# Lévy group and density measures

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# Overview

Main topics:

- density measures
- Lévy group

They have applications e.g. in number theory and, more recently, theory of social choice.

We will show that a normalized finitely additive measure on $\mathbb{N}$ extends density if and only if it is preserved by permutations from the Lévy group. We will also present a new characterization of the Lévy group via statistical convergence.

# Asymptotic density

The asymptotic density of $A \subseteq \mathbb{N}$ is defined by

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

if this limit exists, where

$$A(n) = |A \cap \{1, 2, \dots, n\}|.$$

$\mathcal{D}$ = the set of all subsets of $\mathbb{N}$ having asymptotic density

Drawback: Some sets do not have asymptotic density.
Is it possible to extend $d$ to a finitely additive measure?

# Density measure

We will call a finitely additive normalized measure on $\mathbb{N}$ briefly a *measure*.

## Definition

A *density measure* is a finitely additive measure on $\mathbb{N}$ which extends the asymptotic density; i.e., it is a function $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ satisfying the following conditions:

(a) $\mu(\mathbb{N}) = 1$;

(b) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$;

(c) $\mu|_{\mathcal{D}} = d$.

# References for density measures

The term density measures was probably coined by Dorothy Maharam [M].

Density measures were studied by many authors, e.g.

- Blass, Frankiewicz, Plebanek and Ryll–Nardzewski [BFPRN]
- van Douwen [vD]
- Šalát and Tijdeman in [ŠT].

# Existence of density measures

Existence of density measures is usually proved using either Hahn-Banach theorem or ultrafilters.

If $\mathcal{F}$ is any free ultrafilter on $\mathbb{N}$ then

$$\mu_{\mathcal{F}}(A) = \mathcal{F}\text{-}\lim \frac{A(n)}{n}$$

is a density measure

$\mathcal{F}\text{-}\lim a_n = L \Leftrightarrow \{n \in \mathbb{N}; |a_n - L| < \varepsilon\} \in \mathcal{F}$ for each $\varepsilon > 0$

# Lévy group

## Definition

The *Lévy group* $\mathcal{G}$ is the group of all permutations $\pi$ of $\mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{\left|\{k;\ k \leq n < \pi(k)\}\right|}{n} = 0. \tag{1.1}$$

$$\pi \in \mathcal{G} \ \Leftrightarrow\ \lim_{n\to\infty} \frac{A(n) - (\pi A)(n)}{n} = 0 \text{ for all } A \subseteq \mathbb{N} \tag{1.2}$$

# Equivalent characterization of $\mathcal{G}$

$$\pi \in \mathcal{G} \;\Leftrightarrow\; \operatorname*{limstat}_{n\to\infty} \frac{\pi(n)}{n} = 1 \tag{1.3}$$

Recall that $\operatorname*{limstat}_{n\to\infty} x_n = L$ iff for every $\varepsilon > 0$ the set

$$A_\varepsilon = \{n;\ |x_n - L| \geq \varepsilon\}$$

has zero asymptotic density ($d(A_\varepsilon) = 0$).

$\mathcal{F}$-lim for $\mathcal{F} = \{A \subseteq \mathbb{N}; d(A) = 1\}$

# $\mathcal{G}$-invariance

**Theorem**

*A measure $\mu$ on $\mathbb{N}$ is a density measure if and only if it is $\mathcal{G}$-invariant, i.e., $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi \in \mathcal{G}$.*

# $\mathcal{G}$-invariance

We use van Douwen's result [vD, Theorem 1.12]:

**Theorem**

*A measure $\mu$ on $\mathbb{N}$ is a density measure if and only if $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi : \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{\pi(n)}{n} = 1. \tag{2.1}$$

$$(2.1) \Rightarrow (1.3)$$
$$\mathcal{G}\text{-invariant} \Rightarrow \text{density measure}$$

This implication follows also from a result of Blümlinger and Obata [BO, Theorem 2].

