$\mathcal{I}$-CONVERGENCE AND EXTREMAL $\mathcal{I}$-LIMIT POINTS

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Abstract. The notion of $\mathcal{I}$-convergence was introduced in the paper [18]. This notion includes the notion of the statistical convergence which has been intensively investigated in last twenty years. In the present paper we will give some of its basic properties and we will deal with extremal $\mathcal{I}$-limit points.

Introduction

The notion of $\mathcal{I}$-convergence from the paper [18] corresponds to a generalization of the statistical convergence from [27]. We will show that it is equivalent, in some sense, to the μ-statistical convergence from the paper [9].

Note that the notion of the statistical convergence was introduced in papers [11], [37] and developed in papers [7], [8], [9], [10], [13], [14], [15], [17], [18], [24], [27], [32]. Some applications of the notions of the statistical convergence and the $\mathcal{I}$-convergence in the number theory are given in [35], [36].

In the present paper we will prove some basic results on convergence fields of the $\mathcal{I}$-convergence to complete results from [18]. The notions of $\mathcal{I}\text{-}\liminf x$, $\mathcal{I}\text{-}\limsup x$ will be introduced and their basic properties will be given.

Definitions

Recall the notion of the asymptotic density of a set $A \subset \mathbb{N}$ ($\mathbb{N}$ - the set of positive integers). (see [4], [5], [12], [29, p. 95–96]). For $A \subset \mathbb{N}$ and $n \in \mathbb{N}$ we put

$$d_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k), \quad \delta_n(A) = \frac{1}{S_n}\sum_{k=1}^{n} \frac{\chi_A(k)}{k}$$

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(χ_A is the characteristic function of the set A and $S_n = \sum\limits_{k=1}^{n} \frac{1}{k}$). Then the numbers

$$\underline{d}(A) = \liminf_{n\to\infty} d_n(A), \quad \overline{d}(A) = \limsup_{n\to\infty} d_n(A)$$

are called *lower,* resp. *upper asymptotic density of* A. If there exists the limit $d(A) = \lim\limits_{n\to\infty} d_n(A)$, then $d(A)$ is called the *asymptotic density of the set* A.

Analogously the numbers

$$\underline{\delta}(A) = \liminf_{n\to\infty} \delta_n(A), \quad \overline{\delta}(A) = \limsup_{n\to\infty} \delta_n(A)$$

are called *lower,* resp. *upper logarithmic density of* A. The limit, if there exists, $\delta(A) = \lim\limits_{n\to\infty} \delta_n(A)$ is called the *logarithmic density of the set* A. It is known that

$$\underline{d}(A) \le \underline{\delta}(A) \le \overline{\delta}(A) \le \overline{d}(A) \tag{1}$$

holds for every set $A \subset \mathbb{N}$ ([29, p. 95]).

Hence, if there exists $d(A)$ then there exists $\delta(A)$ and these two numbers are equal. Obviously the numbers $\underline{d}(A)$, $\overline{d}(A)$, $\underline{\delta}(A)$, $\overline{\delta}(A)$ are in the interval $[0, 1]$.

Recall the well-known result

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\left(\frac{1}{n}\right) \qquad (n \to \infty), \tag{2}$$

where γ is the Euler's constant. It follows from (2) that in the definition of $\delta_n(A)$, S_n can be replaced by $\log n$.

Recall some other notions.

Definition A. A sequence $x = (x_n)_1^\infty$ of real numbers is said to be *statistically convergent* to $\xi \in \mathbb{R}$ ($\mathbb{R}$ - the set of reals) if for each $\varepsilon > 0$ $d(A(\varepsilon)) = 0$ holds, where $A(\varepsilon) = \{n : |x_n - \xi| \ge \varepsilon\}$.

Definition B. Let $X \ne \emptyset$. A family $\mathcal{S} \subset 2^X$ of subsets of X is said to be an *ideal* in X if

(i) $\emptyset \in \mathcal{S}$;
(ii) $A, B \in \mathcal{S}$ imply $A \cup B \in \mathcal{S}$;
(iii) $A \in \mathcal{S}$, $B \subset A$ imply $B \in \mathcal{S}$.

(see [19, p. 34])

Definition C. Let $X \ne \emptyset$. A non-void family $\mathcal{F} \subset 2^X$ is said to be a *filter* in X if

(j) $\emptyset \notin \mathcal{F}$

(jj) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{S}$;
(jjj) $A \in \mathcal{F}$, $A \subset B$ imply $B \in \mathcal{F}$.

(see [25, p. 44])

The next proposition gives a simple connection between introduced notions.

Proposition A. *Let $\mathcal{S}$ be a proper ideal in X (i.e. $X \notin \mathcal{S}$), $X \neq \emptyset$. Then a family of sets*

$$\mathcal{F}(\mathcal{S}) = \{M \subset X : \exists A \in \mathcal{S} : M = X \setminus A\}$$

is a filter in X (the filter associated *with the ideal $\mathcal{S}$).*

Definition D. A proper ideal $\mathcal{S}$ is said to be *admissible* if $\{x\} \in \mathcal{S}$ for each $x \in X$.

We will use the notion of the porosity in a metric space which is introduced in the following way (see [38, p. 183–212], [39]).

Let (Y, ϱ) be a metric space, $M \subset Y$. Let $B(y, \delta)$ be a ball centered at $y \in Y$ with the radius $\delta > 0$, i.e. $B(y, \delta) = \{x \in Y : \varrho(x, y) < \delta\}$. For $y \in Y$ and $\delta > 0$ put

$$\gamma(y, \delta, M) = \sup\{t > 0 : \exists z \in B(y, \delta) : [B(z, t) \subset B(y, \delta)] \wedge [B(z, t) \cap M = \emptyset]\}.$$

If there is no such $t > 0$, we put $\gamma(y, \delta, M) = 0$.

The numbers

$$\underline{p}(y, M) = \liminf_{\delta \to 0} \frac{\gamma(y, \delta, M)}{\delta}, \quad \overline{p}(y, M) = \limsup_{\delta \to 0} \frac{\gamma(y, \delta, M)}{\delta}$$

are called the *lower* and *upper porosity of the set M at y*. If for each $y \in Y$ we have $\overline{p}(y, M) > 0$ then M is said to be *porous* in Y. Obviously every set porous in Y is nowhere dense in Y.

If $\overline{p}(y, M) \geq c > 0$ then M is said to be *c-porous* at y. If $\overline{p}(y, M) \geq c > 0$ for each $y \in Y$ then M is said to be c-porous in Y.

If $\underline{p}(y, M) > 0$ then M is said to be *very porous* at y. If M is very porous at y for each $y \in Y$, then M is said to be very porous in Y. The concept of *very c-porous* set at y and very c-porous set in Y can be defined analogously.

If $\underline{p}(y, M) = \overline{p}(y, M) (= p(y, M))$ then the number $p(y, M)$ is called the porosity of M at y. If $p(y, M) = 1$ then M is said to be *strongly porous* at y. The set M is said to be strongly porous in Y if it is strongly porous at each $y \in Y$.

1. $\mathcal{I}$-convergence of sequences of real numbers - examples

Recall the notion of $\mathcal{I}$-convergence ([18]).

Definition 1.1 ([18])**.** Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$. The sequence $x = (x_n)$ of reals is said to be *$\mathcal{I}$-convergent* to $\xi \in \mathbb{R}$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - \xi| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

If $x = (x_n)$ is $\mathcal{I}$-convergent to ξ then we write $\mathcal{I}\text{-}\lim x = \xi$ or $\mathcal{I}\text{-}\lim x_n = \xi$. The number ξ is *$\mathcal{I}$-limit* of the sequence x.

In the paper [18] some basic properties of $\mathcal{I}$-convergence are given. E.g., it is discussed the question which of axioms of convergence (see [23]) $\mathcal{I}$-convergence fulfils.

Further we will give some examples of ideals and corresponding $\mathcal{I}$-convergences.

(I) Put $\mathcal{I}_0 = \{\emptyset\}$. $\mathcal{I}_0$ is the minimal ideal in $\mathbb{N}$. A sequence $x = (x_n)$ is $\mathcal{I}_0$-convergent if and only if it is constant.

(II) Let $\emptyset \neq M \subset \mathbb{N}$, $M \neq \mathbb{N}$. Put $\mathcal{I}_M = 2^M$. Then $\mathcal{I}_M$ is a proper ideal in $\mathbb{N}$. A sequence $x = (x_n)$ is $\mathcal{I}_M$-convergent if and only if it is constant on $\mathbb{N} \setminus M$, i.e. if there is $\xi \in \mathbb{R}$ such that $x_n = \xi$ for each $n \in \mathbb{N} \setminus M$. (Obviously the example (I) is a special case of (II) for $M = \emptyset$).

