LÉVY GROUP AND DENSITY MEASURES

MARTIN SLEZIAK AND MILOŠ ZIMAN

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1. Introduction

In this talk I would like to speak about density measures and Lévy group. Density measures are extensions of asymptotic density to the whole power set $\mathcal{P}(\mathbb{N})$. Lévy group $\mathcal{G}$ is a certain group of permutations of $\mathbb{N}$. (Precise definitions will follow immediately.)

Both the Lévy group and the density measures have found applications in number theory and, more recently, in the theory of social choice (see e.g. Fey [Fe], Lauwers [L]).

The main result is showing that the a finitely additive measure on integers extends density if and only if it is $\mathcal{G}$-invariant. (We will later see that this property characterizes the Lévy group as well.) We will also present a new characterization of the Lévy group via statistical convergence.

Let us start with defining the two central notions of this talk.

2. Preliminaries

2.1. **Density measures.** Recall that the asymptotic density of $A \subseteq \mathbb{N}$ is defined by

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

if this limit exists. The asymptotic density is one of standard tools for measuring the size of subsets of $\mathbb{N}$. However, the drawback of the asymptotic density is that not every subset of $\mathbb{N}$ has the asymptotic density. (The above limit need not exist.) Therefore it is very natural to ask whether it is possible to extend the asymptotic density to a finitely additive measure on $\mathbb{N}$. The set of all subsets of $\mathbb{N}$ having asymptotic density will be denoted by $\mathcal{D}$.

Definition 2.1. A *density measure* is a finitely additive measure on $\mathbb{N}$ which extends the asymptotic density; i.e., it is a function $\mu : \mathcal{P}(\mathbb{N}) \to [0, 1]$ satisfying the following conditions:

(a) $\mu(\mathbb{N}) = 1$;
(b) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$;
(c) $\mu|_{\mathcal{D}} = d$.

(For the sake of brevity we will call the functions $\mu : \mathcal{P}(\mathbb{N}) \to [0, 1]$ fulfilling the conditions (a) and (b) from the preceding definition *measures*.)

The term density measures was probably coined by Dorothy Maharam [M]. They were studied (among many others) by Blass, Frankiewicz, Plebanek and Ryll–Nardzewski in [BFPRN], van Douwen in [vD] or Šalát and Tijdeman in [ŠT].

The existence of density measures is usually proved either using Hahn-Banach theorem or using ultrafilters. We will use the following approach several times:

For any $\mathcal{F} \in \beta\mathbb{N}^*$ the function

$$\mu_{\mathcal{F}}(A) = \mathcal{F}\text{-lim}\,\frac{A(n)}{n}$$

is a density measure (see e.g. [BŠ, Theorem 8.33], [HJ, p.207]).

(Recall that $\mathcal{F}\text{-lim}\, a_n = L \Leftrightarrow \{n \in \mathbb{N}; |a_n - L| < \varepsilon\} \in \mathcal{F}$.)

2.2. **Lévy group.** We will also use a group of permutations of $\mathbb{N}$ which is related to the asymptotic density.

Definition 2.2. The *Lévy group* $\mathcal{G}$ is the group of all permutations π of $\mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{\big|\{k;\, k \le n < \pi(k)\}\big|}{n} = 0. \tag{2.1}$$

We will use the following characterizations of Lévy group [B, Lemma 2].

A permutation $\pi \in \mathcal{G}$ if and only if

$$\lim_{n\to\infty} \frac{A(n) - (\pi A)(n)}{n} = 0 \tag{2.2}$$

for all $A \subseteq \mathbb{N}$.

We have found an interesting connection between the Lévy group and statistical convergence. This new characterization of $\mathcal{G}$ has proved to be useful in the proof of the main theorem.

Let us first recall the definition of statistical convergence.

We say that $\operatorname*{limstat}_{n\to\infty} x_n = L$ iff for every $\varepsilon > 0$ the set

$$A_\varepsilon = \{n;\, |x_n - L| \ge \varepsilon\}$$

has zero asymptotic density ($d(A_\varepsilon) = 0$).

The statistical limit is in fact $\mathcal{F}$-limit for the filter $\mathcal{F}$ consisting of all sets with $d(A) = 1$. (This filter is not an ultrafilter, hence there exist sequences without statistical limit. Of course, the statistical convergence can be formulated using ideals an $\mathcal{I}$-convergence too.)

Theorem 2.3. *A permutation $\pi : \mathbb{N} \to \mathbb{N}$ belongs to $\mathcal{G}$ if and only if*

$$\operatorname*{limstat}_{n\to\infty} \frac{\pi(n)}{n} = 1. \tag{2.3}$$

3. $\mathcal{G}$-invariance

The main result of this talk is the following theorem.

Theorem 3.1. *A measure μ on $\mathbb{N}$ is a density measure if and only if it is $\mathcal{G}$-invariant, i.e., $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi \in \mathcal{G}$.*

The proof employs van Douwen's result [vD, Theorem 1.12]:

Theorem 3.2. *A measure μ on $\mathbb{N}$ is a density measure if and only if $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi : \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{\pi(n)}{n} = 1. \tag{3.1}$$

With this result at hand, one half of Theorem 3.1 is easy. Clearly, if a permutation π fulfills (3.1), then it fulfills also (2.3). This yields the implication:
π is $\mathcal{G}$-invariant $\Rightarrow$ π is a density measure.

Let us note that this result can also be deduced from Blümlinger and Obata [BO, Theorem 2], where it was proved by different means. This theorem deals with linear functionals on the space $\mathcal{D}$ of Césaro summable sequences, but it can be applied to our situation. They have shown that every $\mathcal{G}$-invariant linear functional on $\mathcal{D}$ is a multiple Césaro mean.

