Ideal convergence and other generalized limits

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Abstract

This talk is mainly concerned with two generalizations of convergence of sequences called $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence. We will mention some other generalizations of limit which are related to $\mathcal{I}$-convergence, e.g Banach limit and statistical convergence.

1 Generalizations of limit

The notion of limit is one of the central notions in mathematical analysis. No wonder it was generalized by mathematicians in various ways.

One of natural generalizations of limit is to define an operator extending the usual limit which assigns a value to some non-convergent sequences too.

For example if we want define an extended limit in a such way that the sequence

$$(1, 0, 1, 0, \ldots)$$

has a limit, one would expect that this limit to be $\frac{1}{2}$. Example of an operator satisfying this condition is the Cesàro mean. The Cesàro mean of the sequence (a_n) is the sequence

$$b_n = \frac{a_1 + \ldots + a_n}{n}$$

of arithmetic means of first n elements. It can be verified that if (a_n) is convergent then (b_n) converges to the same limit. The limit of Cesàro mean of the sequence $(1, 0, 1, 0, \ldots)$ is $\frac{1}{2}$, as we expected. We see, that the operator $\varphi(a_n) = \lim\limits_{n\to\infty} b_n$ is an operator which extends the usual limit. (This is also known as $(C, 1)$-convergence. Cesàro summability or $(C, 1)$-summability is analogous generalization of a sum of a series. Summability methods like this are studied in summability theory.)

Stefan Banach proved in [4] that limit can be extended to an operator on all bounded sequences. He proved the existence of so called Banach limit, i.e., a continuous linear functional $\varphi : \ell_\infty \to \mathbb{R}$ defined on the set ℓ_∞ of all bounded real sequences such that for any real sequences $x = (x_n)$, $y = (y_n)$ it holds:

- $\varphi(c.x + d.y) = c.\varphi(x) + d.\varphi(y)$; (linearity)
- if $x \geq 0$, then $\varphi(x) \geq 0$; (positivity)
- $\varphi(x) = \varphi(Sx)$, where Sx is the shift operator defined by $S(x_n) = (x_{n+1})$. (shift-invariance)

- If x is a convergent sequence, then $\varphi(x) = \lim x$. (extends the usual limit)

The existence of Banach limit is proved in mathematical analysis usually by Hahn-Banach theorem. (This proof can be found e.g. in [23], [8] or [17].) We will prove the existence of Banach limit using $\mathcal{I}$-convergence.

We could note that the sequence $x = (1, 0, 1, 0, \ldots)$ has a very special property. Let φ be any Banach limit. Observe that $x + Sx = (1, 1, 1, 1, \ldots)$. Thus $\varphi(x) + \varphi(Sx) = 1 = 2.\varphi(x)$ and $\varphi(x) = \frac{1}{2}$ for any Banach limit φ. A bounded sequence with this property, that every Banach limit has the same value, is called *almost convergent*.

The following characterization of almost convergent sequences is due to Lorentz [16]. (For an alternative proof see [5].)

A sequence (x_n) is almost convergent if and only if

$$\lim_{p\to\infty} \frac{x_n + \ldots + x_{n+p-1}}{p} = L$$

uniformly in n.

The above limit can be rewritten in detail as

$$(\forall\varepsilon > 0)(\exists p_0)(\forall p > p_0)(\forall n) \left| \frac{x_n + \ldots + x_{n+p-1}}{p} - L \right| < \varepsilon.$$

The shift-invariance cannot be fulfilled along with the property $\varphi(x_n.y_n) = \varphi(x_n).\varphi(y_n)$ at the same time. Simply note that for the sequence $x = (1, 0, 1, 0, \ldots)$ it holds $x.Sx = (0, 0, \ldots)$ and $\varphi(x).\varphi(Sx) \neq 0 = \varphi(x.Sx)$. (In fact, if we want the limit operator φ to preserves the multiplication, then from $x.x = x$ we get $\varphi(x)^2 = \varphi(x)$ and $\varphi(x) \in \{0, 1\}$.) We are going to introduce another generalization of convergence which preserves multiplication of sequences.

In general, Banach limit is not determined uniquely. There exist many sequences which can have various Banach limits. (We will see this later.) In a certain sense, most sequences are not almost convergent (see [7]).

2 $\mathcal{I}$-convergence

2.1 Definition

We next introduce the $\mathcal{I}$-convergence which is a generalization of the ordinary convergence. We follow the introductory article [14]. Other good introductory articles are e.g. [18] and [13].

