SUBSPACES OF PSEUDORADIAL SPACES

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Abstract. We prove that every topological space (T_0-space, T_1-space) can be embedded in a pseudoradial space (in a pseudoradial T_0-space, T_1-space). This answers the Problem 3 in [2]. We describe the smallest coreflective subcategory **A** of **Top** such that the hereditary coreflective hull of **A** is the whole category **Top**.

1. Introduction

Pseudoradial or chain-net spaces were introduced by H. Herrlich in [6]. In the paper [2] A. V. Arhangel'skiĭ, R. Isler and G. Tironi asked whether every topological space is a subspace of a pseudoradial space. The question was asked again in Nyikos' survey [10].

In [11] J. Zhou proved that under the assumption $\mathfrak{p} = \mathfrak{c}$ every countable T_2-prime space is a subspace of a pseudoradial T_2-space and, as a consequence, he obtained that every space of countable tightness embeds in a pseudoradial space.

In this paper we show that every topological space (T_0-space, T_1-space) can be embedded in a pseudoradial space (in a pseudoradial T_0-space, T_1-space). This follows from the fact that any topological power of the Sierpiński doubleton is a pseudoradial (T_0-)space.

We also give a characterization of coreflective subcategories **A** of **Top** for which every space can be embedded in a space that belongs to **A**.

2. Preliminaries and notations

The classes of spaces investigated in this paper are closed under the formation of topological sums and quotient spaces. In the categorical language, they are coreflective subcategories of the category **Top** of topological spaces. We recall some properties of coreflective subcategories of **Top** which seem to be useful for our investigations (see e.g. [7], [1]).

Let **A** be a full and isomorphism-closed subcategory of **Top**. Then **A** is *coreflective* if and only if it is closed under the formation of topological sums and

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quotient spaces. If **B** is a class of topological spaces (a subcategory of **Top**) then by CH(**B**) we denote the *coreflective hull* of **B** i.e. the smallest coreflective subcategory of **Top** containing **B**. CH(**B**) consists of all quotients of sums of spaces which belong to **B**.

Let **A** be a subcategory of **Top** and let S**A** denote the subcategory of **Top** consisting of all subspaces of spaces from **A**. Then the following result is known (see [9,Remark 2.4.4(5)] or [3,Proposition 3.1]).

Proposition 2.1. *If* **A** *is a coreflective subcategory of* **Top***, then* S**A** *is also a coreflective subcategory of* **Top***. (*S**A** *is the coreflective hereditary hull of* **A***.)*

Given a topological space X and a point $a \in X$, denote by X_a the space constructed by making each point, other then a, isolated with a retaining its original neighborhoods. (I.e. a subset $U \subseteq X$ is open in X_a if and only if $a \notin U$ or there exists an open subset V of X such that $a \in V \subseteq U$.)

We say that a coreflective subcategory **A** of **Top** is *nontrivial* if **FG** $\subseteq$ **A**. (**FG** denotes the class of all finitely generated topological spaces.) In [3,Proposition 3.5] it is shown that

Proposition 2.2. *If* **A** *is a nontrivial hereditary coreflective subcategory of* **Top***, then for each* $X \in$ **A** *and each* $a \in X$ *the prime factor* X_a *of* X *at* a *belongs to* **A**.

Cardinals are initial ordinals where each ordinal is the (well-ordered) set of its predecessors. We denote the class of all ordinal numbers by On, the class of all infinite cardinals by Cn and the class of all regular cardinals by RCn.

Transfinite sequence is a net defined on an infinite ordinal. In particular, a transfinite sequence defined on the ordinal α is said to be an *α-sequence.*

A topological space X is said to be a *prime space* if it contains precisely one accumulation point.

Finally, let $t(X)$ denote the tightness of X and α be an infinite cardinal. By $\mathbf{Gen}(\alpha)$ we denote the subcategory of **Top** consisting of all spaces X with $t(X) \leq \alpha$. It is well known that $\mathbf{Gen}(\alpha)$ is a coreflective subcategory of **Top**. Moreover, it is the coreflective hull of the class of all prime spaces P with $\operatorname{card} P \leq \alpha$.

