Subspaces of pseudoradial spaces

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Abstract

We prove that every topological space (T_0-space, T_1-space) can be embedded in a pseudoradial space (in a pseudoradial T_0-space, T_1-space). This answers the Problem 3 in [2]. We describe the smallest coreflective subcategory **A** of **Top** such that the hereditary coreflective hull of **A** is the whole category **Top**.

1 Preliminaries

I'd like to present some results of my paper Subspaces of pseudoradial spaces [13]. I will start with defining some basic notions we will need later.

Transfinite sequence $(x_\xi)_{\xi<\alpha}$ is a net defined on an infinite ordinal. We could say that it is a sequence indexed by ordinals less than α and the convergence (limit) of transfinite sequences in topological spaces is defined similarly as the usual convergence of sequences: A transfinite sequence $(x_\xi)_{\xi<\alpha}$ converges to the point $x \in X$ provided that:

$$\underset{U\in\mathcal{T}}{\forall} x \in U \quad \underset{\eta}{\exists}\, \eta < \alpha \quad \underset{\xi}{\forall}\, \eta < \xi < \alpha \quad x_\xi \in U.$$

A topological space X is said to be *pseudoradial* if a subset $V \subseteq X$ is closed in X whenever V is closed with respect to limits of transfinite sequences of elements of V. Pseudoradial spaces are also called chain-net spaces. We denote by $\widetilde{A}$ the smallest set such that $A \subseteq \widetilde{A}$ and $\widetilde{A}$ is closed with respect to limits of transfinite sequences. We will call the set $\widetilde{A}$ the pseudoradial closure of A. Obviously, a topological space X is pseudoradial if and only if for each subset $A \subset X$ the equality $\overline{A} = \widetilde{A}$ holds.

An important example of a pseudoradial space is the space $C(\alpha)$, where α is a regular cardinal. $C(\alpha)$ is the topological space on the set $\alpha \cup \{\alpha\}$ such that a subset $U \subseteq \alpha \cup \{\alpha\}$ is open in $C(\alpha)$ if and only if $U \subseteq \alpha$ or $\operatorname{card} C(\alpha) \setminus U < \alpha$. Equivalently a base for the topology of $C(\alpha)$ is $\mathcal{B} = \{B_\xi; \xi < \alpha\}$ with $B_\xi = \{\eta \in \alpha \cup \{\alpha\} : \eta \geq \xi\}$.

Next we recall the notion of *coreflective subcategories* of **Top**. We will assume that all subcategories are full and isomorphism-closed, so a subcategory **A** of **Top** corresponds to a subclass of the class of all topological spaces (**A** is determined by the class of its objects). Then a coreflective subcategory of **Top** is a class of topological spaces which is closed under the formation of topological sums and quotient spaces. It can be seen directly from this definition that any intersection of coreflective subcategories of **Top** is again a coreflective subcategory of **Top**. Hence for each class **A** of topological spaces there exists the smallest coreflective subcategory of **Top** containing **A**. It is called the *coreflective hull* of **A** and denoted by CH(**A**). CH(**A**) contains precisely the quotient spaces of topological sums of spaces from **A**.

If a coreflective subcategory of **Top** is closed also under the formation of subspaces, it is called *hereditary.* Let **A** be a subcategory of **Top** and let S**A** denote the subcategory of **Top** consisting of all subspaces of spaces from **A**. It is known that if **A** is a coreflective subcategory of **Top**, then S**A** is a coreflective subcategory of **Top** as well.

We denote by **PsRad** the subcategory of all pseudoradial spaces. It is known that this subcategory is coreflective in **Top** and **PsRad** $=$ CH($\{C(\alpha); \alpha$ is a regular cardinal$\}$). Hence the subcategory **SPsRad** of all subspaces of pseudoradial spaces is coreflective as well.

We say that a topological space is a prime space, if it has only one accumulation point. The category **Top** is coreflective hull of prime spaces. (Every topological space X can be obtained as a quotient space of its prime factors X_a. X_a is the space constructed by making each point, other then a, isolated with a retaining its original neighborhoods. I.e. a subset $U \subseteq X$ is open in X_a if and only if $a \notin U$ or there exists an open subset V of X such that $a \in V \subseteq U$.)

2 Overview of results

In the paper [2] the authors asked whether every topological space is a subspace of a pseudoradial space. We show that this is true. This result can be formulated using coreflective subcategories of **Top** like this: **SPsRad** $=$ **Top**.