We first need to recall the definitions of some other notions.

Definition 2.1. A non-empty subset $\mathcal{I}$ of $\mathcal{P}(\mathbb{N})$ is called an *ideal* on $\mathbb{N}$ if

(i) $B \in \mathcal{I}$ whenever $B \subseteq A$ for some $A \in \mathcal{I}$ (closed under subsets)

(ii) $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ (closed under unions)

An ideal is called *proper* if $\mathbb{N} \notin \mathcal{I}$. An ideal is called *maximal* if it is a proper ideal which is maximal with respect to inclusion. An ideal is called *admissible* if it is proper and contains all finite subsets.

Reader should be warned at this point, that many concepts mentioned in this exposition are more frequently defined using limit along a filter. Filter is a dual notion to ideal - it is closed under supersets and intersections. It holds that $\{\mathbb{N} \setminus A; A \in \mathcal{I}\}$ is a filter if and only if $\mathcal{I}$ is ideal. I.e. there is a one-to-one correspondence between ideals and filters. As this talk follows [14], we will use ideals in all definitions. But we will point out where appropriate that some notions are more familiar for filters, we try to include references when possible.

Because of this duality every result or definition using ideals can be reformulated using filters and vice-versa. Just a short comment on terminology: Maximal ideals correspond to ultrafilters, admissible ideals correspond to free filters.

We will need the following characterization of maximal ideals (more frequently introduced for ultrafilters).

Lemma 2.2. *An ideal $\mathcal{I}$ is maximal if and only if for any subset $A \subseteq \mathbb{N}$ it holds either $A \in \mathcal{I}$ or $\mathbb{N} \setminus A \in \mathcal{I}$.*

Proof. If an ideal $\mathcal{I}$ satisfies the above condition then by adding any new set A to ideal we see that any ideal containing A and $\mathcal{I}$ contains $A \cup (\mathbb{N} \setminus A)$, hence it is not proper.

If an ideal doesn't satisfy this condition, i.e. there is a set A with $A \notin \mathcal{I}$ and $\mathbb{N} \setminus A \notin \mathcal{I}$, then $\mathcal{I}' = \{C \subseteq A' \cup B; A' \subseteq A, B \in \mathcal{I}\}$ is a proper ideal. $\square$

It is easy to see that $\emptyset \in \mathcal{I}$ for any ideal. Ideal can be viewed as a way to describe which sets will be considered "small". (Filter is collection of all "large" sets.) It is known that Axiom of Choice implies that any proper ideal is contained in a maximal ideal. (This result is more common in the form: Any filter is contained in an ultrafilter.)

Definition 2.3. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a proper ideal in $\mathbb{N}$. The sequence $x = (x_n)$ of reals is said to be *$\mathcal{I}$-convergent* to $L \in \mathbb{R}$, if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \{n : |x_n - L| \geq \varepsilon\}$$

belongs to $\mathcal{I}$.

If $x = (x_n)$ is $\mathcal{I}$-convergent to L then we write $\mathcal{I}\text{-}\lim x = L$ or $\mathcal{I}\text{-}\lim x_n = L$. The number L is *$\mathcal{I}$-limit* of the sequence x.

The definition of $\mathcal{I}$-convergence can be interpreted very simply - it states that the set $A(\varepsilon)$ of "bad" indices is small. For the usual convergence small means finite, here small set means a set in $\mathcal{I}$.

For the introduction of limit along a filter (also called $\mathcal{F}$-limit) see e.g. [2, p.122, Definition 8.23], [12, p.206, Definition 2.7], [18], [1] (and many others - some set-theoretical expositions define this notion in the chapter on filters).

We will deal only with the case of real sequences, but without any changes the same definition can be used for metric spaces (as it was defined in [14]) or topological spaces (see [15] or [21]).

We will introduce examples of several ideals and corresponding $\mathcal{I}$-convergences. Also other questions, such as properties and concepts analogous to known properties of the usual convergence or the statistical convergence are studied.

2.2 Examples of some interesting ideals

Fréchet ideal and the usual convergence. We should first describe the relation to the usual convergence. The ideal $\mathcal{I}_f$ consisting of all finite sets will be called *Fréchet ideal*. Clearly, $\mathcal{I}_f$-convergence is precisely the usual convergence.

Since $\mathcal{I}_f \subseteq \mathcal{I}$ for any admissible ideal $\mathcal{I}$, the usual convergence implies $\mathcal{I}$-convergence.

Asymptotic density and statistical convergence.