(III) Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ convergence is the usual convergence.

(IV) Put $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Then $\mathcal{I}_d$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_d$-convergence is the statistical convergence.

(V) Put $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_\delta$-convergence is called *logarithmic statistical convergence.* If $\mathcal{I}_{\mathrm{d}}\text{-}\lim x_n = \xi$ then also $\mathcal{I}_\delta\text{-}\lim x_n = \xi$ (see (1)). This implication is not invertible.

(VI) Let $u(A)$ be a uniform density of the set A (see [3]). Put $\mathcal{I}_u = \{A \subset \mathbb{N} : u(A) = 0\}$. Then $\mathcal{I}_u$ is an admissible ideal.

(VII) Examples (IV) and (V) can be generalized as follows: Let $f \colon \mathbb{N} \to \mathbb{R}^+ = (0, \infty)$ be such a function that $\sum\limits_{k=1}^{\infty} f(k) = +\infty$. Put

$$\mathcal{I}^f = \{A \subset \mathbb{N} : \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} \chi_A(k) f(k)}{\sum\limits_{k=1}^{n} f(k)} = 0\}.$$

Then $\mathcal{I}^f$ is an admissible ideal in $\mathbb{N}$. Note that $\mathcal{I}^f$ and $\mathcal{I}_f$ are different ideals (cf. [22], [1], [21]).

(VIII) A wide class of $\mathcal{I}$-convergences can be obtained in the following manner: Let $T = (t_{nk})$ be a non-negative regular matrix (cf. [30, p. 8]). Then for $A \subset \mathbb{N}$ we put

$$d_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{nk}\chi_A(k) \quad (n = 1, 2, \dots),$$

χ_A being the characteristic function of A. If there exists

$$d_T(A) = \lim_{n\to\infty} d_T^{(n)}(A)$$

then it is called the T-*density* of A (cf. [24]). By the regularity of T we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} = 1$$

and so $d_T(A) \in [0, 1]$. Put $\mathcal{I}_{d_T} = \{A \subset \mathbb{N} : d_T(A) = 0\}$. Then $\mathcal{I}_{d_T}$ is an admissible ideal in $\mathbb{N}$. The corresponding $\mathcal{I}_{d_T}$-convergence contains as special case the φ-convergence of Schoenberg ([37]) (choosing $t_{nk} = \frac{\varphi(k)}{n}$ for $k \mid n$ and $t_{nk} = 0$ for $k \nmid n$, φ being the Euler's function).

(IX) Let υ be a finite additive measure defined on a class $\mathcal{U}$ of subsets of $\mathbb{N}$ (cf. [4], [5], [9], [12], [20], [33], [34]) which contains all finite subsets of $\mathbb{N}$ and $\upsilon(\{n\}) = 0$ for every $n \in \mathbb{N}$, further $\upsilon(A) \le \upsilon(B)$ if $A, B \in \mathcal{U}$, $A \subset B$. Then $\mathcal{I}_\upsilon = \{A \subset \mathbb{N} : \upsilon(A) = 0\}$ is an admissible ideal in $\mathbb{N}$. The $\mathcal{I}_d$- and $\mathcal{I}_\delta$-convergences are included in $\mathcal{I}_\upsilon$-convergence. Further for υ we can take the measure density of R. C. Buck (cf. [4], [5]).

(X) Let $\mu_m\colon 2^{\mathbb{N}} \to [0, 1]$ $(m = 1, 2, \dots)$ be finitely additive measures defined on $2^{\mathbb{N}}$. If there exists $\mu(A) = \lim_{m\to\infty} \mu_m(A)$, then $\mu(A)$ is called the measure of A. Obviously $\mu(A)$ is a finitely additive measure defined on a class $\mathcal{E} \subset 2^{\mathbb{N}}$. So $\mathcal{I}_\mu = \{A \subset \mathbb{N} : \mu(A) = 0\}$ is an admissible ideal in $\mathbb{N}$. For μ_m we can take d_m, δ_m, $d_T^{(m)}$.

(XI) Let $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$ be a decomposition of $\mathbb{N}$ (i.e. $D_k \cap D_l = \emptyset$ for $k \ne l$). Assume that D_j $(j = 1, 2, \dots)$ are infinite sets (e.g we can choose $D_j = \{2^{j-1}(2s-1) : s \in \mathbb{N}\}$ for $j = 1, 2, \dots$). Denote by $\mathcal{I}$ the class of all $A \subset \mathbb{N}$ such that A intersects only a finite number of D_j. It is easy to see that $\mathcal{I}$ is an admissible ideal.

(XII) In [13] the concept of density ϱ of sets $A \subset \mathbb{N}$ is axiomatically introduced. Using this concept we can define the ideal $\mathcal{I}_\varrho = \{A \subset \mathbb{N} : \varrho(A) = 0\}$. So we obtain $\mathcal{I}_\varrho$-convergence as a generalization of the statistical convergence.

(XIII) Let $g\colon \mathbb{N} \to (0,\infty)$, $\sum_{n=1}^{\infty} g(n) = \infty$. Define $\mathcal{I}_g = \{A \subset \mathbb{N} : \sum_{n=1}^{\infty} g(n)\chi_A(n) < +\infty\}$. Then $\mathcal{I}_g$ is an admissible ideal (cf. [22]). Putting $g(n) = \frac{1}{n}$ $(n = 1, 2, \dots)$ we get ideal $\mathcal{I}_c = \{A \subset \mathbb{N} : \sum_{a\in A} a^{-1} < +\infty\}$ mentioned already in [18]. This ideal is closely related to the convergence of subseries of the harmonic series (cf. [31]).

To the end of this section we observe that the μ-statistical convergence of J. Connor [9] is in a sense equivalent to our $\mathcal{I}$-convergence. In what follows we suppose that μ is a finite additive measure defined on a field Γ of subsets of $\mathbb{N}$, such that $\mu(\{k\}) = 0$ for every $k \in \mathbb{N}$ and if $A, B \in \Gamma$, $A \subset B$, then $\mu(A) \le \mu(B)$. Put $\mathcal{I} = \{A \in \Gamma : \mu(A) = 0\}$. Then it is easy to see that $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{F}(\mathcal{I}) = \{B \subset \mathbb{N} : \mu(B) = 1\}$.

Conversely, if $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$, we put $\Gamma = \mathcal{I} \cup \mathcal{F}(\mathcal{I})$. Then Γ is a field of subsets of $\mathbb{N}$. Define $\mu\colon \Gamma \to \{0,1\}$ as follows: $\mu(M) = 0$ if $M \in \mathcal{I}$, $\mu(M) = 1$ if $M \in \mathcal{F}(\mathcal{I})$. This definition is correct since $\mathcal{I} \cap \mathcal{F}(\mathcal{I}) = \emptyset$. Further $\mu(\{k\}) = 0$ since $\mathcal{I}$ is admissible, the monotonicity and additivity of μ can be also easily checked.

2. Convergence field of $\mathcal{I}$-convergence

Let $\mathcal{I}$ be an admissible ideal. Put $F_0(\mathcal{I})$ for the set of all real $\mathcal{I}$-convergent sequences. The set $F_0(\mathcal{I})$ is said to be the convergence field of the $\mathcal{I}$-convergence, $F_0(\mathcal{I}) \subset w$ where w is the set of all real sequences. We will give some properties of the field $F_0(\mathcal{I})$.

Theorem 2.1. *Let $\mathcal{I}$ be an admissible ideal.*

(i) *If $\lim_{n\to\infty} x_n = \xi$ then $\mathcal{I}\text{-}\lim x_n = \xi$;*
(ii) *If $\mathcal{I}\text{-}\lim x_n = \xi$, $\mathcal{I}\text{-}\lim y_n = \eta$, then $\mathcal{I}\text{-}\lim(x_n + y_n) = \xi + \eta$;*
(iii) *If $\mathcal{I}\text{-}\lim x_n = \xi$, $\mathcal{I}\text{-}\lim y_n = \eta$, then $\mathcal{I}\text{-}\lim(x_n y_n) = \xi\eta$.*

Proof. (*i*) The statement is an easy consequence of the inclusion $\mathcal{I}_f \subset \mathcal{I}$.

