# LÉVY GROUP AND DENSITY MEASURES

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Abstract. We will deal with finitely additive measures on integers extending the asymptotic density. We will study their relation to the Lévy group $\mathcal{G}$ of permutations of $\mathbb{N}$. Using a new characterization of the Lévy group $\mathcal{G}$ we will prove that a finitely additive measure extends density if and only if it is $\mathcal{G}$-invariant.

## Introduction

Lévy group $\mathcal{G}$ is a group of permutations of positive integers which is tightly linked to the notion of asymptotic density. The connection between this group and asymptotic density (as well as several related notions) was studied e.g. by Blümlinger [3], Obata [12, 4]. Some other groups related to (extensions of) asymptotic density were also studied, we can mention recent papers of Nathanson and Parikh [11] or Giuliano Antonini and Paštéka [7].

In this paper we will study the connection between the Lévy group and finitely additive measures on integers extending the asymptotic density. We will call such measures density measures. The term density measures was probably coined by Dorothy Maharam [10]. They were studied (among many others) by Blass, Frankiewicz, Plebanek and Ryll–Nardzewski in [2], van Douwen in [15] or Šalát and Tijdeman in [14].

Both the Lévy group and the density measures have found applications in number theory and, more recently, in the theory of social choice (see e.g. Fey [5], Lauwers [9]).

The main purpose of this paper is to show that the density measures are precisely the finitely additive measures which are $\mathcal{G}$-invariant. The $\mathcal{G}$-invariant measures were studied by Blümlinger in [3]. Blümlinger and Obata deal with the $\mathcal{G}$-invariant extensions of Cesáro mean in [4].

We also obtain an interesting characterization of the Lévy group in terms of statistical convergence.

## 1. Preliminaries

We start by defining the two central notions of this paper – the Lévy group and the density measures – and mentioning a few necessary facts about them.

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**Definition 1.1.** The *asymptotic density* of a set $A \subseteq \mathbb{N}$ is defined by $d(A) = \lim\limits_{n\to\infty} \frac{A(n)}{n}$, where $A(n) = \big|A \cap [1,n]\big|$. We denote the collection of sets having asymptotic density by $\mathcal{D}$.

A *density measure* is a finitely additive measure on $\mathbb{N}$ which extends the asymptotic density; i.e., it is a function $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ satisfying the following conditions:

(a) $\mu(\mathbb{N}) = 1$;
(b) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$;
(c) $\mu|_{\mathcal{D}} = d$.

(Throughout the paper a *measure* will mean a set function on $\mathcal{P}(\mathbb{N})$ fulfilling (a) and (b).)

Density measures can be constructed using a limit along an ultrafilter. The set of all free ultrafilters on $\mathbb{N}$ will be denoted by $\beta\mathbb{N}^*$. For $\mathcal{F} \in \beta\mathbb{N}^*$ and a bounded sequence $(x_n)$ we denote by $\mathcal{F}\text{-}\lim x_n$ the limit of this sequence along the ultrafilter $\mathcal{F}$ (see [1, p.122, Definition 8.23], [8, p.206, Definition 2.7] for definition and basic properties of a limit along an ultrafilter).

For any $\mathcal{F} \in \beta\mathbb{N}^*$ the function

$$\mu_{\mathcal{F}}(A) = \mathcal{F}\text{-}\lim \frac{A(n)}{n}$$

is a density measure (see e.g. [1, Theorem 8.33], [8, p.207]). We will use this construction of a density measure several times. Another possibility to show the existence of density measures relies on Hahn-Banach theorem.

**Definition 1.2.** The *Lévy group* $\mathcal{G}$ is the group of all permutations $\pi$ of $\mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{\big|\{k;\, k \le n < \pi(k)\}\big|}{n} = 0. \tag{1.1}$$

We will need the following characterization of the Lévy group.

**Lemma 1.3.** [3, Lemma 2] *A permutation $\pi$ of $\mathbb{N}$ belongs to $\mathcal{G}$ if and only if*

$$\lim_{n\to\infty} \frac{A(n) - (\pi A)(n)}{n} = 0 \tag{1.2}$$

*for each $A \subseteq \mathbb{N}$.*

For more information about the Lévy group see [3] and [12].

