December 14, 2012

R. Engelking: General Topology

I started to make these notes from [E1] and only later the newer edition [E2] got into my hands. I don't think that there were too much changes in numbering between the two editions, but if you're citing some results from either of these books, you should check the book, too.

Introduction

Algebra of sets. Functions

Cardinal numbers

For every cardinal number $\mathfrak{m}$, the number $2^{\mathfrak{m}}$, also denoted by $\exp \mathfrak{m}$, is defined as the cardinality of the family of all subsets of a set X satisfying $|X| = \mathfrak{m}$.

To every well-ordered set X an ordinal number α is assigned; it is called the *order type* of X.

Order relations. Ordinal numbers

Any ordinal number can be represented as $\lambda + n$ where λ is a limit ordinal number and $n \in \mathbb{N}$. The number $\lambda + n$ is even (odd) if n is even (odd).

A subset A of set X directed by $\leq$ is *cofinal* in X if for every $x \in X$ there exists an $a \in A$ such that $x \leq a$. Cofinal subsets of linearly ordered sets and of ordered sets are defined similarly.

The axiom of choice

Suppose we are given a set X and a property $\mathcal{P}$ pertaining to subsets of X; we say that $\mathcal{P}$ is a property of *finite character* if the empty set has this property and a set $A \subset X$ has property $\mathcal{P}$ if and only if all finite subsets of A have this property.

Lemma (Teichmüller-Tukey lemma). *Suppose we are given a set X and a property $\mathcal{P}$ of subsets of X. If $\mathcal{P}$ is a property of finite character, then every set $A \subset X$ which has property $\mathcal{P}$ is contained in a set $B \subset X$ which has property $\mathcal{P}$ and is maximal in the family of all subset of X that have $\mathcal{P}$ ordered by $\subset$.*

Real numbers

1 Topological spaces

1.1 Topological spaces. Open and closed sets. Bases. Closure and interior of a set

A family $\mathcal{B} \subset \mathcal{O}$ is called a *base for a topological space* $(X, \mathcal{O})$ if every non-empty open subset of X can be represented as the union of a subfamily of $\mathcal{B}$.

(B1) For any $U_1, U_2 \in \mathcal{B}$ and every point $x \in U_1 \cap U_2$ there exists a $U \in \mathcal{B}$ such that $x \in U \subset U_1 \cap U_2$.

(B2) For every $x \in X$ there exists a $U \in \mathcal{B}$ such that $x \in U$.

If for some $x \in X$ and an open set $U \subset X$ we have $x \in U$, we say that U is a *neighbourhood* of x.

A family $\mathcal{B}(x)$ of neighbourhoods of x is called a *base for topological space* $(X, \mathcal{O})$ *at the point* x if for any neighbourhood V of x there exists a $U \in \mathcal{B}(x)$ such that $x \in U \subset V$.

The smallest cardinal number of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base for a topological space $(X, \mathcal{O})$, is called the *weight of the topological space* $(X, \mathcal{O})$ and is denoted by $w(X, \mathcal{O})$.

A family $\mathcal{P} \subset \mathcal{O}$ is called a *subbase for a topological space* $(X, \mathcal{O})$ if the family of all finite intersections $U_1 \ldots U_k$, where $U_i \in \mathcal{P}$ for $i = 1, 2, \ldots, k$, is a base for $(X, \mathcal{O})$.

base for topology $\rightarrow$ base at point

union of bases at point = base for topology

The *character of a point* x in a topological space $(X, \mathcal{O})$ is defined as the smallest cardinal number of the form $|\mathcal{B}(x)|$, where $\mathcal{B}(x)$ is a base for $(X, \mathcal{O})$ at the point x; this cardinal number is denoted by $\chi(x, (X, \mathcal{O}))$. The *character of a topological space* $(X, \mathcal{O})$ is defined as the supremum of all numbers $\chi(x, (X, \mathcal{O}))$ for $x \in X$; this cardinal number is denoted by $\chi((X, \mathcal{O}))$.

$\chi(X) \leq \aleph_0$=*first-countable*

$w(X) \leq \aleph_0$=*second-countable*

Let $(X, \mathcal{O})$ be a topological space and suppose that for every $x \in X$ a base $\mathcal{B}(x)$ for $(X, \mathcal{O})$ at x is given; the collection $\{\mathcal{B}(x)\}_{x \in X}$ is called a *neighbourhood system for the topological space* $(X, \mathcal{O})$. We shall show that any neighbourhood system $\{\mathcal{B}(x)\}_{x \in X}$ has the following properties:

(BP1) For every $x \in X$, $\mathcal{B}(x) \neq \emptyset$ and for every $U \in \mathcal{B}(x)$, $x \in U$.

(BP2) If $x \in U \in \mathcal{B}(y)$, then there exists a $V \in \mathcal{B}(x)$ such that $V \subset U$.

(BP3) For any $U_1, U_2 \in \mathcal{B}(x)$ there exists a $U \in \mathcal{B}(x)$ such that $U \subset U_1 \cap U_2$.

Corollary. *(1.1.2) If U is an open set and $U \cap A = \emptyset$, then also $U \cap \overline{A} = \emptyset$.*

Theorem. *(1.1.3) The closure operator has the following properties:*

(CO1) $\overline{\emptyset} = \emptyset$

(CO2) $A \subset \overline{A}$

(CO3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(CO4) $\overline{(\overline{A})} = \overline{A}$

Theorem. *(1.1.5) For every $A \subset X$ we have* $\operatorname{Int} A = X \setminus \overline{X \setminus A}$.

Theorem. *(1.1.6) The interior operator has the following properties:*

(IO1) $\operatorname{Int} X = X$

(IO2) $\operatorname{Int} A \subset A$

(IO3) $\operatorname{Int} A \cap B = \operatorname{Int} A \cap \operatorname{Int} B$

(IO4) $\operatorname{Int}(\operatorname{Int} A) = \operatorname{Int} A$

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two topologies on X and $\mathcal{O}_2 \subset \mathcal{O}_1$, then we say that topology $\mathcal{O}_1$ is *finer* than the topology $\mathcal{O}_2$, or that topology $\mathcal{O}_2$ is *coarser* than $\mathcal{O}_1$.

A family $\{A_s\}_{s\in S}$ of subsets of a topological space X is a *locally finite* if for every point $x \in X$ there exists a neighbourhood such that the set $\{s \in S : U \cap A_s \neq \emptyset\}$ is finite. If every point $x \in X$ has a neighbourhood that intersects at most one set of a given family, then we say that the family is *discrete*.

