Continuity and sequential continuity

Continuity at \( x \):

\[
(\forall \varepsilon > 0) (\exists \delta > 0) |y - x| \leq \delta \Rightarrow |f(y) - f(x)| \leq \varepsilon
\]

Continuity in topological space: Let \( O(x) \) denotes a system of open neighborhoods of a point \( x \).

\[
(\forall E \in O(f(x)))(\exists D \in O(x)) f[D] \subseteq E
\]

Of course, we can take a local base for \( O(x) \). The last condition corresponds to \( x \in D \Rightarrow f(x) \in E \), thus it reduces to (C) for the usual base on \( \mathbb{R} \).

Theorem 1. (AC) A function \( f: \mathbb{R} \to \mathbb{R} \) is continuous (at \( x \in \mathbb{R} \)) if and only if it is sequential continuous (at \( x \in \mathbb{R} \)).

The usual proof includes AC in the implication \( \Leftarrow \). We show here that the same holds for the global continuity without the AC.

Lemma 1. Let \( X, Y \) be topological spaces. If \( f: X \to Y \) is continuous, then it is sequential continuous.

Proof. Let \( x_n \to x \). For any neighborhood \( V \) of \( f(x) \) the set \( f^{-1}(V) \) contains all but finitely many members of the sequence \( (x_n)_{n=1}^{\infty} \). Therefore \( V \) contains all but finitely many members of \( (f(x_n))_{n=1}^{\infty} \). This means that \( f(x_n) \to f(x) \). \( \square \)

Lemma 2. If \( f: \mathbb{R} \to \mathbb{R} \) is sequential continuous at \( x \in \mathbb{R} \), then \( f|_{\mathbb{Q} \cup \{x\}} \) is continuous at \( x \).

Proof. The proof follows the same lines as the AC-proof, but we use the fact that we can obtain (explicitly, without AC) \( \mathbb{Q} \) as the set of all members of a sequence. Therefore we have choice function \( \chi: \mathcal{P}(\mathbb{Q}) \setminus \{\emptyset\} \to \mathbb{Q} \) given by \( f(A) = q_{n_0} \), where \( n_0 = \min\{n \in \mathbb{N}; q_n \in A\} \) and \( (q_n)_{n=1}^{\infty} \) is the enumeration of \( \mathbb{Q} \).

Assume that \( f|_{\mathbb{Q} \cup \{x\}} \) is not continuous at \( x \). Then there exists \( \varepsilon > 0 \) such that (C) doesn’t hold. This means that for each \( n \in \mathbb{N} \) there exists \( y \in \mathbb{Q} \) with \( |x - y| < \frac{1}{n} \) such that \( |f(x) - f(y)| > \varepsilon \). We put \( y_n = \chi(\{y; |x - y| < \frac{1}{n} \land |f(x) - f(y)| > \varepsilon\}) \).

Thus we obtain a sequence \( (y_n) \) with \( y_n \to x \) and \( f(y_n) \not\to f(x) \), a contradiction. \( \square \)

Theorem 2. A function \( f: \mathbb{R} \to \mathbb{R} \) is (globally!) continuous if and only if it is sequentially continuous.\(^1\)

Proof. Lemma 1 yields the implication \( \Rightarrow \).

On the other hand, if \( f \) is sequentially continuous at each point, then from Lemma 2 we have that for any \( x \) the restriction on the set \( \mathbb{Q} \cup \{x\} \) is continuous.