3. Subspaces of pseudoradial spaces

We start with the definition of pseudoradial and β-sequential space.

A topological space X is said to be *pseudoradial* if, for any subset A of X, A is closed whenever together with any transfinite sequence it contains all its limits. Let β be an infinite cardinal. A space X is said to be *β-sequential* if, for any subset A of X, A is closed whenever together with any α-sequence such that $\alpha \leq \beta$ it contains all its limits.

Observe, that if X is a β-sequential space, then X is pseudoradial and if $\beta \leq \gamma$ and X is β-sequential then X is γ-sequential.

It is useful to characterize β-sequential spaces using β-sequential closure. Let X be a topological space and $A \subseteq X$. The *β-sequential closure* of A is the smallest set $\widetilde{A}$ such that $A \subseteq \widetilde{A}$ and $\widetilde{A}$ is closed with respect to limits of α-sequences for every $\alpha \leq \beta$. Obviously, if A is a subset of X, then $\widetilde{A} \subseteq \overline{A}$ and if $A \subseteq B \subseteq X$, then $\widetilde{A} \subseteq \widetilde{B}$. The following characterization of β-sequential spaces is well known and easy to see.

Proposition 3.1. *A topological space X is β-sequential if and only if for any subset A of X $\widetilde{A} = \overline{A}$.*

We denote by **PsRad** the (full) subcategory of **Top** consisting of all pseudoradial spaces, by **Psrad**(β) the subcategory consisting of all β-sequential spaces and by **SPsrad**(β) the subcategory of all subspaces of β-sequential spaces. It is well known that **PsRad** and **Psrad**(β) are coreflective subcategories of **Top** and, consequently, **SPsrad**(β) is also coreflective in **Top**.

Denote by $C(\alpha)$ the topological space on $\alpha \cup \{\alpha\}$ such that a subset $U \subseteq \alpha \cup \{\alpha\}$ is open in $C(\alpha)$ if and only if $U \subseteq \alpha$ or $\operatorname{card}(C(\alpha) \setminus U) < \alpha$. It is known that $\mathbf{Psrad}(\beta) = \mathrm{CH}(\{C(\alpha); \alpha \leq \beta; \alpha \in \mathrm{RCn}\})$ and $\mathbf{PsRad} = \mathrm{CH}(\{C(\alpha); \alpha \in \mathrm{RCn}\})$.

Next we want to prove that for any infinite cardinal α $\mathbf{Gen}(\alpha) \subseteq \mathbf{SPsrad}(2^{\alpha})$. As a consequence we obtain that every topological space is a subspace of a pseudoradial space.

Denote by S the Sierpiński space, i.e. the space defined on the set $\{0, 1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space.

Proposition 3.2. *If β is an infinite cardinal, then the topological power S^{β} of the space S is a β-sequential space.*

Proof. Let γ be the smallest cardinal such that S^{γ} is not β-sequential. We want to show that $\gamma > \beta$. Since for any cardinal $\alpha \leq \omega_0$ S^{α} is a sequential space (it is first-countable), $\gamma > \omega_0$. Assume that $\gamma \leq \beta$. According to Proposition 3.1 there exists a subset U of S^{γ} with $\overline{U} \setminus \widetilde{U} \neq \emptyset$ (by $\widetilde{U}$ we denote the β-sequential closure of U).

Let $t \in \overline{U} \setminus \widetilde{U}$, $A = \{\eta \in \gamma; t(\eta) = 0\}$ and $\varkappa = \operatorname{card} A$. Clearly, $A \neq \emptyset$ and $\varkappa \leq \gamma$. Consider the subspace $K = \{s \in S^{\gamma};$ for each $\eta \in \gamma \setminus A$ $s(\eta) = 1\}$. The space K is a closed subspace of S^{γ}, $t \in K$ and, obviously, K is homeomorphic to the space $S^{\varkappa}$.