(*ii*) Let $\varepsilon > 0$. Then the inclusion $\{n : |(x_n + y_n) - (\xi + \eta)| \ge \varepsilon\} \subset \{n : |x_n - \xi| \ge \frac{\varepsilon}{2}\} \cup \{n : |y_n - \eta| \ge \frac{\varepsilon}{2}\}$ holds and the statement follows.

(*iii*) It follows from the assumption of (*iii*) that $B = \{n : |x_n - \xi| < 1\} \in \mathcal{F}(\mathcal{I})$. Obviously $|x_n y_n - \xi\eta| \le |x_n||y_n - \eta| + |\eta||x_n - \xi|$. For $n \in B$ we have $|x_n| < |\xi| + 1$ and it follows

$$|x_n y_n - \xi\eta| \le (|\xi| + 1)|y_n - \eta| + |\eta||x_n - \xi|. \tag{3}$$

Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$0 < 2\delta < \frac{\varepsilon}{|\xi| + |\eta| + 1}. \tag{4}$$

The sets $M_1 = \{n : |x_n - \xi| < \delta\}$ and $M_2 = \{n : |y_n - \eta| < \delta\}$ belong to $\mathcal{F}(\mathcal{I})$. Obviously $B \cap M_1 \cap M_2 \in \mathcal{F}(\mathcal{I})$ and for each $n \in B \cap M_1 \cap M_2$ we have from (3) and (4)

$$|x_n y_n - \xi\eta| < \varepsilon.$$

Hence $\{n : |x_n y_n - \xi\eta| \geq \varepsilon\} \in \mathcal{I}$ and (iii) holds. □

It is known that the $\mathcal{I}$-convergence determines another type of convergence, so-called $\mathcal{I}^*$-convergence, which is connected with sets of the filter $\mathcal{F}(\mathcal{I})$ (see [18]). Recall the definition of the convergence $\mathcal{I}^*$.

Definition 2.1. Let $\mathcal{I}$ be an admissible ideal. A sequence $x = (x_n)$ is said to be $\mathcal{I}^*$*-convergent to* ξ, if there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \ldots\}$ such that $\lim_{k\to\infty} x_{m_k} = \xi$ (shortly $\mathcal{I}^*\text{-}\lim x_n = \xi$).

The notion of $\mathcal{I}^*$-convergence was introduced in [18] for a sequence in a metric space. We will deal with real sequences. In [18] it is proved the implication

$$\mathcal{I}^*\text{-}\lim x_n = \xi \quad \Rightarrow \quad \mathcal{I}\text{-}\lim x_n = \xi. \tag{5}$$

This implication in general cannot be inverted. This shows the following example.

Example 2.1. Let $\mathcal{I}$ be the ideal introduced in example (XI). Define $x = (x_n)$ as follows: For $n \in D_j$ we put $x_n = \frac{1}{j}$ $(j = 1, 2, \ldots)$. Then obviously $\mathcal{I}\text{-}\lim x_n = 0$. We show that $\mathcal{I}^*\text{-}\lim x_n = 0$ does not hold.

Suppose in contrary that $\mathcal{I}^*\text{-}\lim x_n = 0$. Then there is a set $M \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{m\to\infty, m\in M} x_m = 0. \tag{6}$$

By the definition of $\mathcal{F}(\mathcal{I})$ we have $M = \mathbb{N} \setminus H$, where $H \in \mathcal{I}$. By the definition of $\mathcal{I}$ there is a $p \in \mathbb{N}$ such that

$$H \subset D_1 \cup \ldots \cup D_p.$$

But then M contains the set D_{p+1} and so $x_m = \frac{1}{p+1}$ for infinitely many m's from M. This contradicts (6).

In the paper [18] it is proved Theorem 3.2 which gives a condition for the equivalence of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence. The $\mathcal{I}$-convergence is equivalent to $\mathcal{I}^*$-convergence if and only if the ideal $\mathcal{I}$ has the property (AP). (The ideal $\mathcal{I}$ has the property (AP) if for any sequence $\{A_1, A_2, \ldots\}$ of mutually disjoint sets of $\mathcal{I}$ there is a sequence $\{B_1, B_2, \ldots\}$ of sets such that each symmetric difference $A_j \triangle B_j$ $(j = 1, 2, \ldots)$ is finite and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.) As a consequence for real sequences we have that $\mathcal{I}$-convergence is equivalent to $\mathcal{I}^*$-convergence if and only if the ideal $\mathcal{I}$ has the property (AP).

The $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence can be regarded as some summation (limitation) methods. Let $F_0(\mathcal{I})$ $(F_0(\mathcal{I}^*))$ be the convergence field of the method $\mathcal{I}$-convergence ($\mathcal{I}^*$-convergence). I.e. $F_0(\mathcal{I}) = \{x = (x_n) \in w : \text{there is } \mathcal{I}\text{-}\lim x_n \in \mathbb{R}\}$, $F_0(\mathcal{I}^*) = \{x = (x_n) \in w : \text{there is } \mathcal{I}^*\text{-}\lim x_n \in \mathbb{R}\}$. In general we have the inclusion

$$F_0(\mathcal{I}^*) \subset F_0(\mathcal{I}) \tag{7}$$

and the equality in (7) holds if and only if the ideal $\mathcal{I}$ has the property (AP).

Further we shall deal with bounded sequences. We will deal with sets $\ell_\infty \cap F_0(\mathcal{I})$ and $\ell_\infty \cap F_0(\mathcal{I}^*)$, where ℓ_∞ is the space of all bounded sequences endowed with sup-norm. We introduce sets $F(\mathcal{I}) \subset \ell_\infty$ and $F(\mathcal{I}^*) \subset \ell_\infty$ as follows: $F(\mathcal{I}) = \{x = (x_n) \in \ell_\infty : \text{there is } \mathcal{I}\text{-}\lim x_n \in \mathbb{R}\}$, $F(\mathcal{I}^*) = \{x = (x_n) \in \ell_\infty : \text{there is } \mathcal{I}^*\text{-}\lim x_n \in \mathbb{R}\}$.

We saw that $F(\mathcal{I})$ is a linear subspace (subring) of the linear space (ring) ℓ_∞. It follows from Theorem 2.1. An analogous assertion holds for the set $F(\mathcal{I}^*)$.

In what follows we show that properties of convergence fields $F(\mathcal{I})$ and $F(\mathcal{I}^*)$ depends on the ideal $\mathcal{I}$.

Let Z be the class of all admissible ideals in $\mathbb{N}$. The class Z is partially ordered by inclusion. If $Z_0 \subset Z$ is a non-void linearly ordered subset of Z, then it is easy to verify that $\bigcup Z_0$ is an admissible ideal in $\mathbb{N}$ which is an upper bound of Z_0. It follows from Zorn Lemma that in Z there is a maximal admissible ideal. The following lemma gives a characterization of a maximal admissible ideal.

Lemma 2.1. Let $\mathcal{I}_0$ be an ideal in $\mathbb{N}$ which contains all singletons. Then $\mathcal{I}_0$ is maximal admissible if and only if

$$(A \in \mathcal{I}_0) \vee (\mathbb{N} \setminus A \in \mathcal{I}_0) \tag{8}$$

holds for each $A \subset \mathbb{N}$.

Proof. 1) Let an ideal $\mathcal{I}_0$ fulfil (8) for each set $A \subset \mathbb{N}$. By contradiction we show that $\mathcal{I}_0$ is maximal admissible. Let $\mathcal{I}_0 \subsetneqq \mathcal{I}_1$, $\mathcal{I}_1$ - an admissible ideal in $\mathbb{N}$. Then there is $A \subset \mathbb{N}$ such that $A \in \mathcal{I}_1 \setminus \mathcal{I}_0$. Since $A \notin \mathcal{I}_0$, according to (8) we have $\mathbb{N} \setminus A \in \mathcal{I}_0$. Consequently $A \in \mathcal{I}_1$, $\mathbb{N} \setminus A \in \mathcal{I}_1$ and also $\mathbb{N} \in \mathcal{I}_1$ - a contradiction.

2) Let $\mathcal{I}_0$ be a maximal admissible ideal in $\mathbb{N}$. We prove (8). By contradiction. Let $A \subset \mathbb{N}$ be such a set that

$$(A \notin \mathcal{I}_0) \wedge (\mathbb{N} \setminus A \notin \mathcal{I}_0). \tag{9}$$

Put $\mathcal{K} = \{X \subset \mathbb{N} : X \cap A \in \mathcal{I}_0\}$. We show
a) $\mathcal{K} \supset \mathcal{I}_0$;
b) $\mathcal{K}$ is an admissible ideal in $\mathbb{N}$.

a) Let $X \in \mathcal{I}_0$. Then $X \cap A \subset X$ and $X \cap A \in \mathcal{I}_0$, hence $X \in \mathcal{K}$.

b) Obviously $\mathbb{N} \notin \mathcal{K}$ and $\mathcal{K}$ contains each singleton $\{n\}$, $n \in \mathbb{N}$. If $X_1, X_2 \in \mathcal{K}$, then $X_1 \cap A, X_2 \cap A \in \mathcal{I}_0$ and $(X_1 \cup X_2) \cap A = (X_1 \cap A) \cup (X_2 \cap A) \in \mathcal{I}_0$, consequently $X_1 \cup X_2 \in \mathcal{K}$.