Theorem. *(1.1.11) For every locally finite family $\{A_s\}_{s\in S}$ we have the equality $\overline{\bigcup_{s\in S} A_s} = \bigcup_{s\in S} \overline{A_s}$.*

Corollary. *(1.1.12) Let $\mathcal{F}$ be a locally finite family and $F = \bigcup \mathcal{F}$. If all members of $\mathcal{F}$ are closed, then F is a closed set and if all members of $\mathcal{F}$ are clopen, then F is clopen.*

Theorem. *(1.1.13) If $\{A_s\}_{s\in S}$ is locally finite (discrete), then the family $\{\overline{A}_s\}_{s\in S}$ also is locally finite (discrete).*

Theorem. *(1.1.14) If $w(X) \leq \mathfrak{m}$, then for every family $\{U_s\}_{s\in S}$ of open subsets of X there exists a set $S_0 \subset S$ such that $|S_0| \leq \mathfrak{m}$ and $\bigcup_{s\in S_0} U_s = \bigcup_{s\in S} U_s$.*

Theorem. *(1.1.15) If $w(X) \leq \mathfrak{m}$, then for every base $\mathcal{B}$ for X there exists a base $\mathcal{B}_0$ such that $|\mathcal{B}_0| \leq \mathfrak{m}$ and $\mathcal{B}_0 \subset \mathcal{B}$.*

Remark. (1.1.16) Let us note that in the proof of Theorem 1.1.14 we did not use the fact that the members of $\mathcal{B}$ are open (cf. the notion of network defined in Section 3.1).

Theory of real numbers (as equivalence classes) was proposed independently by Ch. Méray and G. Cantor.

(Exercise 1.1.C) A subset U of a topological space satisfying the condition $U = \operatorname{Int} \overline{U}$ is called an open domain.

1.2 Methods of generating topologies

Proposition. *(1.2.1) Suppose we are given a set X and a family $\mathcal{B}$ of subsets of X which has properties (B1)-(B2). Let $\mathcal{O}$ be the family of all subsets of X that are unions of subfamilies of $\mathcal{B}$, i.e., let*

$$U \in \mathcal{O} \text{ if and only if } U = \bigcup \mathcal{B}_0 \text{ for a subfamily } \mathcal{B}_0 \text{ of } \mathcal{B}.$$

The family $\mathcal{O}$ is a topology on X. The family $\mathcal{B}$ is a base for the topological space $(X, \mathcal{O})$.

Example. (1.2.2) Real numbers with topology defined by base $\langle a, b) = K$ - Sorgenfrey line.

Example. (1.2.4) $L = \{(x, y) \in \mathbb{R}^2 \; y \geq 0\}$. For points of the line $y = 0$ we define bases by circles touching it and for $y \neq 0$ as usual. We get Niemytzki plane.

1.3 Boundary of a set and derived set. Dense and nowhere dense sets. Borel sets

Boundary of A: $\operatorname{Fr} A = \overline{A} \cap \overline{X \setminus A} = \overline{A} \setminus \operatorname{Int} A$

Theorem. *(1.3.2) The boundary operator has the following properties:*

(i) $\operatorname{Int} A = A \setminus \operatorname{Fr} A$

(ii) $\overline{A} = A \cup \operatorname{Fr} A$

(iii) $\operatorname{Fr}(A \cup B) \subset \operatorname{Fr} A \cup \operatorname{Fr} B$

(iv) $\operatorname{Fr}(A \cap B) \subset \operatorname{Fr} A \cap \operatorname{Fr} B$

(v) $\operatorname{Fr}(X \setminus A) = \operatorname{Fr} A$

(vi) $X = \operatorname{Int} A \cup \operatorname{Fr} A \cup \operatorname{Int}(X \setminus A)$

(vii) $\operatorname{Fr} \overline{A} \subset \operatorname{Fr} A$

(viii) $\operatorname{Fr} \operatorname{Int} A \subset \operatorname{Fr} A$

(ix) *A is open if and only if* $\operatorname{Fr} A = \overline{A} \setminus A$

(x) *A is closed if and only if* $\operatorname{Fr} A = A \setminus \operatorname{Int} A$

(xi) *A is clopen if and only if* $\operatorname{Fr} A = \emptyset$

A point x in a topological space X is called an *accumulation point* of a set $A \subset X$ if $x \in \overline{A \setminus \{x\}}$. The set of all accumulation points of A is called the *derived set* of A and is denoted by A^d.

Theorem. *(1.3.4) The derived set has the following properties:*

(i) $\overline{A} = A \cup A^d$

(ii) *If $A \subset B$, then $A^d \subset B^d$.*

(iii) $(A \cup B)^d = A^d \cup B^d$

(iv) $\bigcup_{s \in S} A_s^d \subset (\bigcup_{s \in S} A_s)^d$

A set $A \subset X$ is called *dense* in X if $\overline{A} = X$.
A set $A \subset X$ is called *co-dense* in X if $X \setminus A$ is dense.
A set $A \subset X$ is called *nowhere dense* in X if $\overline{A}$ is co-dense.
A set $A \subset X$ is called *dense in itself* if $A \subset A^d$.

Proposition. *(1.3.5) The set A is dense in X if and only if every non-empty open subset of X contains points of A.*

The set A is co-dense in X if and only if every non-empty open subset of X contains points of complement of A.

The set A is nowhere dense in X if and only if every non-empty open subset of X contains a non-empty open set disjoint from A.

Theorem. *(1.3.6) If A is dense in X, then for every open $U \subset X$ we have $\overline{U} = \overline{U \cap A}$.*

The *density of a space* X is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X. If $d(X) \leq \aleph_0$, then we say that the space X is *separable.*

Theorem. *(1.3.7) For every topological space X we have $d(X) \leq w(X)$.*

Corollary. *(1.3.8) Every second-countable space is separable.*

Borel sets, F_σ, G_δ
Complement of F_σ set is G_δ set.

1.4 Continuous mappings. Closed and open mappings. Homeomorphisms

Proposition. *(1.4.1) Let X and Y be topological spaces and f a mapping of X to Y. The following conditions are equivalent:*

(i) The mapping f is continuous.

(ii) Inverse images of all members of a subbase for Y are open in X.

(iii) Inverse images of all members of a base for Y are open in X.

(iv) There are neighborhood systems $\{B(x)\}_{x\in X}$ and $\{D(y)\}_{y\in Y}$ for X and Y respectively, such that for every $x \in X$ and $V \in D(f(x))$ there exists a $U \in B(x)$ satisfying $f(U) \subset V$.

(v) For every $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.

(vi) For every $B \subset Y$ we have $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

(vii) For every $B \subset Y$ we have $f^{-1}(\operatorname{Int} B) \subset \operatorname{Int} f^{-1}(B)$.

Let us observe in connection with the above theorem, that if $f\colon X \to Y$ then for any F_σ (G_δ) $B \subset Y$ the inverse image $f^{-1}(B)$ is an F_σ-set (G_δ-set). Inverse image of Borel sets in Y are Borel sets in X. (1.4.G)

Theorem. *(1.4.7) If a sequence (f_i) of continuous functions from X to $\mathbb{R}$ or I is uniformly convergent to a real-valued function f, then f is a continuous function from X to $\mathbb{R}$. If all f_i's are functions to I, then $f\colon X \to I$.*

Proposition. *(1.4.8) Suppose we are given a set X, a family $\{Y_s\}_{s\in S}$ of topological spaces and a family of mappings $\{f_s\}_{s\in S}$, where f_s is a mapping of X to Y_s. In the class of all topologies on X that makes all f_s's continuous there exists a coarsest topology; this is the topology $\mathcal{O}$ generated by the base consisting of all sets of the form*

$$\bigcap_{i=1}^{k} f_{s_i}^{-1}[V_i],$$

where $s_1, s_2, \ldots, s_k \in S$ and V_i is an open subset of Y_{s_i} for $i = 1, 2, \ldots, k$.

