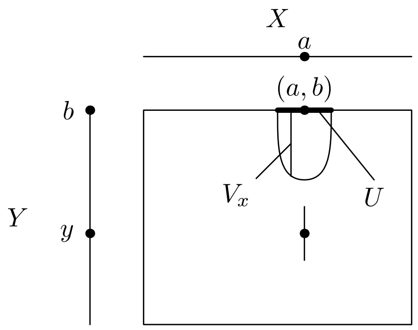
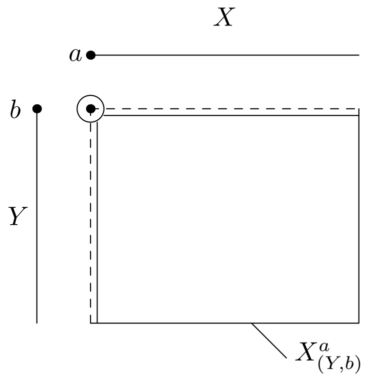
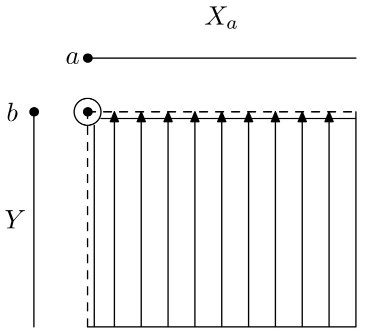


Figure 2: The space $X \bigtriangleup_b Y$

Let $X^a_{(Y,b)}$ be the subspace of $X \bigtriangleup_b Y$ on the subset $\{(a, b)\} \cup (X \setminus \{a\}) \times (Y \setminus \{b\})$ and X_a be the prime factor of X at a. It was shown in [Č2] that, for any space Y in which the subset $\{b\}$ is closed but not open, the map $q\colon X^a_{(Y,b)} \to X_a$ given by $q(x, y) = x$ is quotient.





The subspace $X^a_{(Y,b)}$ The quotient map $X^a_{(Y,b)} \to X_a$

This yields the following proposition:

Proposition 2.2. *Let* $\mathbf{B}$ *be an HAD-class in an epireflective subcategory* $\mathbf{A}$ *of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$. *Let for any* $X \in \mathbf{B}$ *there exist* $Y \in \mathbf{B}$ *and a non-isolated point* $b \in Y$ *with* $\{b\}$ *being closed but not open in* Y *such that* $X \triangle_b Y$ *belongs to* $\mathbf{A}$. *Then* $\mathbf{B}$ *is closed under the formation of prime factors.*

Many familiar epireflective subcategories are closed under the operation $\triangle$, but it can be shown that there is an epireflective subcategory of $\mathbf{Top}$ which is not closed under $\triangle$.

However, we were able to find a condition on the space Y (and the point $b \in Y$) from an epireflective subcategory $\mathbf{A}$, which implies $X \triangle_b Y \in \mathbf{A}$ for every $X \in \mathbf{A}$. We will not introduce here this condition, since its definition is rather cumbersome. We will rather mention several special cases: this condition is fulfilled if Y is a Hausdorff prime space, any infinite space with the cofinite topology, any nondiscrete zero-dimensional space. (By a prime space we mean a topological space with unique non-isolated point.) This leads to the following results:

Corollary 2.3. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{B}$ *be an HAD-class in* $\mathbf{A}$. *If* $\mathbf{B}$ *contains an infinite space with the cofinite topology or it contains a non-discrete zero-dimensional space* (*in particular a prime* T_2*-space*), *then* $\mathbf{B}$ *is closed under prime factors.*

Corollary 2.4. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *with* $I_2 \notin \mathbf{A}$ *and* $\mathbf{B}$ *be an HAD-class in* $\mathbf{A}$. *If* $\mathbf{B}$ *contains a prime space, then* $\mathbf{B}$ *is closed under prime factors.*

We have also shown that if $\mathbf{B}$ contains a non-discrete space, then $\mathbf{B}$ contains a prime space. This implies that

Theorem 2.5. *If* $\mathbf{A}$ *is an epireflective subcategory of* $\mathbf{Top}$ *such that* $\mathbf{A} \subseteq \mathbf{Haus}$ *and* $\mathbf{B}$ *is an HAD-class in* $\mathbf{A}$, *then* $\mathbf{B}$ *is closed under prime factors.*

I wasn't able to show that any HAD-class in any epireflective subcategory of $\mathbf{Top}$ is closed under prime factors or to find a counterexample.