Let $X \in \mathcal{K}$ and $X_1 \subset X$. Then $X_1 \cap A \subset X \cap A \in \mathcal{I}_0$ and $X_1 \cap A \in \mathcal{I}_0$. Hence $X_1 \in \mathcal{K}$.

We showed that $\mathcal{K}$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{K} \supset \mathcal{I}_0$. It follows from the maximality of $\mathcal{I}_0$ that $\mathcal{K} = \mathcal{I}_0$. Since $(\mathbb{N} \setminus A) \cap A = \emptyset \in \mathcal{I}_0$ we have $\mathbb{N} \setminus A \in \mathcal{K}$ - a contradiction with respect to (9). □

Theorem 2.2. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Then $F(\mathcal{I}) = \ell_\infty$ if and only if $\mathcal{I}$ is a maximal admissible ideal in $\mathbb{N}$.*

Proof. 1) Let $\mathcal{I}$ be a maximal admissible ideal in $\mathbb{N}$. Let $x = (x_n) \in \ell_\infty$. We show that there exists $\mathcal{I}\text{-}\lim x_n \in \mathbb{R}$.

Since $x \in \ell_\infty$, there are numbers $a, b \in \mathbb{R}$ such that $a \leq x_n \leq b$ hold for each $n \in \mathbb{N}$. Put $A_1 = \{n : a \leq x_n \leq \frac{a+b}{2}\}$, $B_1 = \{n : \frac{a+b}{2} \leq x_n \leq b\}$. Then $A_1 \cup B_1 = \mathbb{N}$. Since $\mathcal{I}$ is an admissible ideal both sets A_1, B_1 cannot belong to $\mathcal{I}$. Thus at least one of them does not belong to $\mathcal{I}$. Denote it by D_1 and the interval corresponding to it denote by J_1. So we have the set D_1 and the interval J_1 such that $D_1 = \{n : x_n \in J_1\} \notin \mathcal{I}$.

We can construct (by induction) a sequence of closed intervals $J_1 \supset J_2 \supset \ldots \supset J_n \supset \ldots$, $J_n = [a_n, b_n]$, $\lim_{n\to\infty}(b_n - a_n) = 0$ and sets $D_k = \{n : x_n \in J_k\} \notin \mathcal{I}$ $(k = 1, 2, \ldots)$.

Let $\xi \in \bigcap_{l=1}^{\infty} J_k$ and $\varepsilon > 0$. Construct the set $M(\varepsilon) = \{n : |x_n - \xi| < \varepsilon\}$. For sufficiently large m we have $J_m = [a_m, b_m] \subset (\xi - \varepsilon, \xi + \varepsilon)$. Since $D_m \notin \mathcal{I}$ we see that $M(\varepsilon) \notin \mathcal{I}$. Since $\mathcal{I}$ is a maximal ideal, according to Lemma 2.1 we have $\mathbb{N} \setminus M(\varepsilon) \in \mathcal{I}$ and $\{n : |x_n - \xi| \geq \varepsilon\} \in \mathcal{I}$. Hence $\mathcal{I}\text{-}\lim x_n = \xi$.

2) Suppose that $\mathcal{I}$ is not maximal. It follows from Lemma 2.1 that there is a set $M = \{m_1 < m_2 < \ldots\}$ such that $M \notin \mathcal{I}$ and $\mathbb{N} \setminus M \notin \mathcal{I}$. Define the sequence $x = (x_n)$ as follows: $x_n = \chi_M(n)$ $(n = 1, 2, \ldots)$. Then $x \in \ell_\infty$ and $\mathcal{I}\text{-}\lim x_n$ does not exists. Indeed, for every $\xi \in \mathbb{R}$

and sufficiently small $\varepsilon > 0$ the set $\{n : |x_n - \xi| \geq \varepsilon\}$ is equal to M or $\mathbb{N} \setminus M$ or to $\mathbb{N}$ and neither of these sets belong to $\mathcal{I}$. □

Remark 2.1. The previous theorem cannot be extended for unbounded sequences. This is shown in the following:

Proposition 2.1. *Let $\mathcal{I}$ be an admissible ideal. Then there exists an unbounded sequence of real numbers for which $\mathcal{I}\text{-}\lim x_n$ does not exist.*

Proof. Put $x_n = n$ $(n = 1, 2, \dots)$. Obviously $\mathcal{I}\text{-}\lim x_n$ does not exist. □

In what follows we will deal with topological properties of convergence fields $F(\mathcal{I})$ and $F(\mathcal{I}^*)$ in ℓ_∞.

Theorem 2.3. *Suppose that $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$. Then $F(\mathcal{I})$ is a closed linear subspace of ℓ_∞ (ℓ_∞ - endowed with the sup-norm).*

Proof. Let $x^{(m)} = (x_j^{(m)})_{j=1}^\infty \in F(\mathcal{I})$ $(m = 1, 2, \dots)$, $\lim\limits_{m\to\infty} x^{(m)} = x$, $x = (x_j)$ in ℓ_∞, i.e. $\lim\limits_{m\to\infty} \|x^{(m)} - x\| = 0$. We prove $x \in F(\mathcal{I})$.

By the assumption there exist $\mathcal{I}\text{-}\lim x^{(m)} = \xi_m \in \mathbb{R}$, $(m = 1, 2, \dots)$.

The proof will be realized in two steps:

1) We prove that $(\xi_m)_1^\infty$ is a Cauchy sequence (so that there exists $\lim\limits_{m\to\infty} \xi_m = \xi \in \mathbb{R}$).

2) We prove that $\mathcal{I}\text{-}\lim x = \xi$.

1) Let $\eta > 0$. From $\lim\limits_{m\to\infty} x^{(m)} = x$ we deduce that $(x^{(m)})_1^\infty$ is a Cauchy sequence in ℓ_∞. Therefore there is an $m_0 \in \mathbb{N}$ such that for each $u, v > m_0$ we have

$$\|x^{(u)} - x^{(v)}\| < \frac{\eta}{3}. \tag{10}$$

Fix $u, v > m_0$. Note that sets $U(\frac{\eta}{3}) = \{j : |x_j^{(u)} - \xi_u| < \frac{\eta}{3}\}$, $V(\frac{\eta}{3}) = \{j : |x_j^{(v)} - \xi_v| < \frac{\eta}{3}\}$ belong to $\mathcal{F}(\mathcal{I})$, thus their intersection is non-void. For any element $s \in U(\frac{\eta}{3}) \cap V(\frac{\eta}{3})$ we have

$$|x_s^{(u)} - \xi_u| < \frac{\eta}{3}, \quad |x_s^{(v)} - \xi_v| < \frac{\eta}{3}. \tag{11}$$

It follows from (10) and (11)

$$|\xi_u - \xi_v| \leq |\xi_u - x_s^{(u)}| + |x_s^{(u)} - x_s^{(v)}| + |x_s^{(v)} - \xi_v| < \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta.$$

Hence $(\xi_m)_1^\infty$ is a Cauchy sequence and so there exists $\xi = \lim\limits_{m\to\infty} \xi_m \in \mathbb{R}$.