The topology $\mathcal{O}$ is called the topology generated by the family of mappings $\{f_s\}_{s\in S}$.

Proposition. *(1.4.9) A mapping f of a topological space X to a topological space Y whose topology is generated by a family of mappings $\{f_s\}_{s\in S}$, where f_s is a mapping of Y to Y_s, is continuous if and only if the composition $f_s f$ is continuous for every $s \in S$.*

Theorem. *(1.4.10) If Y is a continuous image of X, then $d(Y) \leq d(X)$.*

Corollary. *(1.4.11) Continuous images of separable spaces are separable.*

A continuous mapping $f\colon X \to Y$ is called a *closed* (an *open*) mapping if for every closed (open) set $A \subset X$ the image $f[A]$ is closed (open) in Y. Mappings which are simultaneously closed and open are called *closed-and-open* mappings.

Theorem. *(1.4.12) A mapping $f\colon X \to Y$ is closed (open) if and only if for every $B \subset Y$ and every open (closed) set $A \subset X$ which contains $f^{-1}(B)$, there exists an open (a closed) set $C \subset Y$ containing B and such that $f^{-1}(C) \subset A$.*

Theorem. *(1.4.13) A mapping $f\colon X \to Y$ is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ which contains $f^{-1}(y)$, there exists in Y a neighbourhood V of the point y such that $f^{-1}(V) \subset U$.*

Theorem. *(1.4.14) A mapping $f\colon X \to Y$ is open if and only if there exists a base $\mathcal{B}$ for X such that $f[U]$ is open in Y for every $U \in \mathcal{B}$.*

Theorem. *(1.4.16) For every open mapping $f\colon X \to Y$ and every $x \in X$ we have $\chi(f(x), Y) \leq \chi(x, X)$. If, moreover, $f[X] = Y$, then $w(Y) \leq w(X)$ and $\chi(Y) \leq \chi(X)$.*

Example. (1.4.17) $X = R$, $Y = R/N$, $f\colon X \to Y$ is closed and onto. We get $\chi(Y) > \aleph_0$ and $w(Y) > \aleph_0$, while $w(X) = \chi(X) = \aleph_0$.

$A(\alpha)$ = space on a set with cardinality α, topology= all subsets that do not contain x_0 and all subsets of X that have finite complement.

1.5 Axioms of separations

Theorem. *(1.5.1) For every T_0-space X we have $|X| \leq \exp w(X)$.*

Proposition. *(1.5.2) Suppose we are given a set X and a collection $\{\mathcal{B}(x)\}_{x\in X}$ of families of subsets of X which has properties (BP1)-(BP3). If in addition the collection $\{\mathcal{B}(x)\}_{x\in X}$ has the following property*

(BP4) For every pair of distinct points $x, y \in X$ there exist open set $U \in \mathcal{B}(x)$ and $V \in \mathcal{B}(y)$ such that $U \cap V = \emptyset$,

then the space X with the topology generated by the neighbourhood system $\{\mathcal{B}(x)\}_{x\in X}$ is a Hausdorff space.

Theorem. *(1.5.3) For every Hausdorff space X we have $|X| \leq \exp\exp d(X)$ and $|X| \leq [d(X)]^{\chi(X)}$.*

Theorem. *(1.5.4) For any pair f, g of continuous mappings of a space X into Hausdorff space Y the set $\{x \in X : f(x) = g(x)\}$ is closed.*

A topological space X is called a T_3-*space* or *regular* space, if X is a T_1-space and for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exist open sets U, V such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

Proposition. *(1.5.5) A T_1-space X is a regular space if and only if for every $x \in X$ and every neighbourhood V of x in a fixed subbase $\mathcal{P}$ there exists a neighbourhood U of x such that $\overline{U} \subset V$.*

Theorem. *(1.5.6) For every regular space we have $w(X) \leq \exp d(X)$.*

A topological space is called a $T_{3\frac{1}{2}}$-space or *Tychonoff space* or *completely regular space*, if X is T_1-space and for every $x \in X$ and every closed set $F \subset X$, $x \notin F$ there exists a continuous function $f\colon X \to I$ such that $f(x) = 0$ and $f(F) = 1$.

A topological space is called T_4*-space* or *normal space*, if X is a T_1-space and for every pair of disjoint closed subsets $A, B \subset X$ there exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Theorem (Urysohn's lemma)**.** *(1.5.10) For every pair A, B of disjoint closed subsets of a normal space X there exists a continuous function $f\colon X \to I$ such that $f(A) = 0$ and $f(B) = 1$.*

Corollary. *(1.5.11) A subset A of a normal space X is a closed G_δ-set if and only if there exists a continuous function $f\colon X \to I$ such that $A = f^{-1}(0)$.*

Corollary. *(1.5.12) A subset A of a normal space X is an open F_σ-set if and only if there exists a continuous function $f\colon X \to I$ such that $A = f^{-1}((0, 1\rangle)$.*

Two subsets A and B of a topological space X are called *completely separated* if there exists a continuous function $f\colon X \to I$ such that $f(A) = 0$ and $f(B) = 1$. We say that f separates sets A and B.

A subset A of a topological space X is called *functionally closed*[1] if $A = f^{-1}(0)$ for some $f\colon X \to I$. Every functionally closed set is closed. The complement of functionally closed set is called *functionally open*.

One readily verifies that a T_1-space X is completely regular if and only if the family of all functionally open sets is a base for X. In a normal space functionally closed (open) sets coincide with closed G_δ-sets (open F_σ-sets).

Theorem. *(1.5.13) Any disjoint functionally closed sets A, B in a topological space X are completely separated; moreover, there exists a continuous function $f\colon X \to I$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

Lemma. *(1.5.14) If X is a T_1-space and for every closed set $F \subset X$ and every open $W \subset X$ that contains F there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $F \subset \bigcup_{i=1}^{\infty} W_i$ and $\overline{W_i} \subset W$ for $i = 1, 2, \ldots$, then the space X is normal.*

One can easily check that the condition in the above lemma is not only sufficient but also necessary for normality of a T_1-space X.

Theorem. *(1.5.15) Every second-countable regular space is normal.*

Theorem. *(1.5.16) Every countable regular space is normal.*

Example. (1.5.17) Sorgenfrey line K is a normal space.

[1] The terms "functionally closed set" and "functionally open set" adopted here seem more suitable than the terms "*zero-set*" and "*cozero-set*" which are generally used.

A family $\{A_s\}_{s\in S}$ of subsets of a set X is called a *cover* of X if $\bigcup_{s\in S} A_s = X$. If X is a topological space and all sets A_s are open (closed), we say that the cover $\{A_s\}$ is *open (closed).* A family $\{A_s\}_{s\in S}$ is called *point-finite (point-countable)* if for every $x \in X$ the set $\{s \in S : x \in A_s\}$ is finite (countable). Clearly every locally finite cover is point-finite ($\not\Leftarrow$).

