Lévy group and density measures

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September 3–5, 2007
Main topics:
▶ density measures
▶ Lévy group

They have applications e.g. in number theory and, more recently, theory of social choice.

We will show that a normalized finitely additive measure on $\mathbb{N}$ extends density if and only if it is preserved by permutations from the Lévy group. We will also present a new characterization of the Lévy group via statistical convergence.
The asymptotic density of $A \subseteq \mathbb{N}$ is defined by

$$d(A) = \lim_{n \to \infty} \frac{A(n)}{n}$$

if this limit exists, where

$$A(n) = |A \cap \{1, 2, \ldots, n\}|.$$

$\mathcal{D}$ = the set of all subsets of $\mathbb{N}$ having asymptotic density

Drawback: Some sets do not have asymptotic density. Is it possible to extend $d$ to a finitely additive measure?
Density measure

We will call a finitely additive normalized measure on \( \mathbb{N} \) briefly a measure.

**Definition**

A *density measure* is a finitely additive measure on \( \mathbb{N} \) which extends the asymptotic density; i.e., it is a function \( \mu : \mathcal{P}(\mathbb{N}) \to [0, 1] \) satisfying the following conditions:

(a) \( \mu(\mathbb{N}) = 1 \);
(b) \( \mu(A \cup B) = \mu(A) + \mu(B) \) for all disjoint \( A, B \subseteq \mathbb{N} \);
(c) \( \mu|_{\mathcal{D}} = d \).
The term density measures was probably coined by Dorothy Maharam [M].

Density measures were studied by many authors, e.g.

- Blass, Frankiewicz, Plebanek and Ryll–Nardzewski [BFPRN]
- van Douwen [vD]
- Šalát and Tijdeman in [ŠT].
Existence of density measures

Existence of density measures is usually proved using either Hahn-Banach theorem or ultrafilters.

If $\mathcal{F}$ is any free ultrafilter on $\mathbb{N}$ then

$$\mu_{\mathcal{F}}(A) = \mathcal{F}\text{-lim} \frac{A(n)}{n}$$

is a density measure

$$\mathcal{F}\text{-lim} a_n = L \iff \{ n \in \mathbb{N}; |a_n - L| < \varepsilon \} \in \mathcal{F} \text{ for each } \varepsilon > 0$$
Lévy group

**Definition**

The Lévy group \( G \) is the group of all permutations \( \pi \) of \( \mathbb{N} \) satisfying

\[
\lim_{n \to \infty} \frac{\{|k; k \leq n < \pi(k)\}|}{n} = 0. \tag{1.1}
\]

\[
\pi \in G \iff \lim_{n \to \infty} \frac{A(n) - (\pi A)(n)}{n} = 0 \text{ for all } A \subseteq \mathbb{N}. \tag{1.2}
\]
Equivalent characterization of $\mathcal{G}$

\[
\pi \in \mathcal{G} \iff \limstat_{n \to \infty} \frac{\pi(n)}{n} = 1
\]  

(1.3)

Recall that $\limstat_{n \to \infty} x_n = L$ iff for every $\varepsilon > 0$ the set $A_\varepsilon = \{n; |x_n - L| \geq \varepsilon\}$ has zero asymptotic density ($d(A_\varepsilon) = 0$).

$\mathcal{F}$-lim for $\mathcal{F} = \{A \subseteq \mathbb{N}; d(A) = 1\}$
**Theorem**

A measure $\mu$ on $\mathbb{N}$ is a density measure if and only if it is $\mathcal{G}$-invariant, i.e., $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi \in \mathcal{G}$. 