2) Let $\varepsilon > 0$. Choose v_0 such that for $v > v_0$ we have simultaneously

$$|\xi_v - \xi| < \frac{\varepsilon}{3}, \quad \|x^{(v)} - x\| \le \frac{\varepsilon}{3}. \tag{12}$$

Then for each $n \in \mathbb{N}$ we have

$$|x_n - \xi| \le |x_n - x_n^{(v)}| + |x_n^{(v)} - \xi_v| + |\xi_v - \xi|. \tag{13}$$

Let $A(\varepsilon) = \{n : |x_n - \xi| \ge \varepsilon\}$, $CA(\varepsilon) = \{n : |x_n - \xi| < \varepsilon\}$, $A_v(\frac{\varepsilon}{3}) = \{n : |x_n^{(v)} - \xi_v| \ge \frac{\varepsilon}{3}\}$, $CA_v(\frac{\varepsilon}{3}) = \{n : |x_n^{(v)} - \xi_v| < \frac{\varepsilon}{3}\}$. It follows from (12) and (13) for $n \in CA_v(\varepsilon)$ the inequality $|x_n - \xi| < \varepsilon$ and the inclusion

$$CA_v\left(\frac{\varepsilon}{3}\right) \subset CA(\varepsilon) \tag{14}$$

holds. Note that $A_v(\frac{\varepsilon}{3}) \in \mathcal{I}$. If we take complements of sets in (14) we have $A(\varepsilon) \in \mathcal{I}$ and the proof of 2) is finished. □

We can summarize our results concerning the convergence fields $F(\mathcal{I})$, $F(\mathcal{I}^*)$. We know that $F(\mathcal{I}^*) \subset F(\mathcal{I})$ (see (7)) and the equality $F(\mathcal{I}^*) = F(\mathcal{I})$ holds if and only if $\mathcal{I}$ satisfies the condition (AP). Further $F(\mathcal{I}) = \ell_\infty$ if and only if $\mathcal{I}$ is a maximal ideal. Thus if $\mathcal{I}$ is not maximal and does not satisfy the condition (AP) then

$$F(\mathcal{I}^*) \subsetneqq F(\mathcal{I}) \subsetneqq \ell_\infty.$$

Now we show that for every admissible ideal $\mathcal{I}$ the set $F(\mathcal{I}^*)$ is dense in $F(\mathcal{I})$.

Theorem 2.4. *For every admissible ideal $\mathcal{I}$ in $\mathbb{N}$ we have*

$$\overline{F(\mathcal{I}^*)} = F(\mathcal{I}).$$

($\overline{F(\mathcal{I}^*)}$ is the closure of the set $F(\mathcal{I}^*)$ in ℓ_∞).

Proof. We have $F(\mathcal{I}^*) \subset F(\mathcal{I})$. Since $F(\mathcal{I})$ is a closed subspace of ℓ_∞ (Theorem 2.3) we get $\overline{F(\mathcal{I}^*)} \subset F(\mathcal{I})$. It suffices to prove $F(\mathcal{I}) \subset \overline{F(\mathcal{I}^*)}$.

Put $B(z,\delta) = \{x \in \ell_\infty : \|x - z\| < \delta\}$ for $z \in \ell_\infty$, $\delta > 0$ (ball in ℓ_∞). It suffices to prove that for each $y \in F(\mathcal{I})$ and $0 < \delta < 1$ we have

$$B(y,\delta) \cap F(\mathcal{I}^*) \neq \emptyset. \tag{15}$$

Put $L = \mathcal{I}\text{-}\lim y$. Choose an arbitrary $\eta \in (0,\delta)$. Then

$$A(\eta) = \{n : |y_n - L| \ge \eta\} \in \mathcal{I}.$$

Define $x = (x_n)_1^\infty$ as follows: $x_n = y_n$ if $n \in A(\eta)$ and $x_n = L$ otherwise.

Then obviously $x \in \ell_\infty$, $\mathcal{I}^*\text{-}\lim x = L$ and $x \in B(y,\eta)$. So (15) holds and the proof is finished. □

It is well known fact that if W is a closed linear subspace of a linear normed space X and $X \neq W$, then W is a nowhere dense set in X. This fact evokes the question about the porosity of W. We will give the answer to this question in general and show some applications to convergence fields $F(\mathcal{I})$ and $F(\mathcal{I}^*)$.

Lemma 2.2. Suppose that X is a linear normed space and W is its closed linear subspace, $W \neq X$. Let

$$s(W) = \sup\{\delta > 0 : \exists B(y,\delta) \subset B(Q,1) \setminus W\}$$

(Q being the zero element of X). Then $s(W) = \frac{1}{2}$.

Proof. We proceed indirectly. Suppose that $s(W) > \frac{1}{2}$. Then there is a $\delta > \frac{1}{2}$ such that for suitable y we have

$$B(y,\delta) \subset B(Q,1) \setminus W. \tag{16}$$

There are two possible cases:

$$(\alpha)\ \|y\| > \frac{1}{2}, \quad (\beta)\ \|y\| \le \frac{1}{2}.$$

(α) In this case for every c, $\frac{1}{2} < c < \delta$ we have

$$y + \frac{c}{\|y\|} y \in B(y,\delta).$$

Simultaneously we have

$$\|y + \frac{c}{\|y\|} y\| = \left(1 + \frac{c}{\|y\|}\right) \|y\| = \|y\| + c > 1.$$

Hence $y + \frac{c}{\|y\|} y \notin B(Q,1)$, which contradicts (16).

(β) In this case $Q \in B(y,\delta) \cap W$ which contradicts (16) too.

Hence $s(W) \le \frac{1}{2}$.

Let $v \in X \setminus W$ and put $\alpha = \inf_{u \in W} \|v - u\|$. Obviously $\alpha > 0$. Without loss of generality we can assume $\alpha = \frac{1}{2}$ (if $\alpha \neq \frac{1}{2}$ we can take $\frac{1}{2\alpha} v$ instead of v). For $\varepsilon \in (0, \frac{1}{2})$ there exists $u_0 \in W$ such that $\frac{1}{2} \le \|v - u_0\| < \frac{1}{2} + \varepsilon$ by definition of α. Put $y = v - u_0$ and $\delta = \frac{1}{2} - \varepsilon$. We show that (16) holds for them.

If $z \in B(y,\delta)$ then $\|z - y\| < \frac{1}{2} - \varepsilon$ and $\|z\| \le \|z - y\| + \|y\| < (\frac{1}{2} - \varepsilon) + (\frac{1}{2} + \varepsilon) = 1$, i.e $B(y,\delta) \subset B(Q,1)$.

Suppose that $z \in B(y,\delta) \cap W$ then $\|z - y\| < \frac{1}{2} - \varepsilon$. On the other hand $\|z - y\| = \|z - (v - u_0)\| = \|(z + u_0) - v\| \ge \frac{1}{2}$, since $z + u_0 \in W$. We get a contradiction, hence $B(y,\delta) \cap W = \emptyset$.

If $\varepsilon \to 0^+$ then $\delta \to \frac{1}{2}^-$ and we get $s(W) = \frac{1}{2}$. □

Theorem 2.5. *Suppose that X is a linear normed space and W is its closed linear subspace, $W \neq X$. Then W is a very porous set in X, in more detail*

a) If $x \in X \setminus W$ then $p(x, W) = 1$,
b) If $x \in W$ then $p(x, W) = \frac{1}{2}$.

Proof. The part *a)* is an easy consequence of the closedness of W in X.

We prove *b)*. Since $W \neq X$, there is a $u \in B(Q, 1) \setminus W$ and $\delta > 0$ such that

$$B(u, \delta) \subset B(Q, 1) \setminus W. \tag{17}$$

First we show that

$$\|u\| + \delta \leq 1. \tag{18}$$

We proceed indirectly. Assume that $\|u\| + \delta > 1$. Since $\|u\| < 1$ for a suitable $c > 0$ we have $1 < \|u\| + c\|u\| < \|u\| + \delta$. From this $c\|u\| < \delta$ and so

$$u + cu \in B(u, \delta). \tag{19}$$

On the other hand

$$\|u + cu\| = (1 + c)\|u\| = \|u\| + c\|u\| > 1$$

and so $u + cu \notin B(Q, 1)$, which contradicts (17), (19). Hence (18) holds.

Let $x \in W$, $\varepsilon > 0$. We show that

$$B(x + \varepsilon u, \varepsilon\delta) \subset B(x, \varepsilon) \setminus W, \tag{20}$$

if (17) holds.

For $z \in B(x + \varepsilon u, \varepsilon\delta)$ we put $w = z - x - \varepsilon u$. Then

$$\|w\| = \|z - x - \varepsilon u\| < \varepsilon\delta. \tag{21}$$

Further $z - x = \varepsilon u + w$, hence by (18), (21)

$$\|z - x\| = \|\varepsilon u + w\| \leq \|\varepsilon u\| + \|w\| < \varepsilon\|u\| + \varepsilon\delta \leq \varepsilon.$$

From this we get $z \in B(x, \varepsilon)$.

We show yet $z \notin W$. In the contrary case we have $z - x = \varepsilon u + w \in W$, hence

$$u + \frac{1}{\varepsilon} w \in W. \tag{22}$$

Since $\|\frac{1}{\varepsilon} w\| < \delta$ (see (21)), $\frac{1}{\varepsilon} w \in B(u, \delta)$. But then by (17) we get $u + \frac{1}{\varepsilon} w \notin W$, which contradicts (22).