Theorem. *(1.5.18) For every point-finite open cover $\{U_s\}_{s\in S}$ of a normal space X there exists an open cover $\{V_s\}_{s\in S}$ such that $\overline{V_s} \subset U_s$ for every $s \in S$.*

A topological space X is a *perfectly normal space* if X is a normal space and every closed subset of X is a G_δ-set (equivalently every open subset is F_δ).

Theorem (The Vedenissoff theorem)**.** *(1.5.19) For every T_1-space the following conditions are equivalent:*

(i) The space X is perfectly normal.

(ii) Open subsets of X are functionally open.

(iii) Closed subsets of X are functionally closed.

(iv) For every pair of disjoint closed subsets $A, B \subset X$ there exists a continuous function $f\colon X \to I$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Theorem. *(1.5.20) The class of all T_i-spaces for $i = 1$ and 4 and the class of perfectly normal spaces are invariant under closed mappings.*

1.5.C: A continuous mapping $f\colon X \to X$ is called a *retraction* of X, if $ff = f$; the set of all values of a retraction of X is called a *retract* of X.

Any retract of a Hausdorff space is closed.

1.6 Convergence in topological spaces: Nets and filters. Sequential spaces and Fréchet spaces

We say that the net $S' = \{x_{\sigma'}, \sigma' \in \Sigma'\}$ is *finer* than the net $S = \{x_\sigma, \sigma \in \Sigma\}$ it there exists a function φ of Σ' to Σ with following properties:

(i) For every $\sigma_0 \in \Sigma$ there exists a $\sigma_0' \in \Sigma'$ such that $\varphi(\sigma') \geq \sigma_0$ whenever $\sigma' \geq \sigma_0'$.

(ii) $x_{\varphi(\sigma')} = x_{\sigma'}$ for $\sigma' \in \Sigma'$.

A point x is called a *cluster point of a net* $S = \{x_\sigma, \sigma \in \Sigma\}$ if for every $\sigma_0 \in \Sigma$ there exists a $\sigma \geq \sigma_0$ such that $x_\sigma \in U$.

Proposition. *(1.6.1) If x is a cluster point of the net S' that is finer then S, then x is a cluster point of S. If x is a limit of S, then it is a limits of S'. If x is a cluster point of the net S, then it is a limit of some net S' that is finer than S.*

Proposition. *(1.6.3) The point x belongs to $\overline{A}$ if and only if there exists a net consisting of elements of A and converging to X.*

Corollary. *(1.6.4) A set A is closed if and only if together with any net it contains all its limits.*

Corollary. *(1.6.5) The point x belongs to A^d if and only if there exists a net $S = \{x_\sigma, \sigma \in \Sigma\}$ converging to X, such that $x_\sigma \in A$ and $x_\sigma \neq x$ for every $\sigma \in \Sigma$.*

Proposition. *(1.6.6) A mapping f of a topological space X to a topological space Y is continuous if and only if*

$$f[\lim_{\sigma\in\Sigma} x_\sigma] \subset \lim_{\sigma\in\Sigma} f(x_\sigma)$$

for every net $\{x_\sigma, \sigma \in \Sigma\}$ in the space X.

Proposition. *(1.6.7) A topological space X a Hausdorff space if and only if every net in X has at most one limit.*

Let $\mathcal{R}$ be a family of sets that contains together with A and B the intersection $A \cap B$. By a *filter in* $\mathcal{R}$ we mean a non-empty subfamily $\mathcal{F} \subset \mathcal{R}$ satisfying the following conditions:

(F1) $\emptyset \notin \mathcal{F}$

(F2) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.

(F3) If $A \in \mathcal{F}$ and $A \subset A_1 \in \mathcal{R}$, then $A_1 \in \mathcal{R}$.

A *filter-base* in $\mathcal{R}$ is a non-empty family $\mathcal{G} \subset \mathcal{R}$ such that $\emptyset \notin \mathcal{G}$ and
(FB) If $A_1, A_2 \in \mathcal{G}$, then there exists an $A_3 \in \mathcal{G}$ such that $A_3 \subset A_1 \cap A_2$.

A point x is called a *limit of a filter* $\mathcal{F}$ if every neighborhood of x is a member of $\mathcal{F}$.

A point x is called a *cluster point of a filter* $\mathcal{F}$ if x belongs to closure of every member of $\mathcal{F}$.

We say that a filter $\mathcal{F}'$ is *finer* than a filter $\mathcal{F}$ if $\mathcal{F}' \supset \mathcal{F}$.

Proposition. *(1.6.8) If x is cluster point of the filter $\mathcal{F}'$ that is finer than $\mathcal{F}$, then x is a cluster point of the filter $\mathcal{F}$. If x is a limit of $\mathcal{F}$, then it is a limits of $\mathcal{F}'$. If x is a cluster point of the filter $\mathcal{F}$, then it is a limit of some filter $\mathcal{F}'$ that is finer than S.*

Proposition. *(1.6.9) The point x belongs to $\overline{A}$ if and only if there exists a filter-base consisting of subsets of A converging to x.*

Proposition. *(1.6.10) A mapping f of a topological space X to a topological space Y is continuous if and only if for every filter-base $\mathcal{G}$ in the space X and the filter-base $f[\mathcal{G}] = \{f[A] : A \in \mathcal{G}\}$ in the space Y we have*

$$f[\lim \mathcal{G}] \subset \lim f[\mathcal{G}].$$

Proposition. *(1.6.11) A topological space X is a Hausdorff space if and only if every filter in X has at most one limit.*

Theorem. *(1.6.12) For every net $S = \{x_\sigma, \sigma \in \Sigma\}$ in a topological space X, the family $\mathcal{F}(S)$, consisting of all sets $A \subset X$ with the property that there exists a $\sigma_0 \in \Sigma$ such that $x_\sigma \in A$ whenever $\sigma \geq \sigma_0$, is a filter in the space X and*

$$\lim \mathcal{F}(S) = \lim S.$$

If a net S' is finer than the net S, then the filter $\mathcal{F}(S')$ is finer than the filter $\mathcal{F}(S)$.

Theorem. *(1.6.13) Let $\mathcal{F}$ be a filter in a topological space X; let us denote by Σ the set of all pairs (x, A), where $x \in A \in \mathcal{F}$ and let us define that $(x_1, A_1) \leq (x_2, A_2)$ if $A_2 \subset A_1$. The set Σ is directed by $\leq$, and for the net $S(\mathcal{F}) = \{x_\sigma, \sigma \in \Sigma\}$, where $x_\sigma = x$ for $\sigma = (x, A) \in \Sigma$, we have $\mathcal{F} = \mathcal{F}(S(\mathcal{F}))$ and*

$$\lim S(\mathcal{F}) = \lim \mathcal{F}.$$

sequential space, *Fréchet space*

Theorem. *(1.6.14) Every first-countable space is a Fréchet space and every Fréchet space is a sequential space.*

Proposition. *(1.6.15) A mapping f of a sequential space X to a topological space Y is continuous if and only if $f[\lim x_i] \subset \lim f(x_i)$ for every sequence (x_i) in the space X.*

Proposition. *(1.6.16) If every sequence in a topological space X has at most one limit, then X is a T_1-space. If, moreover, X is first-countable then X is a Hausdorff space.*

Proposition. *(1.6.17) A first-countable space X is a Hausdorff space if and only if every sequence in the space X has at most one limit.*

1.7 Problems

1.7.1 Urysohn spaces and semiregular spaces I

TODO Urysohn space [2]

A topological space X is called a semiregular space if X is a T_2-space and the family of all open domains is a base for X.