Every measure assigns some value to characteristic sequences of subsets of $\mathbb{N}$. We can extend it to a linear functional on ℓ_∞ without violating the $\mathcal{G}$-invariance and then restrict this functional to $\mathcal{D}$. Since this functional is normalized, the restriction is precisely the Césaro mean. In terms of measures, any $\mathcal{G}$-invariant extends the density.

In this short talk we will not go into details of the proof of the opposite implications. We just note that main components of this proof are our characterization of the Lévy group using the statistical convergence (Theorem 2.3) and the following result (see Fridy [Fr, Theorem 1] or Šalát [Š, Lemma 1.1]):

Theorem 3.3. *A sequence* (x_n) *is statistically convergent to* $L \in \mathbb{R}$ *if and only if there exists a set* A *such that* $d(A) = 1$ *and the sequence* x_n *converges to* L *along the set* A*, i.e.,* L *is limit of the subsequence* $(x_n)_{n\in A}$*.*

The basic idea of the proof is that if a permutation π fulfills (2.3) then it can be modified to a new permutation ψ fulfilling (3.1) in such a manner that $\pi(A)$ and $\psi(A)$ differ only in a set having zero density. By the van Douwen's result (Theorem 3.2) the permutation ψ preserves density measure and we can use it to show that the measure of the set A will be preserved by the permutation π as well.

4. Applications

4.1. **Characterization of Lévy group.** By theorem 3.1 every density measure is π-invariant for permutations $\pi \in \mathcal{G}$. It is natural to ask whether there are other permutations with this property. Proposition 4.1 states that this property characterizes Lévy group.

Proposition 4.1. *If* π *is a permutation such that every density measure is* π*-invariant, i.e.,* $\mu(\pi A) = \pi A$ *for every* $A \subset \mathbb{N}$ *and every density measure* μ*, then* $\pi \in \mathcal{G}$*.*

4.2. **An interesting density measure.** We will closed this talk with an interesting example of a density measure which answers several questions posed by van Douwen [vD] and Šalát and Tijdeman [ŠT]. This example of density measure was defined in the paper of Blümlinger [B].

Example 4.2. Let $\mathcal{F}$ be any ultrafilter. By $2\mathcal{F}$ we denote the ultrafilter given by the base $\{2A; A \in \mathcal{F}\}$, i.e., $2\mathcal{F} = \{B \subseteq \mathbb{N}; B \supseteq 2A$ for some $A \in \mathcal{F}\}$. Let us define μ by

$$\mu(A) = 2\,(2\mathcal{F})\text{-}\lim \frac{A(n)}{n} - \mathcal{F}\text{-}\lim \frac{A(n)}{n}.$$

It can be shown that μ is a density measure.

Now let us consider the set $A = \bigcup\limits_{i=1}^{\infty} \{2^{2^i}, 2^{2^i}+1, \ldots, 2.2^{2^i}-1\}$. Note that $A(2.2^{2^i}-1) \geq \frac{1}{2}$ and $A(2^{2^i}-1) \leq \frac{1}{2^{i-3}}$ for any positive integer i. It can be shown that $\overline{d}(A) = \frac{1}{2}$ and $\mu(A) = 1$ for any ultrafilter containing the set $\{2^{2^i}; i \in \mathbb{N}\}$.

This answers the Van Douwen's question [vD, Question 7A.1] whether $\mu(A) \leq \overline{d}(A)$ for every density measure. The same density measure μ is a counterexample to the following claim of Lauwers [L, p.46]:
Every density measure can be expressed in the form

$$\mu_\varphi(A) = \int_{\beta\mathbb{N}^*} \mathcal{F}\text{-}\lim \frac{A(n)}{n} \, \mathrm{d}\varphi(\mathcal{F}), \qquad A \subseteq \mathbb{N} \tag{4.1}$$

for some probability Borel measure φ on the set of all free ultrafilters $\beta\mathbb{N}^*$.
It is easy to note that if this claim were true the answer to van Douwen's question would be positive.

Šalát and Tijdeman have posed another question concerning the density measures [ŠT, p.201]. They ask whether every density measure has the following properties:
a) If $A(n) \leq B(n)$ for all $n \in \mathbb{N}$ then $\mu(A) \leq \mu(B)$ (where $A, B \subseteq \mathbb{N}$).
b) If $\lim\limits_{n\to\infty} \frac{A(n)}{B(tn)} = 1$ then $\mu(A) = t\mu(B)$ (where $A, B \subseteq \mathbb{N}$ and $t \in \mathbb{R}$).
(The authors of [ŠT] conjectured that there exist density measures that do not fulfill a) and b). We will see that this conjecture was right.)

Clearly, any density measure of the form (4.1) has both these properties.

The question a) is closely related to van Douwen's question. Clearly, if a set A fulfills $\overline{d}(A) < \mu(A)$ there is a set B having asymptotic density $d(B) \in (\overline{d}(A), \mu(A))$. Since $d(B) > \overline{d}(A)$, there exists n_0 such that $B(n) \geq A(n)$ for $n > n_0$. Since changing only finitely many elements influences neither asymptotic density nor density measure, any such pair of sets yields a counterexample to the property a).

It is easy to verify that for the set A from the preceding example (and the measure given by an ultrafilter containing $\{2^{2^i}; i \in \mathbb{N}\}$) we get $\mu(2A) = 0$ and $\mu(A) = 1$. This shows that property b) is not valid in general. (A different density measure μ and a set A with $\mu(2A) \neq \frac{1}{2}\mu(A)$ was given by Van Douwen [vD, Example 5.6, Case 2].)

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