Hence we have proved the inclusion (20) under the assumption that $B(u, \delta) \subset B(Q, 1) \setminus W$. But then by definition of $\gamma(x, \varepsilon, W)$ we have

$\gamma(x, \varepsilon, W) \geq \varepsilon\delta$ for each $\delta > 0$ such that (17) holds. From this we get $\gamma(x, \varepsilon, w) \geq \varepsilon s(W)$,

$$\underline{p}(x, W) \geq s(W),$$

where $s(W) = \frac{1}{2}$ is introduced in Lemma 2.2.

Since for every ball $B(y, \delta)$, $\delta > \frac{1}{2}$, $B(y, \delta) \subset B(Q, 1)$ we have $Q \in B(y, \delta)$, the assertion of Theorem 2.5 follows from Lemma 2.2. □

We will apply Theorem 2.5 to the study of the structure of convergence fields $F(\mathcal{I})$, $F(\mathcal{I}^*)$, $\mathcal{I}$ being an admissible ideal in $\mathbb{N}$. We take the linear normed space ℓ_∞ of all bounded real sequences with the sup-norm

$$\|x\| = \sup_{n=1,2,\dots} |x_n|, \quad x = (x_n)_1^\infty \in \ell_\infty.$$

By Theorem 2.2 the convergence field $F(\mathcal{I})$ coincides with ℓ_∞ if and only if $\mathcal{I}$ is a maximal ideal. Hence it is convenient to deal with $F(\mathcal{I})$ under the assumption that $\mathcal{I}$ is not maximal. In this case we have $F(\mathcal{I}) \subsetneqq \ell_\infty$ and by Theorem 2.3 the set $F(\mathcal{I})$ is a closed linear subspace of ℓ_∞.

The following theorem is an easy consequence of Theorem 2.5.

Theorem 2.6. *Suppose that $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$ which is not maximal. Then the following holds:*

(1) *If $x \in \ell_\infty \setminus F(\mathcal{I})$, then $p(x, F(\mathcal{I})) = 1$.*
(2) *If $x \in F(\mathcal{I})$, then $p(x, F(\mathcal{I})) = \frac{1}{2}$.*

Since $F(\mathcal{I}^*) \subset F(\mathcal{I}) = \overline{F(\mathcal{I}^*)}$ (see Theorem 2.4), we get

Corollary 2.1. *Under the condition of Theorem 2.6 we have:*

(1) *If $x \in \ell_\infty \setminus F(\mathcal{I})$, then $p(x, F(\mathcal{I}^*)) = 1$.*
(2) *If $x \in F(\mathcal{I})$, then $p(x, F(\mathcal{I}^*)) = \frac{1}{2}$.*

3. Extremal $\mathcal{I}$-limit points $\mathcal{I}\text{-}\liminf x$ and $\mathcal{I}\text{-}\limsup x$[1]

In the next part of the paper the notions of $\mathcal{I}\text{-}\liminf x$ and $\mathcal{I}\text{-}\limsup x$ will be introduced, and some of their basic properties will be given.

In the paper [13] the notions of the statistical limit point and the statistical cluster point have been introduced. In [15] the authors have introduced notions of extremal statistical limit points (statistical $\liminf x$, statistical $\limsup x$). In the paper [18] the notions of the $\mathcal{I}$-limit point and the $\mathcal{I}$-cluster point of a sequence of elements of a

[1]The notions of $\mathcal{I}\text{-}\limsup x$ and $\mathcal{I}\text{-}\liminf x$ were independently introduced and investigated by K. Demirci: $\mathcal{I}$-limit superior and limit inferior, Math. Communications 6(2001), no. 2, 165–172.

metric space were introduced. These notions generalize notions of the statistical limit point and the statistical cluster point.

Recall that a number ξ is said to be an *$\mathcal{I}$-limit point* of $x = (x_m)$ provided that there is a set $M = \{m_1 < m_2 < \dots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = \xi$. A number ξ is said to be an *$\mathcal{I}$-cluster point* of $x = (x_m)$ if for each $\varepsilon > 0$ we have $\{n \in \mathbb{N} : |x_m - \xi| < \varepsilon\} \notin \mathcal{I}$.

Further we shall deal with an admissible ideal $\mathcal{I}$ and the used symbols have above introduced meanings.

Remark 3.1. Note that for any set $M \subset \mathbb{N}$ at least one of the statements $M \in \mathcal{I}$, $\mathbb{N} \setminus M \in \mathcal{I}$ does not hold.

In what follows we will give a generalization of notions of statistical $\liminf x$ and statistical $\limsup x$ of a real sequence $x = (x_n)$ of [15].

We put for $t \in \mathbb{R}$

$$M_t = \{n : x_n > t\}, \quad M^t = \{n : x_n < t\}.$$

Definition 3.1.

a) If there is a $t \in \mathbb{R}$ such that $M_t \notin \mathcal{I}$, we put

$$\mathcal{I}\text{-}\limsup x = \sup\{t \in \mathbb{R} : M_t \notin \mathcal{I}\}.$$

If $M_t \in \mathcal{I}$ holds for each $t \in \mathbb{R}$ then we put $\mathcal{I}\text{-}\limsup x = -\infty$.

b) If there is a $t \in \mathbb{R}$ such that $M^t \notin \mathcal{I}$, we put

$$\mathcal{I}\text{-}\liminf x = \inf\{t \in \mathbb{R} : M^t \notin \mathcal{I}\}.$$

If $M^t \in \mathcal{I}$ holds for each $t \in \mathbb{R}$ then we put $\mathcal{I}\text{-}\liminf x = +\infty$.

Remark 3.2. If $\mathcal{I} = \mathcal{I}_f$, then the above Definition 3.1 gives usual definition of the notion $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

The next statement is an analogy of Theorem 1.2 of [15].

Theorem 3.1.

(a) $\beta = \mathcal{I}\text{-}\limsup x \in \mathbb{R}$ *if and only if*

$$\{n : x_n > \beta - \varepsilon\} \notin \mathcal{I} \wedge \{n : x_n > \beta + \varepsilon\} \in \mathcal{I} \tag{23}$$

holds for each $\varepsilon > 0$.

(b) $\alpha = \mathcal{I}\text{-}\liminf x \in \mathbb{R}$ *if and only if*

$$\{n : x_n < \alpha + \varepsilon\} \notin \mathcal{I} \wedge \{n : x_n < \alpha - \varepsilon\} \in \mathcal{I} \tag{24}$$

holds for each $\varepsilon > 0$.

Proof. We prove the part (a). The proof of the part (b) is similar.

Let $\varepsilon > 0$. Since $\beta + \varepsilon > \beta = \sup\{t : M_t \notin \mathcal{I}\}$, the number $\beta + \varepsilon$ is not in $\{t : M_t \notin \mathcal{I}\}$ and $\{n : x_n > \beta + \varepsilon\} \in \mathcal{I}$. Further $\beta - \varepsilon < \beta$ and there exists $t' \in \mathbb{R}$ such that $\beta - \varepsilon < t' < \beta$, $t' \in \{t : M_t \notin \mathcal{I}\}$. Hence $\{n : x_n > t'\} \notin \mathcal{I}$ and also $\{n : x_n > \beta - \varepsilon\} \notin \mathcal{I}$. Consequently (23) holds.

On the other hand, suppose that the number β fulfils (23) for each $\varepsilon > 0$. Then, if $\varepsilon > 0$, we have $\beta + \varepsilon \notin \{t : M_t \notin \mathcal{I}\}$ and $\mathcal{I}\text{-}\limsup x \leq \beta + \varepsilon$. Since this holds for every $\varepsilon > 0$, we have

$$\mathcal{I}\text{-}\limsup x \leq \beta. \tag{25}$$

The first of conditions in (23) implies $\mathcal{I}\text{-}\limsup x \geq \beta - \varepsilon$ for each $\varepsilon > 0$, and so we have

$$\mathcal{I}\text{-}\limsup x \geq \beta. \tag{26}$$

Inequalities (25) and (26) imply $\beta = \mathcal{I}\text{-}\limsup x$. □

Theorem 3.2. *The inequality*

$$\mathcal{I}\text{-}\liminf x \leq \mathcal{I}\text{-}\limsup x \tag{27}$$

holds for every sequence $x = (x_n)$ of real numbers.