Let us define a map $g \colon S^{\gamma} \to K$ by

$$g(f)(x) = \begin{cases} f(x), & \text{if } x \in A \\ 1, & \text{if } x \notin A \end{cases}$$

g is continuous and for every $s \in S^{\gamma}$ we get $g(s) \in \widetilde{\{s\}}$, since the constant sequence with each term s converges to $g(s)$. Therefore $\widetilde{g(U)} \subseteq \widetilde{U}$ holds for any subset $U \subseteq S^{\alpha}$. It follows from the continuity of g that $t \in \overline{g(U)}$.

If $\varkappa < \gamma$, then K is β-sequential, therefore $t \in \widetilde{g(U)} \subseteq \widetilde{U}$. Thus, we obtain that $\varkappa = \gamma$.

In this case there exists a homeomorphism $f\colon K \to S^\gamma$ such that $f(t) = t_0$ where $t_0(\eta) = 0$ for each $\eta \in \gamma$. Without loss of generality we can suppose that $K = S^\gamma$ and $t = t_0$. For each $\xi \in \gamma$ let f_ξ denote the element of S^γ given by

$$f_\xi(x) = \begin{cases} 0, & \text{for } x < \xi, \\ 1, & \text{for } x \geq \xi. \end{cases}$$

It is easy to see that the γ-sequence $(f_\xi)_{\xi\in\gamma}$ converges to t_0 in S^γ. Since $t_0 \in \overline{U}$ and $\overline{\{t_0\}} = S^\gamma$, we obtain that $\overline{U} = S^\gamma$ and therefore $f_\xi \in \overline{U}$ for each $\xi \in \gamma$. Put $A_\xi = \{\eta \in \gamma; f_\xi(\eta) = 0\} = \{\eta \in \gamma; \eta < \xi\}$. Then for each $\xi \in \gamma$ $\operatorname{card} A_\xi < \gamma$ and according to the preceding part of proof (the case $\varkappa < \gamma$) $f_\xi \in \widetilde{U}$. Hence, $t_0 \in \widetilde{U}$ contradicting our assumption. Thus, $\gamma > \beta$ and S^β is β-sequential. □

Theorem 3.3. $\mathbf{Gen}(\alpha) \subseteq \mathbf{SPsrad}(2^\alpha)$ *for every infinite cardinal* α.

Proof. Since $\mathbf{Gen}(\alpha)$ is the coreflective hull of the class of all prime spaces P with $\operatorname{card} P \leq \alpha$ and $\mathbf{SPsrad}(2^\alpha)$ is coreflective, it suffices to prove that every prime space P with $\operatorname{card} P \leq \alpha$ belongs to $\mathbf{SPsrad}(2^\alpha)$.

Let P be a prime space and $\operatorname{card} P \leq \alpha$. Then P is a T_0-space and the weight of P $w(P) = \beta \leq 2^\alpha$. It is well known (see e.g. [5,Theorem 2.3.26]) that P is embeddable in S^β. According to Proposition 3.2 S^β is β-sequential and therefore it is also 2^α-sequential. Hence P belongs to $\mathbf{SPsrad}(2^\alpha)$. □

As a consequence of the preceding theorem we obtain:

Theorem 3.4. *Any topological space is a subspace of a pseudoradial space. Moreover, every T_0-space is a subspace of a pseudoradial T_0-space.*

Proof. The first part is an easy consequence of Theorem 3.3. The second part follows from the fact that every T_0-space is subspace of some S^α ([5,Theorem 2.3.26]) and from Proposition 3.2. □

Remark 3.5. According to Theorem 3.4 the coreflective hereditary hull of **PsRad** is **Top** and according to [3,Example 4.9] the coreflective hereditary kernel of **PsRad** is the category **FG** of all finitely generated spaces.

