Continuity and sequential continuity

Continuity at x:

$$(\forall\varepsilon > 0)(\exists\delta > 0)|y - x| \leq \delta \Rightarrow |f(y) - f(x)| \leq \varepsilon \tag{C}$$

Continuity in topological space: Let $O(x)$ denotes a system of open neighborhoods of a point x.

$$(\forall E \in O(f(x)))(\exists D \in O(x))f[D] \subseteq E \tag{D}$$

Of course, we can take a local base for $O(x)$. The last condition corresponds to $x \in D \Rightarrow f(x) \in E$, thus it reduces to (C) for the usual base on $\mathbb{R}$.

Theorem 1. (*AC*) *A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous* ($at\ x \in \mathbb{R}$) *if and only if it is sequential continuous* ($at\ x \in \mathbb{R}$).

The usual proof includes AC in the implication $\Leftarrow$. We show here that the same holds for the global continuity without the AC.

Lemma 1. *Let X, Y be topological spaces. If $f\colon X \to Y$ is continuous, then it is sequential continuous.*

Proof. Let $x_n \to x$. For any neighborhood V of $f(x)$ the set $f^{-1}(V)$ contains all but finitely many members of the sequence $(x_n)_{n=1}^\infty$. Therefore V contains all but finitely many members of $(f(x_n))_{n=1}^\infty$. This means that $f(x_n) \to f(x)$. □

Lemma 2. *If $f\colon \mathbb{R} \to \mathbb{R}$ is sequential continuous at $x \in \mathbb{R}$, then $f|_{\mathbb{Q}\cup\{x\}}$ is continuous at x.*

Proof. The proof follows the same lines as the AC-proof, but we use the fact that we can obtain (explicitly, without AC) $\mathbb{Q}$ as the set of all members of a sequence. Therefore we have choice function $\chi\colon \mathcal{P}(\mathbb{Q}) \setminus \{\emptyset\} \to \mathbb{Q}$ given by $f(A) = q_{n_0}$, where $n_0 = \min\{n \in \mathbb{N}; q_n \in A\}$ and $(q_n)_{n=1}^\infty$ is the enumeration of $\mathbb{Q}$.

Assume that $f|_{\mathbb{Q}\cup\{x\}}$ is not continuous at x. Then there exists $\varepsilon > 0$ such that (C) doesn't hold. This means that for each $n \in \mathbb{N}$ there exists $y \in \mathbb{Q}$ with $|x - y| < \frac{1}{n}$ such that $|f(x) - f(y)| > \varepsilon$. We put $y_n = \chi(\{y; |x - y| < \frac{1}{n} \wedge |f(x) - f(y)| > \varepsilon\})$.

Thus we obtain a sequence (y_n) with $y_n \to x$ and $f(y_n) \not\to f(x)$, a contradiction. □

Theorem 2. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is* (*globally!*) *continuous if and only if it is sequentially continuous.*[1]

Proof. Lemma 1 yields the implication $\Rightarrow$

On the other hand, if f is sequentially continuous at each point, then from Lemma 2 we have that for any x the restriction on the set $\mathbb{Q}\cup\{x\}$ is continuous.

This means that for $\varepsilon > 0$ there exists $\delta > 0$ with

$$(z \in \mathbb{Q}) \wedge (|x - z| < \delta) \Rightarrow |f(x) - f(z)| < \frac{\varepsilon}{2}.$$

[1] The proof is made according to the hint in [H].

For any $y \in \mathbb{R}$ such that $|x-y|<\delta$ there exists δ_y with

$$(z \in \mathbb{Q}) \wedge (|y-z|<\delta_y) \Rightarrow |f(y)-f(z)|<\frac{\varepsilon}{2}.$$

Moreover, $A_y := \{z \in \mathbb{Q}; |z-y|<\delta_y, |z-x|<\delta\} \neq \emptyset$, since $\mathbb{Q}$ is dense subset of $\mathbb{R}$. Therefore we have $z_y := \chi(A_y)$ and

$$|f(x)-f(y)| \leq |f(x)-f(z_y)| + |f(z_y)-f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Question: How can be the above proof generalized?

I think one could prove the following:
If Q is a countable set than there exists a choice function on $\mathcal{P}(Q) \setminus \{\emptyset\}$. (Just use the bijection $\mathbb{N} \to Q$.)
If $f\colon X \to Y$ is sequentially continuous at x then for any countable set $D \subseteq X$ the restriction $f|_{D\cup\{x\}}$ is continuous at x.

But I wasn't able to find the condition for Y such that the above proof would work.

At least this seems to be valid: If X is a countable dense, first countable space and Y be a metric space, then a function $f\colon X \to Y$ is continuous if and only if it is sequentially continuous.

Question: Under AC it holds second countable $\Rightarrow$ countable density. Does it hold without AC?

Dependent (countable) choice

Some choice principles, which are interesting for us:

[H, Definition 3.7]

DC = Principle of dependent choices: For every pair (X, ϱ), where X is a non-empty set and ϱ is a relation on X such that for each $x \in X$ there exists $y \in Y$ with $x\varrho y$, there exists a sequence (x_n) in X with $x_n \varrho x_{n+1}$ for each $n \in \mathbb{N}$.

[H, Theorem 3.8] DC $\Rightarrow$ CC (countable choice)

[H, Definition 3.11]

PIT = Boolean Prime Ideal Theorem: Every boolean algebra such that $0 \neq 1$ has a maximal ideal.

UFT = Ultrafilter Theorem: On any set every filter can be enlarged to an ultrafilter.