## 2. $\mathcal{G}$-invariance

To answer the question of $\mathcal{G}$-invariance of a density measure we use the representation of the Lévy group $\mathcal{G}$ with the help of statistical convergence.

We say that a real sequence $(x_n)$ *converges statistically to* $L$ ($\operatorname*{limstat}\limits_{n\to\infty} x_n = L$) if for every $\varepsilon > 0$ the set

$$A_\varepsilon = \{n;\, |x_n - L| \ge \varepsilon\}$$

has zero density ($d(A_\varepsilon) = 0$).

The following result is well-known (see Fridy [6, Theorem 1] or Šalát [13, Lemma 1.1]).

**Theorem 2.1.** *A sequence $(x_n)$ is statistically convergent to $L \in \mathbb{R}$ if and only if there exists a set $A$ such that $d(A) = 1$ and the sequence $(x_n)$ converges to $L$ along the set $A$, i.e., $L$ is limit of the subsequence $(x_n)_{n\in A}$.*

**Theorem 2.2.** *A permutation $\pi : \mathbb{N} \to \mathbb{N}$ belongs to $\mathcal{G}$ if and only if*

$$\operatorname*{limstat}_{n\to\infty} \frac{\pi(n)}{n} = 1. \tag{2.1}$$

*Proof.* Let $\pi \in \mathcal{G}$. Suppose we are given $\varepsilon > 0$. Denote

$$\begin{aligned} A &= \{k;\, \pi(k) - k > \varepsilon k\} = \{k;\, \pi(k) > (1+\varepsilon)k\}, \\ B &= \{k;\, k - \pi(k) > \varepsilon k\} = \{k;\, \pi(k) < (1-\varepsilon)k\}. \end{aligned}$$

Obviously $C = A \cup B = \{k;\, \left|\frac{\pi(k)}{k} - 1\right| > \varepsilon\}$. So it suffices to show that $d(C) = 0$, i.e., $\lim\limits_{n\to\infty} C(n)/n = 0$.

If $\pi(k) \leq n$ for some $k \in A$, we get $(1+\varepsilon)k \leq \pi(k) \leq n$ and $k \leq \frac{n}{1+\varepsilon}$. Hence $(\pi A)(n) \leq A\left(\left\lfloor\frac{n}{1+\varepsilon}\right\rfloor\right)$ and Lemma 1.3 yields

$$\begin{aligned} \limsup_{n\to\infty} \frac{A(n)}{n} &= \limsup_{n\to\infty} \frac{\pi A(n)}{n} \leq \limsup_{n\to\infty} \frac{A\left(\left\lfloor\frac{n}{1+\varepsilon}\right\rfloor\right)}{n} \\ &= \limsup_{n\to\infty} \frac{A\left(\left\lfloor\frac{n}{1+\varepsilon}\right\rfloor\right)}{\left\lfloor\frac{n}{1+\varepsilon}\right\rfloor} \frac{\left\lfloor\frac{n}{1+\varepsilon}\right\rfloor}{n} \leq \limsup_{n\to\infty} \frac{A(n)}{n} \frac{1}{1+\varepsilon}. \end{aligned}$$

This implies immediately $d(A) = \limsup\limits_{n\to\infty} \frac{A(n)}{n} = 0$.

To show that $d(B) = 0$ we can proceed analogously. Another possibility is to notice that $B = \{k; \pi(k) < (1-\varepsilon)k\} = \pi^{-1}(\{l; l < (1-\varepsilon)\pi^{-1}(l)\}) \subseteq \pi^{-1}(\{l; (1+\varepsilon)l < \pi^{-1}(l)\})$ and repeat the same argument for the permutation $\pi^{-1} \in \mathcal{G}$.

Thus we get $d(C) = d(A) + d(B) = 0$.

To prove the reverse implication assume that $\pi$ satisfies (2.1). As before, taking $\varepsilon > 0$ let us denote $A = \{k;\, \pi(k) > (1+\varepsilon)k\}$. Then $d(A) = 0$.