Theorem 3.4 can be strengthened to the category $\mathbf{Top}_1$ of T_1-spaces as follows.

Let us recall that the *cofinite topology* on an underlying set X is the coarsest T_1 topology on this set. Closed sets in the cofinite topology are finite sets and the whole set X.

For any cardinal number α, let $(S^\alpha)_1$ be the topological space on the set $\{0,1\}^\alpha$ with the topology which is the join of the product topology S^α and the cofinite topology on the set $\{0,1\}^\alpha$. If α is finite, then $(S^\alpha)_1$ is discrete space.

Proposition 3.6. *Let α be an infinite cardinal. The topological space $(S^\alpha)_1$ is α-sequential.*

Proof. The collection

$$\mathcal{B}_1 = \{U_M; M \subseteq \alpha, M \text{ is finite}\}, \text{ where } U_M = \{f \in \{0,1\}^\alpha; f(m) = 0 \text{ for each } m \in M\}$$

is the canonical base for the product topology S^α. Clearly

$$\mathcal{B} = \{U_M \setminus F; M \subseteq \alpha, M \text{ is finite}, F \subseteq \{0,1\}^\alpha, F \text{ is finite}\}$$

is a base for the topology of the space $(S^\alpha)_1$.

We have to show that if $t \in \overline{U} \setminus U$ then $t \in \widetilde{U}$. (By $\widetilde{U}$ we denote the α-sequential closure of U in $(S^\alpha)_1$.) Let us put

$$A_t = \{\eta \in \alpha; t(\eta) = 0\}.$$

Assume that, on the contrary, there exist some $t \in \{0,1\}^\alpha$ and $U \subseteq \{0,1\}^\alpha$ such that $t \in \overline{U} \setminus U$ and $t \notin \widetilde{U}$. Let β be the smallest cardinal number such that $\beta = \operatorname{card} A_t$ for some t and U satisfying $t \in \overline{U} \setminus U$ and $t \notin \widetilde{U}$.

First let β be finite, i.e. let A_t be a finite subset of α. Then U_{A_t} is a neighborhood of t, thus there exists $f_1 \in U \cap U_{A_t}$. Since $U_{A_t} \setminus \{f_1\}$ is a neighborhood of t, there is $f_2 \in U \cap (U_{A_t} \setminus \{f_1\})$. In a similar way we can find for every $n < \omega$, $n \geq 2$, an $f_n \in U \cap (U_{A_t} \setminus \{f_1, \dots, f_{n-1}\})$. We claim that $f_n \to t$.

Every basic neighborhood of t has the form $U_B \setminus F$, where $F \subseteq S^\alpha$ and $B \subseteq A_t$ are finite subsets. U_B contains all terms of the sequence $(f_n)_{n<\omega}$ and by omitting the finite subset F we omit only finitely many of them, since this sequence is one-to-one.

Thus β is not finite and $\omega \leq \beta = \operatorname{card} A_t \leq \alpha$. Let $\{a_\xi; \xi < \beta\}$ be a well-ordering of A_t. Let us define a function $f_\gamma\colon \alpha \to S$ by

$$f_\gamma(x) = \begin{cases} 0, & \text{if } x = a_\xi \text{ for some } \xi < \gamma \\ 1, & \text{otherwise,} \end{cases}$$

for every $\gamma < \beta$.

If $U_B \setminus F$ is a basic neighborhood of f_γ, then $(U_B \setminus F) \cup \{t\}$ is a neighborhood of t. Hence $f_\gamma \in \overline{U}$. Since the cardinality of the set $A_\gamma = \{a_\xi, \xi < \gamma\} = \{\eta \in \beta; f_\gamma(\beta) = 0\}$ is less then β and $f_\gamma \in \overline{U}$, we get $f_\gamma \in \widetilde{U}$.