Proof. If $\mathcal{I}\text{-}\limsup x = +\infty$, then (27) obviously holds. Suppose $\mathcal{I}\text{-}\limsup x < +\infty$. There are two possibilities.

a) $\mathcal{I}\text{-}\limsup x = -\infty$;

b) $-\infty < \mathcal{I}\text{-}\limsup x < +\infty$.

a) In this case we have

$$M_t \in \mathcal{I} \text{ for each } t \in \mathbb{R}. \tag{28}$$

It follows from (28) that $M^t \in \mathcal{F}(\mathcal{I})$ holds for each $t \in \mathbb{R}$ and obviously $M^t \notin \mathcal{I}$ for each $t \in \mathbb{R}$. Hence $\mathcal{I}\text{-}\liminf x = \inf\{t : M^t \notin \mathcal{I}\} = -\infty$ and (27) is proved.

b) We have $\beta = \mathcal{I}\text{-}\limsup x = \sup\{t : M_t \notin \mathcal{I}\}$. If $t > \beta$, then $M_t \in \mathcal{I}$ and $M^t \notin \mathcal{I}$ (Remark 3.1) so we have $\mathcal{I}\text{-}\liminf x = \inf\{t : M^t \notin \mathcal{I}\} \leq \beta$ and (27) holds. □

Recall that the core of a sequence $x = (x_n)$ is said to be the interval $[\liminf x, \limsup x] = \text{core}\{x\}$. In analogy to the st-core$\{x\}$ (see [15]) we can introduce

$$\mathcal{I}\text{-core}\{x\} = [\mathcal{I}\text{-}\liminf x, \mathcal{I}\text{-}\limsup x].$$

Theorem 3.3. *The inequalities*

$$\liminf x \leq \mathcal{I}\text{-}\liminf x \leq \mathcal{I}\text{-}\limsup x \leq \limsup x \tag{29}$$

hold for every real sequence $x = (x_n)$.

Corollary 3.1. *For each real sequence $x = (x_n)$ we have*

$$\mathcal{I}\text{-core}\{x\} \subset \text{core}\{x\}.$$

Proof. We prove only the last inequality of (29). If $\limsup x = +\infty$ then the statement obviously holds. Suppose $L = \limsup x < +\infty$ and $t' > L$. Then the set $M_{t'} = \{n : x_n > t'\}$ is finite and it belongs to $\mathcal{I}$ ($\mathcal{I}$ is an admissible ideal). It follows $\mathcal{I}\text{-}\limsup x = \sup\{t : M_t \notin \mathcal{I}\} \le t'$ and $\mathcal{I}\text{-}\limsup x \le L$. □

Definition 3.2. A sequence $x = (x_n)$ is said to be $\mathcal{I}$*-bounded* if there is a $K > 0$ such that $\{k : |x_k| > K\} \in \mathcal{I}$.

Remark 3.3. If there exists an $\mathcal{I}\text{-}\lim x \in \mathbb{R}$ then $x = (x_n)$ is $\mathcal{I}$-bounded. This statement cannot be conversed. This shows the next Example 3.1.

Theorem 3.4. *The sequence $x = (x_n)$ is $\mathcal{I}$-convergent if and only if*

$$\mathcal{I}\text{-}\liminf x = \mathcal{I}\text{-}\limsup x.$$

If this equality holds then

$$\mathcal{I}\text{-}\lim x = \mathcal{I}\text{-}\liminf x = \mathcal{I}\text{-}\limsup x.$$

Proof. 1) Suppose that there is $\mathcal{I}\text{-}\lim x = L \in \mathbb{R}$. Then for each $\varepsilon > 0$ the set $\{n : |x_n - L| \ge \varepsilon\}$ is in $\mathcal{I}$. Since the property of heredity of the ideal $\mathcal{I}$ we have $\{n : x_n > L + \varepsilon\} \in \mathcal{I}$ and $\{n : x_n < L - \varepsilon\} \in \mathcal{I}$. For each $t \ge L + \varepsilon$ the set M_t is in $\mathcal{I}$, $\sup\{t : M_t \notin \mathcal{I}\} \le L + \varepsilon$ and

$$\mathcal{I}\text{-}\limsup x \le L. \tag{30}$$

Analogously it can be shown

$$\mathcal{I}\text{-}\liminf x \ge L. \tag{30'}$$

It follows from Theorem 3.2, (30) and (30') that

$$L = \mathcal{I}\text{-}\liminf x = \mathcal{I}\text{-}\limsup x.$$

2) Let

$$L = \mathcal{I}\text{-}\liminf x = \mathcal{I}\text{-}\limsup x.$$

Let $\varepsilon > 0$. It follows from the definition of the number $\mathcal{I}\text{-}\limsup x$ that $\{n : x_n \ge L + \varepsilon\} \in \mathcal{I}$. Analogously can be shown $\{n : x_n \le L - \varepsilon\} \in \mathcal{I}$. From the property of additivity of the ideal $\mathcal{I}$ the union of these sets also belongs to $\mathcal{I}$, i.e., $\{n : |x_n - L| \ge \varepsilon\} \in \mathcal{I}$. Hence $L = \mathcal{I}\text{-}\lim x$. □

It is well-known that $\limsup x$ is the greatest limit point of the sequence x. There is a natural question whether this fact holds also for $\mathcal{I}$-convergence, i.e., whether $\mathcal{I}\text{-}\limsup x$ is the greatest $\mathcal{I}$-limit point of the sequence x. In [15] it is shown that, in general, the answer is negative. In [15] the authors showed an example of a sequence $x = (x_n)$ such that the greatest $\mathcal{I}_d$-limit point of x and $\mathcal{I}_d$- $\limsup x$ are different. In this connection we will give an example of a sequence such that the set of all $\mathcal{I}_d$-limit points is non-void, bounded and it has no greatest point. Naturally, $\mathcal{I}_d$- $\limsup x$ exists.

Example 3.1. Put $D_j = \{2^{j-1}(2k-1) : k \in \mathbb{N}\}$ $(j = 1, 2, \dots)$. Obviously $D_j \cap D_k = \emptyset$ for $j \neq k$ and $d(D_j) = \frac{1}{2^j}$ $(j = 1, 2, \dots))$. We define $x = (x_n)_1^\infty$ in the following way: $x_n = 1 - \frac{1}{j}$ for $n \in D_j$ $(j = 1, 2, \dots)$. Then each number $1 - \frac{1}{j}$ $(j = 1, 2, \dots)$ is an $\mathcal{I}_d$-limit point of x and obviously no number greater than 1 is an $\mathcal{I}_d$-limit point of x Further, it follows from the Definition 3.1 that $\mathcal{I}_d$- $\limsup x = 1$.

We show that the number 1 is not $\mathcal{I}_d$-limit point of x. By contradiction. Suppose that 1 is an $\mathcal{I}_d$-limit point of x. Then there is a set $M = \{m_1 < m_2 < \dots\} \subset \mathbb{N}$ such that

$$\overline{d}(M) > 0, \quad \lim_{k\to\infty} x_{m_k} = 1. \tag{31}$$

The definition of x and (31) imply that the set M has a finite intersection with each set D_j $(j = 1, 2, \dots)$. Since $\mathbb{N} = \bigcup\limits_{j=1}^{\infty} D_j$, we have $M = \bigcup\limits_{j=1}^{\infty} (M \cap D_j)$. The equality

$$M = \bigcup_{j=1}^{k} (M \cap D_j) \cup \bigcup_{j=k+1}^{\infty} (M \cap D_j)$$

holds for each fixed k. Hence it follows

$$\overline{d}(M) = \limsup_{n\to\infty} \frac{M(n)}{n} \leq \sum_{j=1}^{k} \limsup_{n\to\infty} \frac{(M \cap D_j)(n)}{n} + \overline{d}(E), \tag{32}$$

where $E = \bigcup\limits_{j=k+1}^{\infty} (M \cap D_j)$.

Since every of the sets $M \cap D_j$ is finite, (32) implies $\overline{d}(M) \leq \overline{d}(E)$. The E is obviously contained in the set of all multiples of 2^k, consequently $\overline{d}(M) \leq 2^{-k}$. The last inequality holds for each $k = 1, 2, \dots$ and we have $d(M) = 0$. A contradiction with respect to (31).

We have shown that the sequence $x = (x_n)$ has no greatest $\mathcal{I}_d$-limit point, but it has $\mathcal{I}_d$- $\limsup x$ equal to 1.

In connection with above example we have for bounded sequences the following result.