If $k \leq n < \pi(k)$, then either $k \in A$ or $k > \frac{n}{1+\varepsilon}$ (otherwise $\pi(k) \leq (1+\varepsilon)k \leq n$, contradicting $n < \pi(k)$). So

$$\begin{gathered} |\{k;\, k \leq n < \pi(k)\}| \leq A(n) + n\left(1 - \frac{1}{1+\varepsilon}\right) + 1 \leq A(n) + n\varepsilon + 1, \\ \limsup_{n\to\infty} \frac{|\{k;\, k \leq n < \pi(k)\}|}{n} \leq \varepsilon + \lim_{n\to\infty} \frac{A(n)}{n} = \varepsilon. \end{gathered}$$

Since $\varepsilon$ can be chosen arbitrarily small, we get

$$\lim_{n\to\infty} \frac{|\{k;\, k \leq n < \pi(k)\}|}{n} = 0,$$

and $\pi \in \mathcal{G}$. □

Van Douwen (see [15, Theorem 1.12]) characterized density measures using invariance with respect to a particular kind of permutations.

**Theorem 2.3.** *A measure $\mu$ on $\mathbb{N}$ is a density measure if and only if $\mu(A)=\mu(\pi A)$ for all $A\subseteq\mathbb{N}$ and all permutations $\pi:\mathbb{N}\to\mathbb{N}$ such that*

$$\lim_{n\to\infty}\frac{\pi(n)}{n}=1. \tag{2.2}$$

Van Douwen proved even more, but here we only need the above result.

One can see easily that if a permutation $\pi$ fulfills (2.2), then it fulfills also (2.1). Using this fact we get

**Proposition 2.4.** *If a measure $\mu$ on $\mathbb{N}$ is $\mathcal{G}$-invariant, then it is a density measure.*

This result can also be deduced from Blümlinger and Obata [4, Theorem 2], where it was proved by different means.

Next we will show that the reverse of this proposition is true as well.

**Proposition 2.5.** *If $\pi\in\mathcal{G}$ and $\mu$ is a density measure, then for each $A\subseteq\mathbb{N}$,*

$$\mu(\pi A)=\mu(A). \tag{2.3}$$

*Proof.* Let $\pi\in\mathcal{G}$, $A\subseteq\mathbb{N}$ and $B=\pi A$. Define

$$A'=A\setminus(A\cap B)=\{a_1<a_2<a_3<\dots\},$$
$$B'=B\setminus(A\cap B)=\{b_1<b_2<b_3<\dots\}.$$

As $A=A'\cup(A\cap B)$ and $B=B'\cup(A\cap B)$, it suffices to prove that $\mu(A')=\mu(B')$.

Without loss of generality we can assume that $A'$ and $B'$ are infinite, otherwise we get $d(A')=d(B')=0$ (if one of the sets is finite, then it has zero density and since $\lim\limits_{n\to\infty}\frac{A'(n)-B'(n)}{n}=\lim\limits_{n\to\infty}\frac{A(n)-B(n)}{n}=0$, the other one has the same density) and $\mu(A)=\mu(A\cap B)=\mu(B)$.

Let us define a permutation $\varphi:\mathbb{N}\to\mathbb{N}$ by

$$\begin{aligned}\varphi(n)&=n && \text{if } n\notin A'\cup B';\\ \varphi(a_i)&=b_i && \text{for } i=1,2,\dots;\\ \varphi(b_i)&=a_i && \text{for } i=1,2,\dots\end{aligned}$$

We claim that $\varphi\in\mathcal{G}$. Indeed, as one of the sets $\{i;\, a_i\le n<b_i\}$, $\{i;\, b_i\le n<a_i\}$ is empty, we get $|\{k;\, k\le n<\varphi(k)\}|=|\{i;\, a_i\le n<b_i\}|+|\{i;\, b_i\le n<a_i\}|=|A'(n)-B'(n)|=|A(n)-B(n)|$ and $\lim\limits_{n\to\infty}\frac{|A(n)-B(n)|}{n}=0$ by Lemma 1.3. Moreover $\varphi(A')=B'$ and $\varphi^{-1}=\varphi$.