[H, Theorem 3.12] UFT $\Leftrightarrow$ PIT

Hamel basis

This part is written according to [A].

Definition 1. Let V be a vector space. We say that B is a *Hamel basis* in V if B is linearly independent and every vector $v \in V$ can be obtained as a linear combination of vectors from B.

Theorem 3. (*AC*) *Every vector space V over field F has a Hamel basis. Moreover, every linearly independent subset W of V can be enlarged to a Hamel bases $B \supseteq W$.*

Sketch of the proof. Union of a chain of linearly independent sets is again linearly independent. Thus by maximality principle there exists a maximal linearly independent set B containing a given subset W. We claim that B is a base.

For any $x \in V$ the set $B \cup \{x\}$ is not linearly independent (by the maximality). Thus there exists $a, a_1, \ldots, a_n \in F$ and $b_1, \ldots, b_n$ with $ax + a_1b_1 + \ldots + a_nb_n = 0$. Thus $x = -\frac{1}{a}(a_1b_1 + \ldots + a_nb_n)$ is a linear combination of vectors from B. □

Theorem 4. *(AC) Every two Hamel bases of a vector space X have the same cardinality.*

Proof. Let B_1, B_2 be two Hamel bases of X. For each $x \in B_1$ let $B_2(x)$ be the (uniquely determined) finite sets of elements of the basis B_2, such that x is the linear combination of these elements. We first show that for every $y \in B_2$ there exists $x \in B_1$ such that $y \in B_2(x)$.

Suppose, on the contrary, that $y \in B_2(x)$ for none x. Then $B_1 \subseteq [B_2 \setminus \{y\}]$ ($[V]$ denotes the linear hull of a set $V \subseteq X$). Since B_1 is a base, we have $[B_2 \setminus \{y\}] = X$ and thus y is a linear combination of elements from $B_2 \setminus \{y\}$. We have shown that B_2 is not linearly independent. This is a contradiction with the assumption that B_2 is a Hamel basis.

We have shown so far $(\forall y \in B_2)(\exists x \in B_1) y \in B_2(x)$. This implies $B_2 = \bigcup_{x \in B_1} B_2(x)$ and $\operatorname{card} B_2 = \operatorname{card}(\bigcup_{x \in B_1} B_2(x)) \leq \operatorname{card} B_1 . \aleph_0 = \operatorname{card} B_1$ (in the last equality we used the fact that $\operatorname{card} B_1$ is infinite.) The same way as we have shown $\operatorname{card} B_2 \leq \operatorname{card} B_1$, we can show the opposite inequality. $\operatorname{card} B_1 \leq \operatorname{card} B_2$. From these two inequalities (by Cantor-Bernstein theorem) we get $\operatorname{card} B_2 = \operatorname{card} B_1$. [2] □

Cauchy equation

We are interested in functions $f\colon \mathbb{R} \to \mathbb{R}$ fulfilling

$$f(x+y) = f(x) + f(y). \tag{1}$$

Note that if we put $x = y = 0$, we get $f(0) = f(0) + f(0)$, so (1) implies

$$f(0) = 0. \tag{2}$$

Observe that $\mathbb{R}$ can be understood as a vector space over the field $\mathbb{Q}$. We will denote this vector space by $V_{\mathbb{R}}(\mathbb{Q})$.

Lemma 3. *Any function $f\colon \mathbb{R} \to \mathbb{R}$ fulfilling (1) is a linear map in the space $V_{\mathbb{R}}(\mathbb{Q})$.*

Proof. We need to show that $f(cx) = cf(x)$, for any $x \in \mathbb{R}$ and any $c \in \mathbb{Q}$, i.e., for any c of the form $c = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$.

From (1) we show by induction that

$$f(cx) = cf(x) \qquad \text{for } c \in \mathbb{N}.$$

[2] This proof is made according to an exercise in [NS]

From (2) we have $0 = f(0) = f(x - x) = f(x) + f(-x)$, thus $f(x) = -f(-x)$ for any $x \in \mathbb{R}$. This implies

$$f(cx) = cf(x) \qquad \text{for } c \in \mathbb{Z}.$$

From this we get

$$p \cdot f(x) = f(px) = f(q \cdot \frac{p}{q}x) = q \cdot f\left(\frac{p}{q}x\right),$$
$$\frac{p}{q}f(x) = f\left(\frac{p}{q}x\right).$$

□

Theorem 5. *Any continuous solution of* (1) *has the form* $f(x) = ax$ *from some* $a \in \mathbb{R}$.

Proof. By Lemma 3 f is a linear map in $V_{\mathbb{R}}(\mathbb{Q})$. Thus it is linear on the linear subspace $\mathbb{Q}$, which is generated by 1. So we have $f(x) = x.f(1) = a.x$ for any $x \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is continuous, the equation $f(x) = ax$ holds for every $x \in \mathbb{R}$. □

Theorem 6. *There exists a non-continuous solution of* (1).

Proof. There exists a Hamel basis B of $V_{\mathbb{R}}(\mathbb{Q})$ containing the independent set $\{1, \sqrt{2}\}$. By putting $f(b) = 1$ for any $b \in B$ we obtain a linear map in $V_{\mathbb{R}}(\mathbb{Q})$ (thus a solution of (1) and (2)) which has not form $f(x) = ax$. Therefore f is not continuous by Theorem 5. □

References

[A] B. Artmann. *Der Zahlbegriff.* Vandenhoeck und Ruprecht, Göttingen, 1983.

[H] H. Herrlich. *The Axiom of Choice.* in preparation.

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