As applications of our results on heredity and closedness under prime factors we can show

Lemma 2.6. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *such that* $I_2 \notin \mathbf{A}$. *If* $\mathbf{B} = \mathrm{HAD}_{\mathbf{A}}(\mathbf{D})$, *where* $\mathbf{D} \subseteq \mathbf{A}$ *is a set of spaces and* $\mathbf{B}$ *contains at least one prime space, then there exists a prime space* $B \in \mathbf{A}$ *such that* $\mathbf{B} = \mathrm{HAD}_{\mathbf{A}}(B) = \mathrm{AD}_{\mathbf{A}}(B)$. *Moreover,* $\mathrm{CH}(B) = \mathrm{HCH}(B)$ *is hereditary.*

Here $\mathrm{CH}(\mathbf{B})$ means the coreflective hull of $\mathbf{B}$ in $\mathbf{Top}$, i.e., the smallest coreflective subcategory of $\mathbf{Top}$ containing $\mathbf{B}$.

$\mathrm{AD}_{\mathbf{A}}(B)$ and $\mathrm{HAD}_{\mathbf{A}}(B)$ have a similar meaning - the smallest (hereditary) AD-class in $\mathbf{A}$ containing $\mathbf{B}$.

Theorem 2.7. *Let* $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{Top}$ *such that* $I_2 \notin \mathbf{A}$. *If* $\mathbf{B}$ *is an HAD-class in* $\mathbf{A}$ *and* $\mathbf{B}$ *contains at least one prime space, then the coreflective hull* $\mathrm{CH}(\mathbf{B})$ *of* $\mathbf{B}$ *in* $\mathbf{Top}$ *is hereditary.*

3 Bireflective subcategories of Top

So far we have only worked with the epireflective subcategories fulfilling the condition $I_2 \notin \mathbf{A}$. The motivation for this condition is the fact, that precisely these epireflective subcategories of **Top** are closed under the formation of prime factors.

Let us note without going into details that our results can be extended to all epireflective subcategories of **Top** using the one-to-one correspondence from [M] between bireflective subcategories of **Top** (i.e., subcategories with $I_2 \in \mathbf{A}$) and non-bireflective subcategories of **Top**.

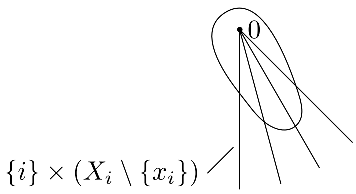
This correspondence is given by

$$\mathbf{A} \mapsto \mathbf{A} \cap \mathbf{Top}_0.$$

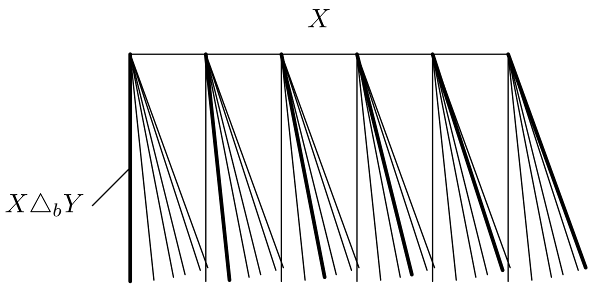
4 Questions

1. Is the construction $X \triangle_b Y$ related to some well-known construction?

It is shown in [S2] that $X \triangle_b Y$ can be embedded into $X \times (\circledast_X(Y,b))$ where $\circledast_I(X_i,x_i)$ denotes the *wedge sum.* This operations consists in identifying points $x_i \in X_i$ into one-points. It is the coproduct in the category of pointed topological spaces.



The wedge sum $\circledast_I(X,x)$



$X \triangle_b Y$ as a subspace of $X \times (\circledast_X Y)$

2. How the counterexample, showing that not every epireflective subcategory of **Top** is closed under $\triangle$, is constructed?

The counterexample used in [S1] is the epireflective hull of a strongly rigid space. A topological space X is called *strongly rigid* if any continuous map $f\colon X \to X$ is either constant or id_X. If a strongly rigid space X fulfills moreover a mild technical condition, then its epireflective hull $\mathrm{EH}(\mathbf{X})$ is not closed under the formation of prime factors.

3. Example of an AD-class in an epireflective subcategory **A** of **Top**, which is not coreflective in **A**.

The class of k_R-spaces in **Tych**. A topological space X is called *k_R-space* if it is completely regular and if every map $f\colon X \to \mathbb{R}$, whose restriction to every compact subset $K \subseteq X$ is continuous, is continuous on X.

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