By Theorems 2.1 and 2.2 there exists a set $F$ such that $d(F)=0$ and

$$\lim_{\substack{n\in\mathbb{N}\setminus F\\ n\to\infty}}\frac{\varphi(n)}{n}=1. \tag{2.4}$$

Set $F':=A'\cap(F\cup\varphi F)$ and $E:=F'\cup\varphi F'$. Clearly $F'\subseteq A'$ and $\varphi F'=B'\cap(F\cup\varphi F)\subseteq B'$. Since the permutations in $\mathcal{G}$ preserve density, we get $d(\varphi F)=0$ and $d(F\cup\varphi F)=0$. Thus $d(F')=d(\varphi F')=0$.

We modify the permutation $\varphi$ a little bit to get a permutation satisfying (2.2).

$$\psi(n)=\begin{cases}n, & n\in E;\\ \varphi(n), & n\notin E.\end{cases}$$

(The equality $\varphi E=E$ holds since $\varphi=\varphi^{-1}$. So by changing the permutation $\varphi$ on the set $E$ to identity map we get again a permutation of $\mathbb{N}$.) If $n\in F$, then either

$n \in E$ and $\psi(n) = n$, or $n \notin A' \cup B'$ and $\psi(n) = \varphi(n) = n$. Hence, using (2.4) we get

$$\lim_{n\to\infty} \frac{\psi(n)}{n} = 1.$$

Moreover $(A' \setminus F') \cap E = (A' \setminus F') \cap \varphi F' \subseteq A' \cap B' = \emptyset$. Thus $A' \setminus F'$ and $E$ are disjoint and $\psi$ coincides with $\varphi$ on the set $A' \setminus F'$. Then $\psi(A' \setminus F') = \varphi(A' \setminus F') = B' \setminus \varphi F'$.

Now, by Theorem 2.3 we get $\mu(A' \setminus F') = \mu(\psi(A' \setminus F')) = \mu(B' \setminus \varphi F')$, and finally

$$\begin{aligned}\mu(A') &= \mu(A' \setminus F') + \mu(F') = \mu(A' \setminus F') \\ &= \mu(B' \setminus \varphi F') = \mu(B' \setminus \varphi F') + \mu(\varphi F') = \mu(B').\end{aligned}$$

□

The last two propositions together give us the main result of this paper.

**Theorem 2.6.** *A measure $\mu$ on $\mathbb{N}$ is a density measure if and only if it is $\mathcal{G}$-invariant, i.e., $\mu(A) = \mu(\pi A)$ for all $A \subseteq \mathbb{N}$ and all permutations $\pi \in \mathcal{G}$.*

## 3. Applications

We proved in Proposition 2.5 that every density measure is $\pi$-invariant for permutations $\pi \in \mathcal{G}$. It is natural to ask whether there are other permutations with this property. Proposition 3.1 states that this property characterizes Lévy group.

**Proposition 3.1.** *If $\pi$ is a permutation such that every density measure is $\pi$-invariant, i.e., $\mu(\pi A) = \pi A$ for every $A \subset \mathbb{N}$ and every density measure $\mu$, then $\pi \in \mathcal{G}$.*

*Proof.* Suppose that there is a permutation $\pi \notin \mathcal{G}$ such that every density measure $\mu$ is $\pi$-invariant. By Lemma 1.3 then there exist a set $A \subseteq \mathbb{N}$ and a sequence $n_k$ with

$$\lim_{k\to\infty} \frac{A(n_k) - (\pi A)(n_k)}{n_k} = a > 0.$$

Then any free ultrafilter $\mathcal{F}$ with $\{n_k; k \in \mathbb{N}\} \in \mathcal{F}$ yields a density measure $\mu(A) = \mathcal{F}\text{-}\lim \frac{A(n)}{n}$ such that $\mu(A) = \mu(\pi A) + a$. □

Using some known facts on the Lévy group we can characterize the pairs of sets having the property $\mu(A) = \mu(B)$ for every density measure $\mu$. We will need the following result.