Let $(X, \mathcal{O})$ be a Hausdorff space. Generate on X a topology $\mathcal{O}' \subset \mathcal{O}$ by the base consisting of all open domains of $(X, \mathcal{O})$ and show that the space $(X, \mathcal{O}')$ is semiregular and has the same open domains as the space $(X, \mathcal{O})$.

1.7.2 Cantor-Bendixson theorem

perfect set= dense in itself and closed
scattered set= contains no non-empty dense in itself subset

Show that if each member of a family $\mathcal{A}$ of subset of a space X is dense in itself, then the union $\bigcup \mathcal{A}$ is dense in itself. Note that if $A \subset X$ is dense in itself, then the closure $\overline{A}$ is dense in itself. Deduce from the above that every topological space can be represented as the union of two disjoint sets, one of which is perfect and the second one is scattered.

A point x of a topological space X is called a *condensation point* of a set $A \subset X$ if every neighborhood of x contains uncountably many points of A; the set of all condensation points of A is denoted by A^0.

Verify that $A^0 \subset A^d$, $A^0 = \overline{A^0}$ and $(A \cup B)^0 = A^0 \cup B^0$. Show that for every subset A of a second-countable space, the difference $A \setminus A^0$ is countable and $(A^0)^0 = A^0$.

Deduce from the above that every second-countable space can be represented as the union of two disjoint sets, of which one is perfect and the other countable (this is the *Cantor-Bendixson theorem*). Cantor and I. Bendixson proved this fact independently in 1883 for subsets of the real line.

[2] TODO

1.7.3 Cardinal functions I

The smallest infinite cardinal number α such that every family of pairwise disjoint non-empty open subsets of X has cardinality $\leq \alpha$ is called the *Souslin number* or *cellularity* and denoted by $c(X)$.

$$w(X) \geq d(X) \geq c(X)$$

The smallest infinite cardinal number α such that every subset of X consisting exclusively of isolated points (i.e. satisfying the equality $A = A \setminus A^d$) has cardinality $\leq \alpha$ is denoted by $hc(X)$.

$$w(X) \geq hc(X) \geq c(X)$$

The smallest infinite cardinal number α such that every closed subset consisting exclusively of isolated points has cardinality $\leq \alpha$ is called the *extent* of the space X and denoted by $e(X)$.

$$w(X) \geq hc(X) \geq e(X)$$

For sake of simplicity, in all problems about cardinal functions, the cardinal functions defined in the main body of the book (weight, character and density, as yet) will be re-defined to assume only infinite values: the new value of $f(X)$ is defined to be $\aleph_0$ if the old value is finite, and to be equal to the old value if this is an infinite cardinal number. (Sometimes topologists say that "there are no finite cardinal numbers in general topology").

If Y is a continuous image of X, then $c(Y) \leq c(X)$ and $hc(Y) \leq hc(X)$. If, moreover, X is a T_1-space, then also $e(Y) \leq e(X)$.

The *tightness of a point* x in a topological space X is the smallest cardinal number $\mathfrak{m} \geq \aleph_0$ with the property that if $x \in \overline{C}$, then there exists a $C_0 \subset C$ such that $|C_0| \leq \mathfrak{m}$ and $x \in \overline{C}_0$; this cardinal number is denoted by $\tau(x, X)$. The *tightness of a topological space* X is the supremum of all numbers $\tau(x, X)$ for $x \in X$; this cardinal number is denoted by $\tau(X)$.

$\tau(x, X) \leq \chi(x, X)$ and $\tau(X) \leq \chi(X)$.

Tightness $\tau(X)$ is equal to the smallest cardinal number $\mathfrak{m} \geq \aleph_0$ with the property that for any $C \subset X$ which is not closed there exists a $C_0 \subset C$ such that $|C_0| \leq \mathfrak{m}$ and $\overline{C}_0 \setminus C \neq \emptyset$.

For every sequential space we have $\tau(X) = \aleph_0$.

2 Operations on topological spaces

2.1 Subspaces

$\tilde{A} = \overline{A} \cap M$ ($\tilde{A}$=in subspace M)

Proposition. *(2.1.3) If the composition gf of mappings $f \colon X \to Y$ and $g \colon Y \to Z$ is closed (open), then the restriction $g|f[X] \colon f[X] \to Z$ is closed (open).*

Proposition. *(2.1.4) If $f \colon X \to Y$ is a closed (an open) mapping, then on any subspace the $L \subset Y$ the restriction $f_L \colon f^{-1}(L) \to L$ is closed (open).*

homeomorphic embedding

Theorem. *(2.1.6) Any subspace of a T_i-space is a T_i-space for $i \le 3\frac{1}{2}$. Normality is hereditary with respect to closed subsets. Perfect normality is a hereditary property.*

Two subsets A and B of a topological space X are called *separated* if $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Two disjoint sets are separated if and only if neither of them contains accumulation points of the other.

Theorem. *(2.1.7) For every T_1-space X the following conditions are equivalent:*

(i) The space X is hereditarily normal.

(ii) Every open subspace of X is normal.

(iii) For every pair of separated sets $A, B \subset X$ there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Hereditarily normal spaces are sometimes called T_5*-spaces,* and members of the narrower class of perfectly normal spaces are called T_6*-spaces.*

Theorem (Tietze-Urysohn theorem)**.** *(2.1.8) Every continuous function from a closed subspace M of a normal space X to I or R is continuously extendable over X.*

Theorem. *(2.1.9) If a continuous mapping f of a dense subset A of a topological space X to a Hausdorff space Y is continuously extendable over X, then the extension is uniquely determined by f.*

Niemytzki plane is not normal.

Proposition. *(2.1.11) If $\{U_s\}_{s \in S}$ is an open cover of a space X and $\{f_s\}_{s \in S}$, where $f_s \colon U_s \to Y$ is a family of compatible continuous mappings, the combination $f = \nabla f_s$ is a continuous mapping of X to Y.*

Corollary. *(2.1.12) A mapping f of a topological space X to a topological space Y is continuous if and only if every point $x \in X$ has a neighborhood U_x such that $f|U_x$ is continuous.*

Proposition. *(2.1.13) The same as preceding proposition for locally finite closed cover.*

Theorem. *(2.1.14) For every countable discrete family $\{F_i\}_{i=1}^{\infty}$ of closed subsets of a normal space X there exists a family $\{U_i\}_{i=1}^{\infty}$ of open subsets of X such that $F_i \subset U_i$ for $i = 1, 2, \dots$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \ne j$.*

Proposition. *(2.1.15) Suppose we are given a topological space X, a cover $\{A_s\}_{s \in S}$ of the space X and a family $\{f_s\}_{s \in S}$ of compatible mappings, where $f_s \colon A_s \to Y$ such that the combination $f = \mathop{\nabla}\limits_{s \in S} f_s \colon X \to Y$ is continuous. If all mappings f_s are open (closed and the family $f_s[A_s]$ is locally finite), then the combination f is open (closed).*

2.1.D: Verify that a subspace M of a topological space X is a retract of X if and only if every continuous mapping defined on M is extendable over X of - equivalently – if and only if there exists a mapping $r \colon X \to M$ such that $r|M = id_M$.