Definition 2.4. Let A be a subset of $\mathbb{N}$. We put $A(n) = |\{k \in A; k \leq n\}|$. The *asymptotic density* of A is the limit

$$d(A) := \lim_{n\to\infty} \frac{A(n)}{n},$$

if this limit exists.

We see that asymptotic density is limit of frequencies of numbers in the sets $\{0, 1, \ldots, n\}$, therefore it is (when it exists) intuitively correct measure of size of subsets of integers.

It can be easily verified that $d(A \cup B) = d(A) + d(B)$ if A and B are disjoint subsets. Using this fact we can show that:

Lemma 2.5. *The set* $\mathcal{I}_d := \{A \subseteq \mathbb{N}; d(A) = 0\}$ *is an ideal.*

$\mathcal{I}_d$-convergence was extensively studied (see e.g. [10] or [19]), it is known as *statistical convergence*.

Asymptotic density and statistical convergence are frequently used in number theory. Asymptotic density is (in some context) appropriate way to describe whether a subset of natural numbers is small or large.

The statistical convergence is perhaps the most important example in connection with the $\mathcal{I}$-convergence, because the research in $\mathcal{I}$-convergence was motivated mainly by known results on statistical convergence.

Van der Waerden's ideal and the ideal $\mathcal{I}_c$. There are also systems of subsets of $\mathbb{N}$ for which proving the properties of ideal is far from trivial. Let us put $\mathcal{W}$ = set of all subsets of $\mathbb{N}$ which don't contain arbitrary long arithmetic progressions. Famous theorem of Van der Waerden implies that $\mathcal{W}$ is an ideal. (The Van der Waerden's result can be rephrased as: $A \cup B$ contains arbitrary long AP's $\Rightarrow$ one of the sets A, B contains arbitrary long AP's.) Another famous theorem of Szemerédi says that $\mathcal{W} \subset \mathcal{D}$.

The problem whether the set of all prime numbers P belongs to $\mathcal{W}$ was long standing open problem. It was recently shown by Tao and Green [11] that primes contains arbitrary long arithmetic progressions.

An open conjecture of Erdös says that every set $\{n_1 < n_2 < \ldots\}$ such that $\sum\limits_{k=1}^{\infty} \frac{1}{n_k}$ diverges contains arbitrary long arithmetic progressions. It can be shown that $\mathcal{I}_c = \{A \subseteq \mathbb{N}; \sum\limits_{k=1}^{\infty} \frac{1}{a_k} < \infty\}$ is an ideal. Thus this conjecture can be restated using these ideals as follows: $\mathcal{W} \subseteq \mathcal{I}_c$.

2.3 Basic properties of $\mathcal{I}$-convergence

The proofs of the following properties are not very hard and they are, to a certain extent, analogous to the proofs in the case of the usual convergence. Therefore these proofs will be omitted, they can be found e.g. in [13].

Proposition 2.6. *Let $\mathcal{I}$ be an admissible ideal.*

(i) If $\lim_{n\to\infty} x_n = L$ then $\mathcal{I}\text{-}\lim x_n = L$.

(ii) If $\mathcal{I}\text{-}\lim x_n$ exists, then $\liminf x_n \le \mathcal{I}\text{-}\lim x_n \le \limsup x_n$.

(iii) The $\mathcal{I}$-limits are unique.

(iv) $\mathcal{I}\text{-}\lim(ax_n + by_n) = a\,\mathcal{I}\text{-}\lim x_n + b\,\mathcal{I}\text{-}\lim y_n$ (provided the $\mathcal{I}$-limits of (x_n) and (y_n) exist).

(v) $\mathcal{I}\text{-}\lim(x_n.y_n) = \mathcal{I}\text{-}\lim x_n.\mathcal{I}\text{-}\lim y_n$ (provided the $\mathcal{I}$-limits of (x_n) and (y_n) exist).

Proposition 2.7. *If $\mathcal{I}$ is a maximal ideal then any bounded sequence is $\mathcal{I}$-convergent.*

Proof. Let (x_n) be a bounded sequence. Choose a_0 and b_0 such that $a_0 \le x_n < b_0$. Put $c_0 := \frac{a_0+b_0}{2}$. Then precisely one of the sets $\{n \in \mathbb{N}; x_n \in \langle a_0, c_0)\}$, $\{n \in \mathbb{N}; x_n \in \langle c_0, b_0)\}$ belongs to the ideal $\mathcal{I}$. (They are complement of each other and the ideal $\mathcal{I}$ is maximal.) We choose $\langle a_1, b_1)$ as that subinterval from $\langle a_0, c_0)$ and $\langle c_0, b_0)$ for which $\{n \in \mathbb{N}; x_n \in \langle a_1, b_1)\}$ doesn't belong to $\mathcal{I}$.