# $\mathcal{G}$-invariance

The proof of the opposite implication uses the following result (Fridy [F, Theorem 1] or Šalát [Š, Lemma 1.1]):

## Theorem

*A sequence $(x_n)$ is statistically convergent to $L \in \mathbb{R}$ if and only if there exists a set $A$ such that $d(A) = 1$ and the sequence $x_n$ converges to $L$ along the set $A$, i.e., $L$ is limit of the subsequence $(x_n)_{n\in A}$.*

# Basic idea of the proof

If $\pi$ fulfills $(1.3)$

$$\pi \in \mathcal{G} \;\Leftrightarrow\; \operatorname*{limstat}_{n\to\infty} \frac{\pi(n)}{n} = 1$$

it can be modified to $\psi$ fulfilling $(2.1)$

$$\lim_{n\to\infty} \frac{\psi(n)}{n} = 1$$

and $\pi A$ and $\psi A$ differ only in a set of zero density.

$$\mu(A) = \mu(\psi A) = \mu(\pi A)$$

# Lévy group and invariance of density measures

## Proposition

*If $\pi$ is a permutation such that every density measure is $\pi$-invariant, i.e., $\mu(\pi A) = \pi A$ for every $A \subseteq \mathbb{N}$ and every density measure $\mu$, then $\pi \in \mathcal{G}$.*

# An interesting density measure

Blümlinger [B]:
$2\mathcal{F} = \{B \subseteq \mathbb{N}; B \supseteq 2A \text{ for some } A \in \mathcal{F}\}$
(the ultrafilter given by the base $\{2A; A \in \mathcal{F}\}$)

$$\mu(A) = 2\,(2\mathcal{F})\text{-}\lim \frac{A(n)}{n} - \mathcal{F}\text{-}\lim \frac{A(n)}{n}$$

is a density measure
Let $A = \bigcup_{i=1}^{\infty} \{2^{2^i}, 2^{2^i}+1, \ldots, 2.2^{2^i}-1\}$ and $\{2^{2^i}; i \in \mathbb{N}\} \in \mathcal{F}$.
Then $\mu(A) = 1$ and $\overline{d}(A) = \frac{1}{2}$.

# An interesting density measure

A negative answer van Douwen [vD, Question 7A.1]:
Does $\mu(A) \leq \overline{d}(A)$ hold for every density measure?
Counterxample to the following claim of Lauwers [L, p.46]:
*Every density measure can be expressed in the form*

$$\mu_\varphi(A) = \int_{\beta\mathbb{N}^*} \mathcal{F}\text{-lim} \frac{A(n)}{n} \,\mathrm{d}\varphi(\mathcal{F}), \qquad A \subseteq \mathbb{N} \tag{3.1}$$

*for some probability Borel measure $\varphi$ on the set of all free ultrafilters $\beta\mathbb{N}^*$.*

# An interesting density measure

Šalát and Tijdeman [ŠT]: Has every density measure the following properties?
a) If $A(n) \leq B(n)$ for all $n \in \mathbb{N}$ then $\mu(A) \leq \mu(B)$ (where $A, B \subseteq \mathbb{N}$).
b) If $\lim\limits_{n\to\infty} \frac{A(n)}{B(tn)} = 1$ then $\mu(A) = t\mu(B)$ (where $A, B \subseteq \mathbb{N}$ and $t \in \mathbb{R}$).
Answer to both these questions is negative.

a) If $\mu(A) > \overline{d}(A)$ and $d(B) \in (\overline{d}(A), \mu(A))$ then $B(n) > A(n)$ for $n > n_0$, but $\mu(A) > d(B) = \mu(B)$.
b) In the preceding example we have $\mu(A) = 1$ and $\mu(2A) = 0$.

# Thanks for your attention!

The preprint [SZ] presented here, as well as the text of this talk and these slides can be found at:
http://thales.doa.fmph.uniba.sk/sleziak/papers/

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