2.1.E: Prove that normality is hereditary with respect to F_σ-sets.

2.1.I: Prove that the Sorgenfrey line is hereditarily separable.

2.2 Sums

Theorem. *(2.2.7) Any sum of T_i-spaces is a T_i-space for $i \leq 6$.*

2.3 Cartesian products

Proposition. *(2.3.1) The family of all sets $\prod\limits_{s \in S} W_s$, where W_s is an open subset of X_s and $W_s \neq X_s$ only for finitely many $s \in S$, is a base for the Cartesian product $\prod\limits_{s \in S} X_s$.*

Moreover, if for every $s \in S$ a base $\mathcal{B}_s$ for X_s is fixed, then the subfamily consisting of those $\prod\limits_{s \in S} W_s$ in which $W_s \in \mathcal{B}_s$ whenever $W_s \neq X_s$, also is a base.

The base for $\prod\limits_{s \in S} X_s$ described in the first part of the above proposition is called the *canonical base* for the Cartesian product.

Proposition. *(2.3.2) If $\{X_s\}_{s \in S}$ is a family of topological spaces and A_s is for every $s \in S$ a subspace of X_s, then the two topologies defined on the set $A = \prod\limits_{s \in S} A_s$, viz, the topology of the Cartesian product of subspaces $\{A_s\}_{s \in S}$ and the topology of a subspace of the Cartesian product $\prod\limits_{s \in S} X_s$, coincide.*

Proposition. *(2.3.3) For every family of sets $\{A_s\}$ where $A_s \subset X_s$ in the Cartesian product $\prod X_s$ we have $\overline{\prod A_s} = \prod \overline{A_s}$.*

Corollary. *(2.3.4) The set $\prod A_s$, where $\emptyset \neq A_s \subset X_s$, is closed in $\prod X_s$ if and only if every A_s is closed in X_s.*

Corollary. *(2.3.5) The set $\prod A_s$, where $\emptyset \neq A_s \subset X_s$, is dense in $\prod X_s$ if and only if every A_s is dense in X_s.*

Projections are open but they aren't closed in general.

Theorem. *(2.3.11) Any Cartesian product of T_i-spaces is a T_i-space for $i \leq 3\frac{1}{2}$. If the Cartesian product $\prod\limits_{s \in S} X_s$ is a non-empty T_i-space, then all X_s's are T_i-spaces for $i \leq 6$.*

Example. (2.3.12) $K \times K$ is not normal, K - the Sorgenfrey line.

Theorem. *(2.3.13) If $w(X_s) \leq \alpha \geq \aleph_0$ for every $s \in S$ and $\operatorname{card} S \leq \alpha$ then $w(\prod_{s \in S} X_s) \leq \alpha$.*

Similarly, if $\chi(X_s) \leq \alpha \geq \aleph_0$ for every $s \in S$ and $\operatorname{card} S \leq \alpha$ then $\chi(\prod_{s \in S} X_s) \leq \alpha$.

Corollary. *(2.3.14) First-countability and second-countability are $\aleph_0$-multiplicative properties.*

Theorem (Hewitt-Marczewski-Pondiczery)**.** *(2.3.15) If $d(X_s) \leq \alpha \geq \aleph_0$ for every $s \in S$ and $\operatorname{card} S \leq 2^\alpha$, then $d(\prod X_s) \leq \alpha$.*

Corollary. *(2.3.16) Separability is a c-multiplicative property.*

Theorem. *(2.3.17) If* $d(X_s) \le \alpha \ge \aleph_0$ *for every* $s \in S$*, then any family of pairwise disjoint non-empty open subsets of the Cartesian product has cardinality* $\le \alpha$*.*

Corollary. *(2.3.18) In the Cartesian product of separable spaces any family of pairwise disjoint non-empty open sets is countable.*

Suppose we are given a topological space X, a family $\{Y_s\}_{s\in S}$ of topological spaces and a family of mappings $\mathcal{F} = \{f_s\}$, where $f_s\colon X \to Y_s$. We say that the family $\mathcal{F}$ *separates points* if for every pair of distinct points $x, y \in X$ there exists a mapping $f_s \in \mathcal{F}$ such that $f_s(x) \neq f_s(y)$. If for every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a mapping $f_s \in \mathcal{F}$ such that $f_s(x) \notin \overline{f_s(F)}$, then we say that the family $\mathcal{F}$ *separates points and closed sets*. Let us note that if X is a T_0-space, then every family $\mathcal{F}$ separating points and closed sets separates points as well.

Lemma. *(2.3.19) If the mapping* $f\colon X \to Y$ *is one-to-one and the one-element family* $\{f\}$ *separates points and closed sets, then* f *is a homeomorphic embedding.*

Theorem (The diagonal theorem)**.** *(2.3.20) If the family* $\mathcal{F} = \{f_s\}_{s\in S}$*, where* $f_s\colon X \to Y_s$*, separates points, then the diagonal* $f = \mathop{\triangle}\limits_{s\in S} f_s\colon X \to \prod\limits_{s\in S} Y_s$ *is a one-to-one mapping. If, moreover, the family* $\mathcal{F}$ *separates points and closed sets, then* f *is a homeomorphic embedding.*

In particular, if there exists an $s \in S$ *such that* f_s *is a homeomorphic embedding, then* f *is a homeomorphic embedding.*

Corollary. *(2.3.21) If* $X_s = X$ *for every* $s \in S$*, then the diagonal* $i = \triangle id_{X_s}\colon X \to \prod X_s$ *is a homeomorphic embedding; hence the diagonal* $\triangle$ *of the Cartesian product* $X^{\mathfrak{m}}$ *is homeomorphic to* X*.*

By the *graph of mapping* f of a space X to a space Y, we mean the subset of Cartesian product $X \times Y$ defined by $G(f) = \{(x, y) \in X \times Y : y = f(x)\}$.

Corollary. *(2.3.22) For every continuous mapping* $f\colon X \to Y$ *the graph* $G(f)$ *is the image of* X *under the homeomorphic embedding* $id_X \triangle f\colon X \to X \times Y$*. The restriction* $p|G(f)$ *of the projection* $p\colon X \times Y \to X$ *is a homeomorphism. If* Y *is a Hausdorff space, then* $G(f)$ *is a closed subset of* $X \times Y$*.*

We say that the space X is *universal* for all spaces having a topological property $\mathcal{P}$ if X has the property $\mathcal{P}$ and every space that has the property $\mathcal{P}$ is embeddable in X.