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**Introduction**

**Main results**

**Applications**

**References**
**$\mathcal{G}$-invariance**

We use van Douwen’s result [vD, Theorem 1.12]:

**Theorem**

A measure $\mu$ on $\mathbb{N}$ is a density measure if and only if $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n \to \infty} \frac{\pi(n)}{n} = 1.$$  \hspace{1cm} (2.1)

$$\Rightarrow \quad (1.3)$$

$\mathcal{G}$-invariant $\Rightarrow$ density measure

This implication follows also from a result of Blümlinger and Obata [BO, Theorem 2].
The proof of the opposite implication uses the following result (Fridy [F, Theorem 1] or Šalát [Š, Lemma 1.1]):

**Theorem**

A sequence \((x_n)\) is statistically convergent to \(L \in \mathbb{R}\) if and only if there exists a set \(A\) such that \(d(A) = 1\) and the sequence \(x_n\) converges to \(L\) along the set \(A\), i.e., \(L\) is limit of the subsequence \((x_n)_{n \in A}\).
Basic idea of the proof

If \( \pi \) fulfills (1.3)

\[
\pi \in \mathcal{G} \iff \lim \text{stat} \frac{\pi(n)}{n} = 1
\]

it can be modified to \( \psi \) fulfilling (2.1)

\[
\lim_{n \to \infty} \frac{\psi(n)}{n} = 1
\]

and \( \pi A \) and \( \psi A \) differ only in a set of zero density.

\[
\mu(A) = \mu(\psi A) = \mu(\pi A)
\]
Proposition

If $\pi$ is a permutation such that every density measure is $\pi$-invariant, i.e., $\mu(\pi A) = \pi A$ for every $A \subseteq \mathbb{N}$ and every density measure $\mu$, then $\pi \in \mathcal{G}$. 
Blümlinger [B]:

\[ 2\mathcal{F} = \{ B \subseteq \mathbb{N}; B \supseteq 2A \text{ for some } A \in \mathcal{F} \} \]

(the ultrafilter given by the base \( \{2A; A \in \mathcal{F}\} \))

\[
\mu(A) = 2 \ (2\mathcal{F})\text{-lim} \frac{A(n)}{n} - \mathcal{F}\text{-lim} \frac{A(n)}{n}
\]

is a density measure

Let \( A = \bigcup_{i=1}^{\infty} \{2^2^i, 2^2^i + 1, \ldots, 2.2^2^i - 1\} \) and \( \{2^2^i; i \in \mathbb{N}\} \in \mathcal{F} \).

Then \( \mu(A) = 1 \) and \( \overline{d}(A) = \frac{1}{2} \).
An interesting density measure

A negative answer van Douwen [vD, Question 7A.1]: Does $\mu(A) \leq \bar{d}(A)$ hold for every density measure?

Counterexample to the following claim of Lauwers [L, p.46]:

*Every density measure can be expressed in the form*

$$\mu_\varphi(A) = \int_{\beta\mathbb{N}^*} \mathcal{F}\text{-lim} \left( \frac{A(n)}{n} \right) \, d\varphi(\mathcal{F}), \quad A \subseteq \mathbb{N} \quad (3.1)$$

*for some probability Borel measure $\varphi$ on the set of all free ultrafilters $\beta\mathbb{N}^*$.*
Šalát and Tijdeman [ŠT]: Has every density measure the following properties?

a) If \( A(n) \leq B(n) \) for all \( n \in \mathbb{N} \) then \( \mu(A) \leq \mu(B) \) (where \( A, B \subseteq \mathbb{N} \)).

b) If \( \lim_{n \to \infty} \frac{A(n)}{B(tn)} = 1 \) then \( \mu(A) = t\mu(B) \) (where \( A, B \subseteq \mathbb{N} \) and \( t \in \mathbb{R} \)).

Answer to both these questions is negative.

a) If \( \mu(A) > \overline{d}(A) \) and \( d(B) \in (\overline{d}(A), \mu(A)) \) then \( B(n) > A(n) \) for \( n > n_0 \), but \( \mu(A) > d(B) = \mu(B) \).

b) In the preceding example we have \( \mu(A) = 1 \) and \( \mu(2A) = 0 \).
Thanks for your attention!

The preprint [SZ] presented here, as well as the text of this talk and these slides can be found at:
http://thales.doa.fmph.uniba.sk/sleziak/papers/

Email: sleziak@fmph.uniba.sk
M. Blümlinger. 
Lévy group action and invariant measures on $\beta \mathbb{N}$. 

A note on extensions of asymptotic density. 

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D. Maharam.
Finitely additive measures on the integers.

T. Šalát.
On statistically convergent sequences of real numbers.
*Mathematica Slovaca, 30:139–150, 1980.*

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