Now we again bisect the interval $\langle a_1, b_1)$ by putting $c_1 = \frac{a_1+b_1}{2}$. If both sets $\{n \in \mathbb{N}; x_n \in \langle a_1, c_1)\}$, $\{n \in \mathbb{N}; x_n \in \langle c_1, b_1)\}$ belonged to $\mathcal{I}$, then their union $\{n \in \mathbb{N}; x_n \in \langle a_1, b_1)\}$ would belong to $\mathcal{I}$. Thus at least one of them is not in $\mathcal{I}$. We choose the corresponding interval for $\langle a_2, b_2)$.

By induction we obtain the monotonous sequences (a_n), (b_n) with the same limit $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n := L$ such that for any $k \in \mathbb{N}$ it holds $\{n \in \mathbb{N}; x_n \in \langle a_k, b_k)\} \notin \mathcal{I}$.

We claim that $\mathcal{I}\text{-}\lim x_n = L$. Indeed, for any $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that $\langle a_m, b_m) \subseteq (L-\varepsilon, L+\varepsilon)$, thus $A(\varepsilon) \subseteq \{n \in \mathbb{N}; x_n \notin \langle a_m, b_m)\}$. The set $\{n \in \mathbb{N}; x_n \notin \langle a_m, b_m)\}$ belongs to $\mathcal{I}$ (its complement doesn't belong to $\mathcal{I}$ and $\mathcal{I}$ is maximal) hence $A(\varepsilon) \in \mathcal{I}$ as well. □

If we modify the definition of the $\mathcal{I}$-limit such that we allow $\pm\infty$ as the value of limit, then every (not only bounded) sequence has $\mathcal{I}$-limit for a maximal ideal $\mathcal{I}$.

Proposition 2.8. *If L is a cluster point of a sequence (x_n) (i.e. there is a subsequence x_{n_k} with $\lim_{k\to\infty} x_{n_k} = L$) then there is an ideal $\mathcal{I}$ with $\mathcal{I}\text{-}\lim x_n = L$.*

Proof. It suffices to prove the existence of a proper ideal with $\mathbb{N} \setminus \{n_k; k \in \mathbb{N}\} \in \mathcal{I}$, or, equivalently, with $\{n_k; k \in \mathbb{N}\} \notin \mathcal{I}$. We can take the ideal generated by $\mathbb{N} \setminus \{n_k; k \in \mathbb{N}\}$ and all finite sets. □

The converse inclusion holds as well, thus the set of all cluster points coincides with the set of all $\mathcal{I}$-limits.

Propositions 2.7 and 2.8 are stated in their "filter form" e.g. in [2] and [12].

Using Proposition 2.8 we can see that for our first example – the sequence $(1, 0, 1, 0, \ldots)$ – we can obtain only values 0 or 1 by taking $\mathcal{I}$-limits. (We've already mentioned that the same holds for any multiplicative limit operator.) This doesn't look very intuitive, but everything can be saved using the Cesàro mean. This is the principle of the following construction of Banach limit.

Construction of Banach limit. Again we should note that the following construction is more common with ultrafilters in literature.

Let $\mathcal{I}$ be a maximal ideal on $\mathbb{N}$. Let (x_n) be a bounded sequence. The Cesàro mean of (x_n) is then bounded as well. Thus it has an $\mathcal{I}$-limit and the functional

$$\varphi(x_n) = \mathcal{I}\text{-}\lim \frac{x_1 + \ldots + x_n}{n}$$

is well defined. We claim that φ is a Banach limit.

Continuity, positivity, linearity and extension of the usual limit are clear. It only remains to prove the shift-invariance. To show this, simply observe that $\frac{x_1+\ldots+x_n}{n} - \frac{(Sx)_1+\ldots+(Sx)_n}{n} = \frac{x_1+\ldots+x_n}{n} - \frac{x_2+\ldots+x_{n+1}}{n} = \frac{x_1-x_{n+1}}{n}$. As the sequence (x_n) is bounded, the last expression converges to 0 and the $\mathcal{I}$-limit of both sequences must be the same.

Since we used maximal ideals in our construction of Banach limit, this proof needs Axiom of Choice.