It suffices to show that all prime spaces belong to **SPsRad**, since **Top** is the coreflective hull of prime spaces.

This question was partly answered by Jin-Yuan Zhou in [14], where he proved under an additional set-theoretical assumption that the answer is affirmative for countable tight spaces. We show the same in ZFC in a similar way as in that paper. We first prove that every prime space can be embedded in a pseudoradial space and this implies (as we have already mentioned) that **Top** $=$ **SPsRad**.

Zhou proved in his paper also something more. He showed (under an additional set-theoretical assumption) that every countable Hausdorff prime space can be embedded in a Hausdorff pseudoradial space. I don't know whether this result holds in ZFC. It is proved that it holds for T_1-spaces in ZFC. It is known that embedding of prime spaces of the form $\omega \cup \{p\}$, $p \in \beta\omega \setminus \omega$, into regular spaces is ZFC-independent (see [4]).

According to the results of the first part of this paper **PsRad** is an example of coreflective subcategory of **Top** such that S**A** $=$ **Top**. We find the smallest subcategory $\mathbf{A}_0$ of **Top** having this property. This can be regarded as a kind of characterization of such subcategories. This is connected with the Problem 7 from [9]: To investigate coreflective subcategories of **Top** such that S**A** $=$ **Top** and the hereditary coreflective kernel of **A** is the subcategory **FG** (i.e. **A** doesn't contain any hereditary coreflective subcategory larger then **FG**).

3 Subspaces of pseudoradial spaces

Let us note that results of this section has been previously observed in [10].

We denote by S the Sierpiński space, i.e. the space defined on the set $\{0,1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space. The key step of proving that each topological space is a subspace of a pseudoradial space is this proposition:

Proposition 3.2. *If β is an infinite cardinal, then the topological power S^β of the space S is a pseudoradial space.*

Proof. We should show that if a point t of S^β belongs to $\overline{U}$ for some subset of U then it also belongs to $\widetilde{U}$. We show it only for $t_0 \in S^\beta$ where $t_0(\eta) = 0$ for each $\eta \in \beta$. The proof for other points of S^β if fairly similar.

So we assume that $t_0 \in \overline{U}$ and $t_0 \notin U$. Let us put for any subset $A \subseteq \beta$

$$f_A(x) = \begin{cases} 0, & x \in A, \\ 1, & x \notin A. \end{cases}$$

(I.e. f_A is the characteristic function of the set $\beta \setminus A$.) Each f_A is a point of S^β and $t_0 = f_\beta$ in this notation. Using transfinite induction we show that $f_A \in \widetilde{U}$ for each $A \subseteq \beta$.

First, let A be a finite subset of β. Then $U_A = \{g \in S^\beta; g(x) = 0$ for each $x \in A\}$ is a neighborhood of t_0, hence there exists a point $g \in U_A \cap U$. The constant sequence $(g)_{\xi<\omega}$ converges to f_A.

Let $\operatorname{card} A = \alpha \leq \beta$ and for each B with $\operatorname{card} B < \alpha$ it holds $f_B \in \widetilde{U}$. The set A can be written as an union of an increasing system $A = \bigcup_{\xi<\alpha} B_\xi$ such that $\operatorname{card} B_\xi < \alpha$. (We can obtain such system using any well-ordering of A.) Obviously the transfinite sequence $(f_{B_\xi})_{\xi<\alpha}$ converges to f_A and since $f_{B_\xi} \in \widetilde{U}$ for each $\xi < \alpha$ and $\widetilde{U}$ is closed with respect to limits of transfinite sequences, we get $f_A \in \widetilde{U}$. □

According to theorem known in general topology each T_0-space can be embedded in a space S^β for some infinite cardinal β. Note that each prime space is a T_0-space. So from this we get:

Theorem 3.4. *Any topological space is a subspace of a pseudoradial space. Moreover, every T_0-space is a subspace of a pseudoradial T_0-space.*

In order to extend this result to the class of T_1-spaces we cannot use the space S^β, because this space is not T_1. Let us recall that the *cofinite topology* on an underlying set X is the coarsest T_1 topology on this set. Closed sets in the cofinite topology are finite sets and the whole set X.

For any cardinal number β, let $(S^\beta)_1$ be the topological space on the set $\{0,1\}^\beta$ with the topology which is the join of the product topology S^β and the cofinite topology on the set $\{0,1\}^\beta$. If β is finite, then $(S^\beta)_1$ is discrete space.