It only remains to show that the sequence f_γ converges to t. Any basic neighborhood of t has the form $U_B \setminus F$, where $B \subseteq A_t$, B and F are finite. Let $\delta_1 = \sup\{\xi : a_\xi \in B\}$ and $\delta_2 = \sup\{\xi : f_\xi \in F\}$. Since F and B are finite, $\delta_1, \delta_2 < \alpha$. Let $\delta = \max\{\delta_1, \delta_2\}$. Then for each $\gamma > \delta$ $f_\gamma \in U_B \setminus F$.

Thus $t \in \widetilde{U}$, a contradiction.

Theorem 3.7. *Every T_1-space is a subspace of a pseudoradial T_1-space.*

Proof. Let X be a T_1-space. Then there exists an embedding $e\colon X \hookrightarrow S^\alpha$ of X into some topological power S^α of S. Since X is T_1, $e\colon X \hookrightarrow (S^\alpha)_1$ is an embedding as well. $(S^\alpha)_1$ is a T_1-space and it is pseudoradial by Proposition 3.6.

4. Coreflective subcategories with S**A** = **Top**

In [8] H. Herrlich and M. Hušek suggested to investigate the coreflective subcategories of **Top** for which the coreflective hereditary kernel is the category **FG** and the coreflective hereditary hull is the whole category **Top**. Let $\mathcal{S}$ denote the collection of all such subcategories of **Top**. We next show that the intersection of any nonempty family of elements of $\mathcal{S}$ belongs to $\mathcal{S}$ and $\mathcal{S}$ has the smallest element.

Recall (see Proposition 2.1) that if **A** is a coreflective subcategory of **Top**, then the coreflective hereditary hull of **A** is S**A**. We first give a characterization of coreflective subcategories of **Top** for which S**A** = **Top**.

Theorem 4.1. *Let* **A** *be a coreflective subcategory of* **Top**. *Then* S**A** = **Top** *if and only if* $S^\alpha \in \mathbf{A}$ *for every infinite cardinal* α.

Proof. Let **A** be a coreflective subcategory of **Top** for which S**A** = **Top** and α be any infinite cardinal. There exists a space $X \in \mathbf{A}$ such that S^α is a subspace of X. For each $a \in \alpha$ let $p_a : S^\alpha \to S$ denote the a-th projection of topological power S^α by onto S. The set $(p_a)^{-1}(0)$ is open in S^α so that there exists an open subset U_a in X such that $U_a \cap S^\alpha = (p_a)^{-1}(0)$. The map $f_a : X \to S$ given by $f_a(x) = 0$ for each $x \in U_a$ and $f_a(x) = 1$ otherwise is a continuous extension of $p_a : S^\alpha \to S$. The map $f : X \to S^\alpha$ with $f_a = p_a \circ f$ for each $a \in \alpha$ is continuous and the restriction $f|_{S^\alpha}$ is the identity map on S^α. Hence f is a retraction and, consequently, f is a quotient map. Thus $S^\alpha \in \mathbf{A}$.

Conversely, if for any cardinal α S^α belongs to **A**, then any prime space belongs to S**A** and since S**A** is a coreflective subcategory of **Top** we obtain that S**A** = **Top**. □

Corollary 4.2. *If* $\{\mathbf{A}_i, i \in I\}$ *is a nonempty collection of coreflective subcategories of* **Top** *such that for each* $i \in I$ $\mathrm{S}\mathbf{A}_i = \mathbf{Top}$ *and* $\mathbf{A} = \bigcap\{\mathbf{A}_i, i \in I\}$, *then* S**A** = **Top**.

If, moreover, for each $i \in I$ *the coreflective hereditary kernel of* $\mathbf{A}_i$ *is* **FG**, *then, obviously, the coreflective hereditary kernel of* **A** *is again* **FG**.