We can now construct a sequence which is not almost convergent. We put $x_n = 0$ for $3^{2k} \leq n < 3^{2k+1}$ and $x_n = 1$ for $3^{2k+1} \leq n < 3^{2k+2}$, $k \in \mathbb{N}$. Let (y_n) be the Cesàro mean of (x_n). By an easy computation one can show that $y_{3^{2k+1}} \leq \frac{1}{3}$ and $y_{3^{2k}} \geq \frac{2}{3}$, therefore at least two different values of a Banach limit can be obtained as an $\mathcal{I}$-limit of y_n for a maximal ideal $\mathcal{I}$.

3 $\mathcal{I}^*$-convergence

We will define one more kind of convergence related to ideals.

Definition 3.1. Let $\mathcal{I}$ be an admissible ideal. We say that a sequence (x_n) of real numbers $\mathcal{I}^*$-converges to L if there exists a set $M = \{m_1 < m_2 < \ldots\}$ such that $\mathbb{N} \setminus M \in \mathcal{I}$ and $\lim\limits_{k\to\infty} x_{m_k} = L$.

Intuitively, this definition can be understood in a such way that there exists a "large" set $M \subseteq \mathbb{N}$ such that x_n converges to L along this set.

$\mathcal{I}^*$ is in some situations better applicable. Relationship of these 2 kinds of convergence is described in the following two results.

Proposition 3.2. *If a sequence is $\mathcal{I}^*$-convergent, then it is $\mathcal{I}$-convergent to the same limit.*

Proof. Let $M = \{m_1 < m_2 < \ldots\}$ be the set such that $\mathbb{N} \setminus M \in \mathcal{I}$ and $\lim\limits_{k\to\infty} x_{m_k} = L$. Then $A(\varepsilon)$ contains only finitely many members of M for any ε. Hence it is union of the set $\mathbb{N} \setminus M$ and a finite set, thus $A(\varepsilon) \in \mathcal{I}$. □

Theorem 3.3 ([14, Theorem 3.3])**.** *$\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are equivalent if and only if the ideal $\mathcal{I}$ satisfies the following condition (AP): For any countable family $\{A_i, i \in \mathbb{N}\}$ of mutually disjoint sets ($A_i \cap A_j = \emptyset$ for $i \neq j$)*

from $\mathcal{I}$ there exists a countable family $\{B_i, i \in \mathbb{N}\}$ such that the symmetric difference $A_i \triangle B_i$ is finite for every $i \in \mathbb{N}$ and $B = \bigcup\limits_{i=1}^{\infty} B_i$ belongs to $\mathcal{I}$.

(AP) stands for additive property, it is similar to σ-additivity - although the ideal is not countable additive, after changing only finitely many elements of each set, we get a system whose union is in $\mathcal{I}$.

It is known, for instance, that the $\mathcal{I}_d$ and $\mathcal{I}_d^*$ convergence are equivalent.

There are examples of ideals for which these 2 types of convergence are not equivalent. Perhaps the simplest example is the example which appears in [14, Theorem 3.1(ii)], [15, Theorem 7] and [13, Example 2.1].

4 Some applications of $\mathcal{I}$-limits

Density measures.

Definition 4.1. A *finitely additive measure* μ on $\mathbb{N}$ is a function $\mu : \mathcal{P}(\mathbb{N}) \to \langle 0, 1\rangle$ such that $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$. Here we moreover require finitely additive measure to be normed, i.e. $\mu(\mathbb{N}) = 1$.

S. Banach proved in [3] that the asymptotic density can be extended to a finitely additive measure on $\mathbb{N}$. (The finitely additive measures on $\mathbb{N}$ extending the asymptotic density are sometimes called *density measures,* [24], [6], [20], [22]). We now show another procedure yielding density measures. (Again, this construction more frequently employs ultrafilters, see e.g. [2, Theorem 8.33], [12, p.207], [6].)

Recall the sequence $\frac{A(n)}{n}$. This sequence is bounded. Thus for any maximal ideal $\mathcal{I}$ there exists the $\mathcal{I}$-limit

$$\mu(A) = \mathcal{I}\text{-}\lim \frac{A(n)}{n}.$$

We claim that μ is a density measure. It is easy to see that μ is defined for all subsets of $\mathbb{N}$ and $\mu(\mathbb{N}) = 1$. The finite additivity follows from the simple fact that for disjoint A, B it holds $(A \cup B)(n) = A(n) + B(n)$.

A density measure can be constructed using any Banach limit instead of $\mathcal{I}$-limit, see [9].

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