Theorem. *(2.3.23) The Tychonoff cube* $I^{\mathfrak{m}}$ *is universal for all Tychonoff spaces of weight* $\mathfrak{m} \ge \aleph_0$*.*

The *Cantor cube of weight* $m \ge \aleph_0$ is the space $D^{\mathfrak{m}}$. The Cantor cube $D^{\aleph_0}$ is called *Cantor set*. Cantor cube is universal space for all zero-dimensional spaces of weight $\mathfrak{m}$.

Theorem. *(2.3.24) For every* $\mathfrak{m} \ge \aleph_0$ *and every* $x \in D^{\mathfrak{m}}$*, we have* $\chi(x, D^{\mathfrak{m}}) = \mathfrak{m}$*.*

Corollary. *(2.3.25) For every* $\mathfrak{m} \ge \aleph_0$ *and every* $x \in I^{\mathfrak{m}}$ *we have* $\chi(x, I^{\mathfrak{m}}) = \mathfrak{m}$*.*

The *Alexandroff cube* of weight $\alpha \geq \aleph_0$ is the space F^α, where F is Sierpiński space.

Theorem. *(2.3.26) The Alexandroff cube F^α is universal for all T_0-spaces of weight $\alpha \geq \aleph_0$.*

Proposition. *(2.3.27) If the Cartesian product $f = \prod f_s$, where $f_s \colon X_s \to Y_s$ and $X_s \neq \emptyset$ for $s \in S$, is closed, then all mappings f_s are closed.*

The converse is not true in general.

Proposition. *(2.3.29) The Cartesian product $f = \prod f_s$, where $f_s \colon X_s \to Y_s$ and $X_s \neq \emptyset$ for $s \in S$, is open if and only if all mappings f_s are open and there exists a finite set $S_0 \subset S$ such that $f_s(X_s) = Y_s$ for $s \in S \setminus S_0$.*

Proposition. *(2.3.30) If mappings $f_1, f_2, \ldots, f_k$, where $f_i \colon X_i \to Y_i$, are closed, Y_1 is a T_1-space and $Y_2, Y_3, \ldots, Y_k$ are T_3-spaces, then the diagonal $f = f_1 \triangle \ldots \triangle f_k$ is closed.*

Converse is not true in general. Proposition 2.3.30 cannot be generalized to infinite diagonals.

Proposition. *(2.3.32) If the diagonal $f = \triangle f_s$ is open, where $f_s \colon X_s \to Y_s$, then all mappings f_s are open.*

The converse is not true, even for finite systems.

Proposition. *(2.3.34) A net x_σ in the Cartesian product $\prod X_s$ converges to x if and only if every $p_s(x_\sigma)$ converges to $p_s(x)$.*

Proposition. *(2.3.35) If $\mathcal{F}$ is a filter in the Cartesian product $\prod X_s$, then for every $s \in S$ the family $\mathcal{F}_s = \{p_s(F) : F \in \mathcal{F}\}$ is a filter in X_s. The filter $\mathcal{F}$ converges to x if and only if the filter $\mathcal{F}_s$ converges to $p_s(x)$ for every $s \in S$.*

Example. (2.3.36) Normality is not a hereditary property.

Example. (2.3.B) $\operatorname{Int}(A \times B) = \operatorname{Int} A \times \operatorname{Int} B$, $\operatorname{Fr}(A \times B) = \operatorname{Fr} A \times \operatorname{Fr} B$
If A_s is an F_σ-set (G_δ-set) and $|S| \leq \aleph_0$, then $\prod A_s$ is and F_σ-set (G_δ-set).

Example. (2.3.C) X is Hausdorff if and only if the diagonal $\triangle$ of the Cartesian product $X \times X$ is closed in $X \times X$.

Example. (2.3.L) If a topological property P is hereditary with respect to both closed subsets and open subsets and is countably multiplicative, then, in the class of Hausdorff spaces, P is hereditary with respect to G_δ sets.

If a topological property P is hereditary with respect to both closed subsets and open subsets and is multiplicative, then if the closed interval I has P, all Tychonoff spaces have P.

2.4 Quotient spaces and quotient mappings

Proposition. *(2.4.2) A mapping f of a quotient space X/E to a topological space Y is continuous if and only if the composition $f \circ q$ is continuous.*

Let $f \colon X \to Y$ be continuous. Let $E(f)$ be equivalence relation on X determined by f. The mapping $f \colon X \to Y$ can be represented as the composition $\overline{f}q$, $\overline{f}$ is continuous.

Proposition. *(2.4.3) For a mapping f of a topological space X onto a topological space Y the following conditions are equivalent:*

(i) The mapping f is quotient.

(ii) The set $f^{-1}(U)$ is open in X if and only if U is open in Y.

(iii) The set $f^{-1}(F)$ is closed in X if and only if F is closed in Y.

(iv) The mapping $\overline{f}\colon X/E(f) \to Y$ is homeomorphism.

Corollary. *(2.4.4) The composition of two quotients mapping is a quotient mapping.*

Corollary. *(2.4.5) If the composition gf of two mappings is quotient, then g is a quotient mapping.*

Corollary. *(2.4.6) If for a continuous mapping $f\colon X \to Y$ there exists a set $A \subset X$ such that $f(A) = Y$ and the restriction $f|A$ is quotient, then f is a quotient mapping.*

Corollary. *(2.4.7) Every one-to-one quotient mapping is a homeomorphism.*

Corollary. *(2.4.8) Closed mappings onto and open mappings onto are quotient mappings.*

Proposition. *(2.4.9) For an equivalence relation E on a topological space X the following conditions are equivalent:*

(i) The natural mapping $q\colon X \to X/E$ is closed (open).

(ii) For every closed (open) set $A \subset X$ the union of all equivalence classes that meet A is closed (open) in X.

(iii) For every open (closed) set $A \subset X$ the union of all equivalence classes that are contained in A is open (closed) in X.

Corollary. *(2.4.10) The quotient mapping $f\colon X \to Y$ is closed (open) if and only if the set $f^{-1}f(A) \subset X$ is closed (open) for every closed (open) $A \subset X$.*

We say that an equivalence relation E on a space X is *closed (open) equivalence relation* if the natural mapping $g\colon X \to X/E$ is closed (open). Decompositions of topological space that correspond to closed (open) equivalence relation are called *upper (lower semicontinuous)*. In this context the word *identification* is also often used, mainly with respect to upper semicontinuous decompositions: we say that the quotient space X/E, where E is the equivalence relation corresponding to the decomposition $\mathcal{E}$, is obtained by identifying each element of $\mathcal{E}$ to a point.

adjunction space = we are given two disjoint topological spaces X and Y and a continuous mapping $f\colon M \to Y$ defined on a closed subset M of the space X. Adjunction space = $(X \oplus Y)/E$.