**Lemma 3.2.** [3, Lemma 3] *Let $A, B \subseteq \mathbb{N}$ such that $A, B, \mathbb{N} \setminus A, \mathbb{N} \setminus B$ are infinite sets. Then there is a $\pi \in \mathcal{G}$ with $B = \pi A$ if and only if $\lim\limits_{n\to\infty} \frac{A(n)-B(n)}{n} = 0$.*

**Proposition 3.3.** *Let $A, B \subseteq \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{A(n)-B(n)}{n} = 0$ if and only if $\mu(A) = \mu(B)$ for every density measure $\mu$.*

*Proof.* Assume that $\lim\limits_{n\to\infty} \frac{A(n)-B(n)}{n} = 0$. If we moreover assume that $A$ and $B$ are neither finite nor cofinite, then by Lemma 3.2 there exists a permutation $\pi \in \mathcal{G}$ with $B = \pi A$ and therefore $\mu(B) = \mu(A)$ by Proposition 2.5. If one of the sets $A$, $B$ is finite, then $d(A) = d(B) = 0$. If one of them is cofinite, then $d(A) = d(B) = 1$, in both cases $\mu(A) = \mu(B)$.

On the other hand if $\mu(B)=\mu(A)$ holds for every density measure $\mu$, then, in particular, for every $\mathcal{F}\in\beta\mathbb{N}^*$ we get $\mathcal{F}\text{-lim}\,\frac{A(n)}{n}=\mathcal{F}\text{-lim}\,\frac{B(n)}{n}$.

This implies that $\mathcal{F}\text{-lim}\,\frac{A(n)-B(n)}{n}=0$ for every $\mathcal{F}\in\beta\mathbb{N}^*$. Thus the only limit point of the sequence $(\frac{A(n)-B(n)}{n})$ is 0 and $\lim\limits_{n\to\infty}\frac{A(n)-B(n)}{n}=0$. □

The above proposition yields an alternative proof of Proposition 3.1: Assume that the equality $\mu(\pi A)=\mu(A)$ holds for every density measure $\mu$. By the above proposition we get $\lim\limits_{n\to\infty}\frac{(\pi A)(n)-A(n)}{n}=0$ and $\pi\in\mathcal{G}$.

The rest of this section will be devoted to a counterexample answering some questions concerning density measures posed in [14] and [15]. This example can be found in Blümlinger's paper [3]. [1]

Let us recall that for any $A\subseteq\mathbb{N}$ the upper asymptotic density is given by $\overline{d}(A)=\limsup\limits_{n\to\infty}\frac{A(n)}{n}$ and the lower asymptotic density is given by $\underline{d}(A)=\liminf\limits_{n\to\infty}\frac{A(n)}{n}$.

**Example 3.4.** Let $\mathcal{F}$ be any free ultrafilter on $\mathbb{N}$. By $2\mathcal{F}$ we denote the ultrafilter given by the base $\{2A; A\in\mathcal{F}\}$, i.e., $2\mathcal{F}=\{B\subseteq\mathbb{N}; B\supseteq 2A \text{ for some } A\in\mathcal{F}\}$. Let us define $\mu$ by

$$\mu(A)=2\,(2\mathcal{F})\text{-lim}\,\frac{A(n)}{n}-\mathcal{F}\text{-lim}\,\frac{A(n)}{n}.$$

This function is shown to be a $\mathcal{G}$-invariant measure in [3, p.5092–5093], hence by Theorem 2.6 it is a density measure. For the sake of completeness we will sketch the proof of this fact.

The estimates $\frac12\frac{A(n)}{n}\le\frac{A(2n)}{2n}\le\frac12+\frac12\frac{A(n)}{n}$ imply

$$\frac12\,\mathcal{F}\text{-lim}\,\frac{A(n)}{n}\le(2\mathcal{F})\text{-lim}\,\frac{A(n)}{n}\le\frac12+\frac12\,\mathcal{F}\text{-lim}\,\frac{A(n)}{n}.$$

From this we obtain $\mu(A)\in[0,1]$.