Corollary 4.3. $\mathbf{A} = \mathrm{CH}(\{S^\alpha; \alpha \in \mathrm{Cn}\})$ *is the smallest coreflective subcategory of* **Top** *such that* S**A** = **Top**. *Obviously, the coreflective hereditary kernel of* **A** *is* **FG** *(since* $\mathbf{FG} \subseteq \mathbf{A} \subseteq \mathbf{PsRad}$*).*

Note that Theorem 4.1, Corollary 4.2 and Corollary 4.3 remain valid after replacing **Top** by $\mathbf{Top}_0$ (the category of T_0-spaces).

We next present another class of (in some sense more convenient) generators of the category $\mathrm{CH}(\{S^\alpha; \alpha \in \mathrm{Cn}\})$.

Let α be an infinite cardinal and $B_\beta = \{\gamma \in \alpha \cup \{\alpha\}; \gamma \geq \beta\}$ for each $\beta \in \alpha$. Then $M(\alpha)$ is the topological space on the set $\alpha \cup \{\alpha\}$ with the topology consisting of all B_β, β being a non-limit ordinal less then α or $\beta = 0$. These spaces have the following useful property:

Proposition 4.4. *Let α be an infinite cardinal and $M(\alpha)$ be a subspace of X. Then there exists a retraction $f\colon X \to M(\alpha)$.*

Proof. For every non-limit ordinal $\beta < \alpha$ denote by U_β the union of all open subsets of X with $U \cap M(\alpha) = B_\beta$ and put $U_0 = X$. Clearly, if $0 \leq \beta < \beta' < \alpha$ then $U_\beta \supsetneqq U_{\beta'}$ and for each $\beta < \alpha$ $U_\beta \cap M(\alpha) = B_\beta$. Define $f\colon X \to M(\alpha)$ by

$$f(x) = \sup\{\beta \in \alpha : x \in U_\beta\}.$$

Obviously, $f^{-1}(B_\beta) = U_\beta$ for non-limit ordinal β. Thus f is continuous. Moreover we have $f(\beta) = \beta$ for $\beta \in M(\alpha)$ and f is a retraction. $\square$

Theorem 4.5. *Let* $\mathbf{A}$ *be a coreflective subcategory of* **Top** *and α be an infinite cardinal. The following statements are equivalent:*

(1) $\mathbf{Psrad}(\alpha) \subseteq \mathrm{S}\mathbf{A}$
(2) $S^\alpha \in \mathbf{A}$
(3) $M(\alpha) \in \mathbf{A}$

Proof. (1) $\Rightarrow$ (2) By Proposition 3.2 $S^\alpha \in \mathbf{Psrad}(\alpha)$. Hence $S^\alpha \in \mathrm{S}\mathbf{A}$, i.e. S^α is a subspace of a space $X \in \mathbf{A}$. Following the proof of Theorem 3.3 we can construct a retraction $f\colon X \to S^\alpha$, thus $S^\alpha \in \mathbf{A}$.

(2) $\Rightarrow$ (3) Let $S^\alpha \in \mathbf{A}$. The weight of the space $M(\alpha)$ is $w(M(\alpha)) = \alpha$, therefore $M(\alpha)$ is a subspace of S^α by [5,Theorem 2.3.26]. Then by Proposition 4.4 there exists a retraction $g\colon S^\alpha \to M(\alpha)$ and $M(\alpha) \in \mathbf{A}$.

(3) $\Rightarrow$ (1) Let $M(\alpha) \in \mathbf{A}$. Clearly, $M(\beta)$ is a subspace of $M(\alpha)$ for every $\beta < \alpha$. ($M(\beta)$ is the subspace on the set $\beta \cup \{\beta\}$.) Thus for every $\beta \leq \alpha$ we have $M(\beta) \in \mathrm{S}\mathbf{A}$ and $C(\beta) = (M(\beta))_\beta \in \mathrm{S}\mathbf{A}$ (using Proposition 2.2). Therefore $\mathbf{Psrad}(\alpha) = \mathrm{CH}(\{C(\beta); \beta \leq \alpha\}) \subseteq \mathrm{S}\mathbf{A}$. $\square$

Corollary 4.6. *Let α be an infinite cardinal number. Then* $\mathrm{CH}(M(\alpha)) = \mathrm{CH}(S^\alpha)$ *and this is the smallest coreflective subcategory of* **Top** *such that* $\mathbf{Psrad}(\alpha) \subseteq \mathrm{S}\mathbf{A}$.