If X is a T_1-space, then there exists an embedding $e\colon X \hookrightarrow S^\beta$ of X into some topological power S^β of S. Since X is T_1, $e\colon X \hookrightarrow (S^\beta)_1$ is an embedding as well. So in order to prove that each T_1-space is a subspace of a T_1-pseudoradial space, it suffices to show that the space $(S^\beta)_1$ is pseudoradial. The proof of this result is only a bit more complicated than the proof of the fact that S^β is pseudoradial and we will omit it.

Theorem 3.6. *Every T_1-space is a subspace of a pseudoradial T_1-space.*

The same result doesn't hold for Hausdorff spaces. A counterexample is any compact Hausdorff space X which is not pseudoradial. If X would be a subspace of a pseudoradial space, it would be itself pseudoradial (closed subspace of a pseudoradial space). We can take e.g. $X = \beta\omega$.

4 Coreflective subcategories with $\mathrm{S}\mathbf{A} = \mathbf{Top}$

We showed that the category **PsRad** is an example of a coreflective subcategory of **Top** such that $\mathrm{S}\mathbf{A} = \mathbf{Top}$. In what follows we find the smallest coreflective subcategory of **Top** with this property.

Theorem 4.1. *Let* $\mathbf{A}$ *be a coreflective subcategory of* **Top**. *Then* $\mathrm{S}\mathbf{A} = \mathbf{Top}$ *if and only if* $S^\alpha \in \mathbf{A}$ *for every infinite cardinal* α.

Proof. Let $\mathbf{A}$ be a coreflective subcategory of **Top** for which $\mathrm{S}\mathbf{A} = \mathbf{Top}$ and α be any infinite cardinal. There exists a space $X \in \mathbf{A}$ such that S^α is a subspace of X. For each $a \in \alpha$ let $p_a \colon S^\alpha \to S$ denote the a-th projection of topological power S^α onto S. The set $(p_a)^{-1}(0)$ is open in S^α so that there exists an open subset U_a in X such that $U_a \cap S^\alpha = (p_a)^{-1}(0)$. The map $f_a \colon X \to S$ given by $f_a(x) = 0$ for each $x \in U_a$ and $f_a(x) = 1$ otherwise is a continuous extension of $p_a \colon S^\alpha \to S$. The map $f \colon X \to S^\alpha$ with $f_a = p_a \circ f$ for each $a \in \alpha$ is continuous and the restriction $f|_{S^\alpha}$ is the identity map on S^α. Hence f is a retraction and, consequently, f is a quotient map. Thus $S^\alpha \in \mathbf{A}$.

Conversely, if S^α belongs to $\mathbf{A}$ for any cardinal α, then any prime space belongs to $\mathrm{S}\mathbf{A}$ and since $\mathrm{S}\mathbf{A}$ is a coreflective subcategory of **Top** we obtain that $\mathrm{S}\mathbf{A} = \mathbf{Top}$. $\square$

Corollary 4.3. $\mathbf{A} = \mathrm{CH}(\{S^\alpha; \alpha$ *is an infinite cardinal*$\})$ *is the smallest coreflective subcategory of* **Top** *such that* $\mathrm{S}\mathbf{A} = \mathbf{Top}$.

We next present another class of generators of the category $\mathrm{CH}(\{S^\alpha; \alpha$ is an infinite cardinal$\})$.

Let α be an infinite cardinal and $B_\beta = \{\gamma \in \alpha \cup \{\alpha\}; \gamma \geq \beta\}$ for each $\beta \in \alpha$. Then $M(\alpha)$ is the topological space on the set $\alpha \cup \{\alpha\}$ with the topology consisting of all B_β, β being a non-limit ordinal less then α or $\beta = 0$. These spaces have the following property:

Proposition 4.4. *Let* α *be an infinite cardinal and* $M(\alpha)$ *be a subspace of* X. *Then there exists a retraction* $f \colon X \to M(\alpha)$.