Theorem 3.5. *Let $x = (x_n) \in \ell_\infty$ and let $\mathcal{I}(\Gamma_x)$ be the set of all $\mathcal{I}$-cluster points of x. Then*

$$\mathcal{I}\text{-}\limsup x = \max \mathcal{I}(\Gamma_x).$$

Remark 3.4. It can be shown for a bounded sequence $x = (x_n)$ the equality

$$\mathcal{I}\text{-}\liminf x = \min \mathcal{I}(\Gamma_x).$$

Proof of Theorem 3.5. Put $L = \mathcal{I}\text{-}\limsup x$. Suppose $L' > L$. First we show that L' is not in $\mathcal{I}(\Gamma_x)$. We have

$$L = \sup S, \quad S = \{t : \{n : x_n > t\} \notin \mathcal{I}\}. \tag{33}$$

Choose $\varepsilon > 0$ such that $L < L' - \varepsilon < L'$. Then $L' - \varepsilon \notin S$ and $\{n : x_n > L' - \varepsilon\} \in \mathcal{I}$. If follows from the definition of $\mathcal{I}$-cluster point that $L' \notin \mathcal{I}(\Gamma_x)$.

We show $L \in \mathcal{I}(\Gamma_x)$. Let $\eta > 0$. It follows from (33) that there is a $t_0 \in \mathbb{R}$ such that $L - \eta < t_0 \leq L$, $t_0 \in S$. Hence

$$\{n : x_n > t_0\} \notin \mathcal{I}. \tag{34}$$

Simultaneously, since $L + \frac{\eta}{2} \notin S$, we have

$$\{n : x_n > L + \frac{\eta}{2}\} \in \mathcal{I}. \tag{34'}$$

It follows from (34) and (34') $\{n : x_n \in (L - \eta, L + \eta)\} \notin \mathcal{I}$ and $L \in \mathcal{I}(\Gamma_x)$. □

Remark 3.5. For unbounded sequences the set $\mathcal{I}(\Gamma_x)$ can be void. Example: $x = 1, -1, 2, -2, \ldots$

4. Maximal ideals in ℓ_∞, $F(\mathcal{I})$ and $F(\mathcal{I}^*)$

Further we will deal with the rings of sequences of real numbers ℓ_∞, $F(\mathcal{I})$ and $F(\mathcal{I}^*)$ from the algebraical point of view. We will give a connection between algebraic maximal (proper) ideals A in these rings and maximal (proper) ideals $\mathcal{I}$ in $\mathbb{N}$.

First recall some well-known facts about ideals in rings (see, e.g. [16], [6], [2]). Let R be a commutative ring with $\overline{1}$. Then A is a maximal proper ideal in R if and only if R/A is a field. Obviously, if $\overline{1} \in A$ then the ideal A is not proper. In the next we will use the following consequences of the axiom of choice.

Lemma 4.1. Every proper ideal A in a commutative ring R is contained in a maximal ideal A_0 in R.

Lemma 4.2. Let $\mathcal{U} = \{U_i\}$ be a family of subsets of X such that no finite subfamily of $\mathcal{U}$ covers X. Then the family $\mathcal{U}$ can be extended to a maximal ideal $\mathcal{I}$ in X.

For sequences $x = (x_n)$, $y = (y_n)$ of real numbers we put $x + y = (x_n + y_n)$ and $x.y = (x_n.y_n)$. Then ℓ_∞, $F(\mathcal{I})$ and $F(\mathcal{I}^*)$ ($\mathcal{I}$ is an admissible ideal in $\mathbb{N}$) are commutative rings with $\overline{1} = (1)$ with $+$ and $\cdot$ defined in this way.

In the next statement we will admit that the maximal ideal $\mathcal{I}$ in $\mathbb{N}$ need not be admissible, i.e., $\mathcal{I}$ fulfils only conditions of Definition B. A detail analysis of the proof of Theorem 2.2 shows that the condition on $\mathcal{I}$ to be admissible is superfluous. In this form Theorem 2.2 is used in the proof of the following theorem.

Theorem 4.1. *Let S be a subring of ℓ_∞ such that:*

(i) *S contains all constant sequences,*
(ii) *if $(x_n) \in S$ and $x_n > K > 0$ for some real K and every $n \in \mathbb{N}$, then the sequence $\left(\frac{1}{x_n}\right) \in S$.*

Then A is a maximal ideal in S if and only if there exists maximal ideal $\mathcal{I}$ in $\mathbb{N}$ such that

$$A = A_{\mathcal{I}} = \{x = (x_n) \in S; \mathcal{I}\text{-}\lim x_n = 0\}.$$

Proof. Suppose that $\mathcal{I}$ is a maximal ideal in $\mathbb{N}$. Obviously $A_{\mathcal{I}}$ is an ideal in S. We show that $A_{\mathcal{I}}$ is maximal. According to Theorem 2.2 there exists $\mathcal{I}\text{-}\lim x_n$ for each $(x_n) \in \ell_\infty$. We can define a homomorphism $\Phi_{\mathcal{I}} \colon S \to \mathbb{R}$ such that $\Phi_{\mathcal{I}}(x) = \mathcal{I}\text{-}\lim x_n$. Since S contains all constant sequences, $\Phi_{\mathcal{I}}$ is surjective. Clearly $A_{\mathcal{I}} = Ker\, \Phi_{\mathcal{I}}$ and $S/A_{\mathcal{I}} = \mathbb{R}$. Thus $A_{\mathcal{I}}$ is a maximal ideal.

We show that every maximal ideal in S is of the form $A_{\mathcal{I}}$ for some maximal ideal $\mathcal{I}$ in $\mathbb{N}$. Let $M(\mathbb{N})$ be the set of all maximal ideals in $\mathbb{N}$. Assume that A is an ideal in S which is not contained in any $A_{\mathcal{I}}$, $\mathcal{I} \in M(\mathbb{N})$. Then for each maximal ideal $\mathcal{I} \in M(\mathbb{N})$ there exists a sequence $(x_n) \in A$ such that $x_n \geq 0$ and $\mathcal{I}\text{-}\lim x_n = L > 0$. Thus there exists $V_{\mathcal{I}} \in \mathcal{F}(\mathcal{I})$ such that $x_n > \frac{L}{2}$ for every $n \in V_{\mathcal{I}}$. We show that there exist a finite set

$$\{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m\} \tag{35}$$

such that $\mathbb{N} \subset \bigcup_{k=1}^{m} V_{\mathcal{I}_k}$. In the opposite case $\{V_{\mathcal{I}}; \mathcal{I} \in M(\mathbb{N})\}$ fulfils assumptions of the Lemma 4.2 and there exists a maximal ideal $\mathcal{I}_0$ with $\{V_{\mathcal{I}}; \mathcal{I} \in M(\mathbb{N})\} \subset \mathcal{I}_0$. Thus we have $V_{\mathcal{I}_0} \in \mathcal{I}_0$ and $V_{\mathcal{I}_0} \in \mathcal{F}(\mathcal{I}_0)$, a contradiction.

For $\mathcal{I}_1, \ldots, \mathcal{I}_m$ from (35), let $x^{(i)}$, $i = 1, \ldots, m$, be the sequence $x^{(i)} = (x_n^{(i)}) \in A$ satisfying $x_n^{(i)} > 0$ and $x_n^{(i)} > \frac{L_i}{2}$ for $n \in V_{\mathcal{I}_i}$ ($L_i = \mathcal{I}_i$- $\lim x_n^{(i)}$). Put $z_n = x_n^{(1)} + \ldots + x_n^{(m)}$. Then $(z_n) \in A$ and $z_n > \min\limits_{1 \le i \le m} \frac{L_i}{2} > 0$ for each $n \in \mathbb{N}$. It follows from *(ii)* that $\left(\frac{1}{z_n}\right) \in S$ and $\overline{1} = \left(z_n \cdot \frac{1}{z_n}\right) \in A$, hence A is not a proper ideal in S. □

Corollary 4.1. *The set $A \subset \ell_\infty(F(\mathcal{I}), F(\mathcal{I}^*))$ is a maximal ideal in $\ell_\infty(F(\mathcal{I}), F(\mathcal{I}^*))$ if and only if there exists a maximal ideal $\mathcal{J}$ in $\mathbb{N}$ such that*

$$A = A_{\mathcal{J}} = \{x = (x_n) \in \ell_\infty(F(\mathcal{I}), F(\mathcal{I}^*)) : \mathcal{J}\text{-}\lim x_n = 0\}.$$

Example 4.1. Note that if $\mathcal{I}$ is the maximal ideal consisting exactly of all subset of $\mathbb{N}$ which does not contain $m \in \mathbb{N}$, we receive the maximal ideal

$$A = \{x = (x_n) \in S : x_m = 0\}$$

in S.

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