Corollary 4.7. $\mathrm{CH}(\{M(\alpha); \alpha \in \mathrm{RCn}\}) = \mathrm{CH}(\{S^\alpha; \alpha \in \mathrm{Cn}\})$.

Corollary 4.8. *Let* $\mathbf{A}$ *be a coreflective subcategory of* **Top**. *Then* $\mathrm{S}\mathbf{A} = \mathbf{Top}$ *if and only if $M(\alpha) \in \mathbf{A}$ for every regular cardinal α.*

For a topological space X and $x \in X$, $t(X, x)$ denotes the tightness of the point x in the topological space X.

For any infinite cardinal α, let $B(\alpha)$ be the topological space on the set $\alpha \cup \{\alpha\}$ with the topology consisting of all sets $B_\beta = \{\gamma \in \alpha \cup \{\alpha\}; \gamma \geq \beta\}$ where $\beta < \alpha$.

Proposition 4.9. *Let* $\mathbf{A}$ *be a coreflective subcategory of* **Top**. *Then* $\mathrm{S}\mathbf{A} = \mathbf{Top}$ *if and only if for every regular cardinal α* $\mathbf{A}$ *contains a space X such that there exists $x \in X$ with $t(X, x) = \alpha$ and for $\alpha = \omega_0$ the prime factor X_x of X at x is, moreover, not finitely generated.*

Proof. One direction follows from $t(M(\alpha), \alpha) = \alpha$ and Theorem 4.5.

Now let $t(X,x)=\alpha$ and $X\in\mathbf{A}$. Then there exists $V\subseteq X$ and $x\in X$ with $\operatorname{card} V=\alpha$, $x\in\overline{V}$ and $x\notin\overline{U}$ for any $U\subseteq V$, $\operatorname{card} U<\alpha$. Let Y be the subspace of X on the set $V\cup\{x\}$. Y belongs to S$\mathbf{A}$ and by Proposition 2.2 Y_x also belongs to S$\mathbf{A}$.

Next we want to prove that $B(\alpha)\in \mathrm{S}\mathbf{A}$.

We claim that the topological space Y_x is finer than $B(\alpha)$. Indeed, if $\operatorname{card}(V\setminus U)<\alpha$ then $x\notin\overline{V\setminus U}$ and U is neighborhood of x in Y, hence U is open in Y_x. Clearly, the set $\{x\}$ is not open in Y.

Since $\operatorname{card} Y=\alpha$, we can assume that Y is a topological space on the set $\alpha\cup\{\alpha\}$ and $x=\alpha$. For every $\gamma<\delta\leq\alpha$ let $S_{\gamma\delta}$ be a Sierpiński topological space on the set $\{\gamma,\delta\}$ with the set $\{\delta\}$ open. A subset $U\subseteq\alpha\cup\{\alpha\}$ is open in $B(\alpha)$ if and only if it is open in Y_x and $U\cap\{\gamma,\delta\}$ is open in $S_{\gamma\delta}$ for every $\gamma<\delta\leq\alpha$ (i.e. U contains with $\gamma\in U$ every $\delta>\gamma$). Thus $B(\alpha)$ is a quotient space of $Y_x\sqcup(\coprod S_{\gamma\delta})$ and $B(\alpha)\in\mathrm{S}\mathbf{A}$.

Then the prime factor $(B(\alpha))_\alpha=C(\alpha)$ belongs to S$\mathbf{A}$ for every regular cardinal α, hence $\mathbf{PsRad}\subseteq\mathrm{S}\mathbf{A}$ and by Theorem 3.4 $\mathrm{S}\mathbf{A}=\mathbf{Top}$. $\square$

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