This means that for \( \varepsilon > 0 \) there exists \( \delta > 0 \) with

\[
(z \in \mathbb{Q}) \land (|x - z| < \delta) \Rightarrow |f(x) - f(z)| < \frac{\varepsilon}{2}.
\]

\(^1\)The proof is made according to the hint in [H].
For any \( y \in \mathbb{R} \) such that \(|x - y| < \delta\) there exists \( \delta_y \) with
\[
(z \in \mathbb{Q}) \land (|y - z| < \delta_y) \Rightarrow |f(y) - f(z)| < \frac{\varepsilon}{2}.
\]
Moreover, \( A_y := \{ z \in \mathbb{Q}; |z - y| < \delta_y, |z - x| < \delta \} \neq \emptyset \), since \( \mathbb{Q} \) is dense subset of \( \mathbb{R} \). Therefore we have \( z_y := \chi(A_y) \) and
\[
|f(x) - f(y)| \leq |f(x) - f(z_y)| + |f(z_y) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Question: How can be the above proof generalized?
I think one could prove the following:
If \( Q \) is a countable set than there exists a choice function on \( \mathcal{P}(Q) \setminus \{\emptyset\} \). (Just use the bijection \( \mathbb{N} \rightarrow Q \).)
If \( f: X \rightarrow Y \) is sequentially continuous at \( x \) then for any countable set \( D \subseteq X \) the restriction \( f|_{D \cup \{x\}} \) is continuous at \( x \).
But I wasn’t able to find the condition for \( Y \) such that the above proof would work.
At least this seems to be valid: If \( X \) is a countable dense, first countable space and \( Y \) be a metric space, then a function \( f: X \rightarrow Y \) is continuous if and only if it is sequentially continuous.
Question: Under \( \text{AC} \) it holds second countable \( \Rightarrow \) countable density. Does it hold without \( \text{AC} \)?

**Dependent (countable) choice**

Some choice principles, which are interesting for us:

[H, Definition 3.7] DC = Principle of dependent choices: For every pair \((X, g)\), where \( X \) is a non-empty set and \( g \) is a relation on \( X \) such that for each \( x \in X \) there exists \( y \in Y \) with \( xgy \), there exists a sequence \((x_n)\) in \( X \) with \( x_n \vDash x_{n+1} \) for each \( n \in \mathbb{N} \).

[H, Theorem 3.8] DC \( \Rightarrow \) CC (countable choice)

[H, Definition 3.11] PIT = Boolean Prime Ideal Theorem: Every boolean algebra such that \( 0 \neq 1 \) has a maximal ideal.

UFT = Ultrafilter Theorem: On any set every filter can be enlarged to an ultrafilter.

[H, Theorem 3.12] UFT \( \iff \) PIT

**Hamel basis**

This part is written according to [A].

**Definition 1.** Let \( V \) be a vector space. We say that \( B \) is a Hamel basis in \( V \) if \( B \) is linearly independent and every vector \( v \in V \) can be obtained as a linear combination of vectors from \( B \).

**Theorem 3.** (\( \text{AC} \)) Every vector space \( V \) over field \( F \) has a Hamel basis. Moreover, every linearly independent subset \( W \) of \( V \) can be enlarged to a Hamel bases \( B \supseteq W \).
Sketch of the proof. Union of a chain of linearly independent sets is again linearly independent. Thus by maximality principle there exists a maximal linearly independent set \( B \) containing a given subset \( W \). We claim that \( B \) is a base.

For any \( x \in V \) the set \( B \cup \{x\} \) is not linearly independent (by the maximality). Thus there exists \( a, a_1, \ldots, a_n \in F \) and \( b_1, \ldots, b_n \) with \( ax + a_1b_1 + \ldots + a_nb_n = 0 \). Thus \( x = -\frac{1}{a}(a_1b_1 + \ldots + a_nb_n) \) is a linear combination of vectors from \( B \).

**Theorem 4.** (AC) Every two Hamel bases of a vector space \( X \) have the same cardinality.

**Proof.** Let \( B_1, B_2 \) be two Hamel bases of \( X \). For each \( x \in B_1 \) let \( B_2(x) \) be the (uniquely determined) finite sets of elements of the basis \( B_2 \), such that \( x \) is the linear combination of these elements. We first show that for every \( y \in B_2 \) there exists \( x \in B_1 \) such that \( y \in B_2(x) \).