It is clear that $\mu(A)=d(A)$ whenever $A$ has density. Finite additivity of $\mu$ follows from the additivity of $\mathcal{F}$-limit. Hence $\mu$ is indeed a density measure for any free ultrafilter $\mathcal{F}$.

Now let us consider the set $A=\bigcup\limits_{i=1}^{\infty}\{2^{2^i},2^{2^i}+1,\dots,2.2^{2^i}-1\}$. Note that $A(2.2^{2^i}-1)\ge\frac12$ and $A(2^{2^i}-1)\le\frac{1}{2^{i-3}}$ for any positive integer $i$. It can be shown that $\overline{d}(A)=\frac12$ and $\mu(A)=1$ for any free ultrafilter containing the set $\{2^{2^i}; i\in\mathbb{N}\}$.

Van Douwen asked in [15, Question 7A.1] whether $\mu(A)\le\overline{d}(A)$ for every density measure. The same question was asked again in the survey [16]. The measure $\mu$ and the set $A$ from the above example answer this question in negative.

This also yields a counterexample to the following claim of Lauwers [9, p.46]: *Every density measure can be expressed in the form*

$$\mu_\varphi(A)=\int_{\beta\mathbb{N}^*}\mathcal{F}\text{-lim}\,\frac{A(n)}{n}\,\mathrm{d}\varphi(\mathcal{F}),\qquad A\subseteq\mathbb{N} \tag{3.1}$$

*for some probability Borel measure $\varphi$ on the set of all free ultrafilters $\beta\mathbb{N}^*$.*
It is easy to note that if this claim were true the answer to van Douwen's question would be positive. (The Lauwers' claim was falsified already by Blümlinger [3]. In

[1] Let us note that the authors originally suggested an example that was much more complicated. The possibility of using the following example was pointed out by a referee.

this paper the set of all measures expressible by (3.1) is denoted by $\mathcal{R}$ and it is shown to be a proper subset of the set of all $\mathcal{G}$-invariant measures.)

Let us note that Lauwers has shown in [9, Lemma 4] that a permutation $\pi$ preserves density measures of the form (3.1) if and only if $\pi \in \mathcal{G}$. The proof is similar to our proof of Proposition 3.1. (The permutations preserving density measures of the form (3.1) are called bounded in [9].)

Šalát and Tijdeman have posed another question concerning the density measures [14, p.201]. They ask whether every density measure has the following properties:

(a) If $A(n) \leq B(n)$ for all $n \in \mathbb{N}$ then $\mu(A) \leq \mu(B)$ (where $A, B \subseteq \mathbb{N}$).

(b) If $\lim\limits_{n\to\infty} \frac{A(n)}{B(tn)} = 1$ then $\mu(A) = t\mu(B)$ (where $A, B \subseteq \mathbb{N}$ and $t \in \mathbb{R}$).

(The authors of [14] conjectured that there exist density measures that do not fulfill (a) and (b). We will see that this conjecture was right.)

Clearly, any density measure of the form (3.1) has both these properties.

It is easy to verify that for the set $A$ from the preceding example (and the measure given by an ultrafilter containing $\{2^{2^i}; i \in \mathbb{N}\}$) we get $\mu(2A) = 0$ and $\mu(A) = 1$. This shows that property b) is not valid in general. (A different density measure $\mu$ and a set $A$ with $\mu(2A) \neq \frac{1}{2}\mu(A)$ was given by van Douwen [15, Example 5.6, Case 2].)

The question (a) is closely related to van Douwen's question. Clearly, if a set $A$ fulfills $\overline{d}(A) < \mu(A)$ there is a set $B$ having asymptotic density $d(B) \in (\overline{d}(A), \mu(A))$. Since $d(B) > \overline{d}(A)$, there exists $n_0$ such that $B(n) \geq A(n)$ for $n > n_0$. Since changing only finitely many elements influences neither asymptotic density nor density measure, any such pair of sets yields a counterexample to the property (a).

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