Proof. For every non-limit ordinal $\beta < \alpha$ denote by U_β the union of all open subsets of X with $U \cap M(\alpha) = B_\beta$ and put $U_0 = X$. Clearly, if $0 \leq \beta < \beta' < \alpha$ then $U_\beta \supsetneqq U_{\beta'}$ and for each $\beta < \alpha$ $U_\beta \cap M(\alpha) = B_\beta$. Define $f \colon X \to M(\alpha)$ by

$$f(x) = \sup\{\beta \in \alpha : x \in U_\beta\}.$$

Obviously, $f^{-1}(B_\beta) = U_\beta$ for non-limit ordinal β. ($x \in U_\beta \Rightarrow f(x) \geq \beta \Rightarrow x \in f^{-1}(B_\beta)$), $x \notin U_\beta \Rightarrow f(x) < \beta \Rightarrow x \notin f^{-1}(B_\beta)$.) Thus f is continuous. Moreover we have $f(\beta) = \beta$ for $\beta \in M(\alpha)$ and f is a retraction. $\square$

Corollary 4.7. $\mathrm{CH}(\{M(\alpha); \alpha$ *is a regular cardinal*$\})$ *is the smallest coreflective subcategory of* **Top** *such that* $\mathrm{S}\mathbf{A} = \mathbf{Top}$.

Proof. If $M(\alpha)$ belongs to **A**, then according to [3, Proposition 3.5] the prime factor $(M(\alpha))_\alpha = C(\alpha)$ belongs to S**A**. Therefore S**A** contains **PsRad** $=$ CH($\{C(\alpha); \alpha$ is a regular cardinal$\}$) and, consequently, S**A** $\supseteq$ **SPsRad** $=$ **Top**.

On the other hand, if S**A** $=$ **Top**, then there exists a topological space X such that $M(\alpha)$ is a subspace of X. According to Proposition 4.4 there exists a retraction $f\colon X \to M(\alpha)$ and $M(\alpha)$ belongs to S**A**. □

Using the spaces $M(\alpha)$ can be proved one more characterization of the smallest coreflective subcategory of **Top** such that S**A** $=$ **Top**:

Proposition 4.9. *Let* **A** *be a coreflective subcategory of* **Top**. *Then* S**A** $=$ **Top** *if and only if for every regular cardinal α* **A** *contains a space X such that there exists $x \in X$ with $t(X, x) = \alpha$ and for $\alpha = \omega_0$ the prime factor X_x of X at x is, moreover, not finitely generated.*

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Questions and notes

1. Which set-theoretical assumption have been used in [4] and [14]?

The result of [14] is proved under $\mathfrak{p} = \mathfrak{c}$, which is a consequence of MA. ($\mathfrak{p}$= minimal cardinality of an infinite subset of ω, which has SFIP and fails to have infinite pseudointersection. SFIP = strong finite intersection property = finite subfamilies have infinite intersection. Pseudointersection of a family = is almost included in each set of this family, $P \subset^* F$ for each $F \in \mathcal{F}$.) It is proved in [4] that CH implies that each $\omega \cup \{p\}$ is a subspace of some regular pseudoradial space and there is a model in which this result doesn't hold.

2. Using these results we can give a simple alternative proof of Kannan's result that there is no roper subcategory of **Top** which is simultaneously coreflective and epireflective (i.e. closed with respect to the formation of topological sums, quotients, subspaces and products).

Let $\mathbf{A}$ be the smallest such category. If D is the 2-point discrete space, then $D \in \mathbf{A}$ and $D^\omega \in \mathbf{A}$. D^ω is not indiscretely generated, therefore $S \in \mathbf{A}$ and $S^\alpha \in \mathbf{A}$ for each infinite cardinal α. Thus we get by Theorem 4.1 that $\mathbf{A} = \mathrm{S}\mathbf{A} = \mathbf{Top}$.

3. Let X be any topological space. Are there any bounds for cardinal invariants of a pseudoradial space Y such that X is a subspace of Y.

Radial character: In the paper [13] it is proved in fact that $\mathbf{Gen}(\alpha) \subseteq \mathbf{SPsRad}(2^\alpha)$, which means that if $t(X) \leq \alpha$ then there exists a pseudoradial space with $\sigma_C(Y) \leq 2^\alpha$ such that X is a subspace of Y. For T_0- and T_1-spaces results of [13] yields only estimate $\sigma_C(Y) \leq 2^{\operatorname{card} X}$. [6, Example 3.1] $\mathbb{N}_p = \omega \cup \{p\}$ is a countable space, which is not subsequential $\Rightarrow \sigma_C(Y) > \aleph_0$.

4. Are pseudoradial spaces precisely the spaces determined by nets which contain cofinal well-ordered subnet?

5. Can we replace T_1 with sober spaces in Theorem 3.6?