Suppose, on the contrary, that \( y \in B_2(x) \) for none \( x \). Then \( B_1 \subseteq [B_2 \setminus \{y\}] \) (\([V]\) denotes the linear hull of a set \( V \subseteq X \)). Since \( B_1 \) is a base, we have \([B_2 \setminus \{y\}] = X \) and thus \( y \) is a linear combination of elements from \( B_2 \setminus \{y\} \). We have shown that \( B_2 \) is not linearly independent. This is a contradiction with the assumption that \( B_2 \) is a Hamel basis.

We have shown so far \( \forall y \in B_2(\exists x \in B_1) y \in B_2(x) \). This implies \( B_2 = \bigcup_{x \in B_1} B_2(x) \) and \( \text{card } B_2 = \text{card} \left( \bigcup_{x \in B_1} B_2(x) \right) \leq \text{card } B_1 \setminus \{y\} = \text{card } B_1 \) (in the last equality we used the fact that \( B_1 \) is infinite.). The same way as we have shown card \( B_2 \leq \text{card } B_1 \), we can show the opposite inequality. card \( B_1 \leq \text{card } B_2 \). From these two inequalities (by Cantor-Bernstein theorem) we get card \( B_2 = \text{card } B_1 \). \( \square \)

**Cauchy equation**

We are interested in functions \( f: \mathbb{R} \to \mathbb{R} \) fulfilling

\[ f(x + y) = f(x) + f(y). \]  

(1)

Note that if we put \( x = y = 0 \), we get \( f(0) = f(0) + f(0) \), so (1) implies

\[ f(0) = 0. \]  

(2)

Observe that \( \mathbb{R} \) can be understood as a vector space over the field \( \mathbb{Q} \). We will denote this vector space by \( V_{\mathbb{R}}(\mathbb{Q}) \).

**Lemma 3.** Any function \( f: \mathbb{R} \to \mathbb{R} \) fulfilling (1) is a linear map in the space \( V_{\mathbb{R}}(\mathbb{Q}) \).

**Proof.** We need to show that \( f(cx) = cf(x) \), for any \( x \in \mathbb{R} \) and any \( c \in \mathbb{Q} \), i.e., for any \( c \) of the form \( c = \frac{p}{q} \), \( p \in \mathbb{Z} \), \( q \in \mathbb{N} \setminus \{0\} \).

From (1) we show by induction that

\[ f(cx) = cf(x) \quad \text{for } c \in \mathbb{N}. \]

\( \square \)This proof is made according to an exercise in [NS]
From (2) we have $0 = f(0) = f(x - x) = f(x) + f(-x)$, thus $f(x) = -f(-x)$ for any $x \in \mathbb{R}$. This implies
$$f(cx) = cf(x) \quad \text{for } c \in \mathbb{Z}.$$ From this we get
$$p \cdot f(x) = f(px) = f(q \cdot \frac{p}{q} x) = q \cdot f \left( \frac{p}{q} x \right),$$
$$\frac{p}{q} f(x) = f \left( \frac{p}{q} x \right).$$

**Theorem 5.** Any continuous solution of (1) has the form $f(x) = ax$ from some $a \in \mathbb{R}$.

**Proof.** By Lemma 3 $f$ is a linear map in $V_{\mathbb{R}}(\mathbb{Q})$. Thus it is linear on the linear subspace $\mathbb{Q}$, which is generated by 1. So we have $f(x) = x.f(1) = a.x$ for any $x \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $f$ is continuous, the equation $f(x) = ax$ holds for every $x \in \mathbb{R}$. 

**Theorem 6.** There exists a non-continuous solution of (1).

**Proof.** There exists a Hamel basis $B$ of $V_{\mathbb{R}}(\mathbb{Q})$ containing the independent set $(1, \sqrt{2})$. By putting $f(b) = 1$ for any $b \in B$ we obtain a linear map in $V_{\mathbb{R}}(\mathbb{Q})$ (thus a solution of (1) and (2)) which has not form $f(x) = ax$. Therefore $f$ is not continuous by Theorem 5. 

**References**