Theorem. *(2.4.13) If M is a closed subspace of X and $\mathcal{E}$ is an upper semicontinuous decomposition of M, then the decomposition of X into elements of $\mathcal{E}$ and one-points set $\{x\}$ with $x \in X \setminus M$ is upper semicontinuous.*

Proposition. *(2.4.14) A quotient space of a quotient space of X is a quotient space of X. More precisely ...*

Proposition. *(2.4.15) If $f\colon X \to Y$ is a quotient mapping, then for any set $B \subset Y$ which is either closed or open, the restriction $f_B\colon f^{-1}(B) \to B$ is a quotient mapping.*

In other words, if E is an equivalence relation on a space X, then for any $A \subset X$ which is either open or closed and satisfies the condition $q^{-1}q(A) = A$, where q is the natural mapping, the mapping $\overline{q|A}\colon A/(E|A) \to q[A] \subset X/E$ is homeomorphism.

Proposition. *(2.4.18) Suppose we are given a topological space X, a cover $\{A_s\}_{s\in S}$ of the space X and a family $\{f_s\}_{s\in S}$ of compatible mappings, where $f_s\colon A_s \to Y$ such that the combination $f = \triangle f_s\colon X \to Y$ is continuous. If there exists a set $S_0 \subset S$ such that the restriction $f_s|A_s\colon A_s \to f_s(A_s)$ are quotient for $s \in S_0$ and $\{f_s(A_s)\}_{s\in S_0}$ is either an open cover of Y or a locally finite closed cover of Y, then the combination f is a quotient mapping.*

Now, suppose we are given a family $\{X_s\}_{s\in S}$ of topological spaces and for every $s \in S$ an equivalence relation E_s on X_s. Letting $\{x_s\}E\{y_s\}$ if and only if $x_s E_s y_s$ for every $s \in S$ we define an equivalence relation E on the Cartesian product $\prod\limits_{s\in S} X_s$; this relation is called the *Cartesian product of relations* $\{E_s\}_{s\in S}$.

Proposition. *(2.4.19) If for every $s \in S$, E_s is an open equivalence relation on a space X_s and $q_s\colon X_s \to X_s/E_s$ is the natural mapping, then the mapping $\overline{\prod\limits_{s\in S} q_s}\colon \prod\limits_{s\in S} X_s/\prod\limits_{s\in S} E_s \to \prod\limits_{s\in S}(X_s/E_s)$ is a homeomorphism.*

Example. (2.4.20) Two quotient maps such their product is not quotient. $X_1 = Y_1 = R\backslash\{\frac{1}{2}, \frac{1}{3}, \ldots\}$ and $f_1 = id_{X_1}$. $X_2 = R$, $Y_2 = R/N$, $f_2\colon X_2 \to Y_2$ is a natural mapping. $f = f_1 \times f_2$ is not a quotient mapping.

Example. (2.4.E) Sum $\oplus f_s$ is quotient if and only if all mappings f_s are quotient.

For every retraction $f\colon X \to X$ the restriction $f|X\colon X \to f(X)$ is a quotient mapping.

2.4.F: $f\colon X \to Y$ if X onto Y is called *hereditarily quotient* if for every $B \subset Y$ the restriction $f_B\colon f^{-1}(B) \to B$ is a quotient mapping.

A mapping $f\colon X \to Y$ of X onto Y is hereditarily quotient if and only if the set $f[\overline{f^{-1}(B)}] \subset Y$ is closed for every $B \subset Y$ or –equivalently - if and only if for every $y \in Y$ and any open $U \subset X$ that contains $f^{-1}(y)$, we have $y \in \operatorname{Int} f[U]$.

Composition of two hereditarily quotient mappings is a hereditarily quotient mapping. Sum of hereditarily quotient mappings is a hereditarily quotient mapping. Proposition 2.4.18 holds also for hereditarily quotient mappings.

Any quotient mapping $f\colon X \to Y$ onto a Fréchet space Y in which every sequence has at most one limit (in particular, onto a Fréchet T_2-space) is hereditarily quotient.

2.5 Limits of inverse systems

Suppose that to every σ in a set Σ directed by the relation $\leq$ corresponds a topological space X_σ, and that for any $\varrho \leq \sigma$ a continuous mapping $\pi^\sigma_\varrho : X_\sigma \to X_\varrho$ is defined; suppose further that $\pi^\varrho_\tau \pi^\sigma_\varrho = \pi^\sigma_\tau$ for $\tau \leq \varrho \leq \sigma$ and that $\pi^\sigma_\sigma = id_{X_\sigma}$. In this situation we say that the family $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is an *inverse system* of the spaces X_σ; the mappings π^σ_ϱ are called *bonding mappings* of the inverse system $\mathbf{S}$. If $\Sigma = N$ with natural order, $\mathbf{S}$ is called *inverse sequence*.

Let $\mathbf{S}$ be an inverse system; an element $\{x_\sigma\}$ of the Cartesian product $\prod_{\sigma\in\Sigma} X_\sigma$ is called at *thread* of $\mathbf{S}$ if $\pi^\sigma_\varrho(x_\sigma) = x_\varrho$ for any $\varrho \leq \sigma$, and the subspace of $\prod X_\sigma$ consisting of all threads of $\mathbf{S}$ is called *limit of the inverse system* and is denoted by $\varprojlim \mathbf{S}$.

Proposition. *(2.5.1) Limit of an inverse system of Hausdorff spaces is a closed subset of the Cartesian product.*

Proposition. *(2.5.2) The limit of an inverse system of T_i-spaces is a T_i-space for $i \leq 3\frac{1}{2}$.*

Example. (2.5.3) Suppose we are given a family $\{X_s\}_{s\in S}$ of topological spaces where $|S| \geq \aleph_0$. Observe that the family Σ of all finite subsets of S is directed by inclusion. Letting $X_\sigma = \prod_{s\in\sigma} X_s$ we obtain inverse system. (π^σ_ϱ is the restriction of elements of X_σ to the subset ϱ of the set σ.) Limit of this system is cartesian product $\prod_{s\in S} X_s$.

Let $X = \varprojlim \mathbf{S}$. A mapping $\pi_\sigma = p_\sigma|X : X \to X_\sigma$ is called the *projection of the limit* of $\mathbf{S}$ to X_σ.

Proposition. *(2.5.5) The family of all sets $\pi^{-1}_\sigma(U_\sigma)$, where U_σ is an open subset of X_σ and σ runs over a subset Σ' cofinal in Σ, is a base for the limit of the inverse system $\mathbf{S}$.*

Moreover, if for every $\sigma \in \Sigma$ a base $\mathcal{B}_\sigma$ for X_σ is fixed, then the subfamily consisting of those $\pi^{-1}_\sigma(U_\sigma)$ in which $U_\sigma \in \mathcal{B}_\sigma$, also is a base.

Proposition. *(2.5.6) For every subspace A of the limit X of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ the family $\mathbf{S}_A = \{\overline{A}_\sigma, \tilde{\pi}^\sigma_\varrho, \Sigma\}$, where $A_\sigma = \pi_\sigma[A]$ and $\tilde{\pi}^\sigma_\varrho(x) = \pi^\sigma_\varrho(x)$ for $x \in \overline{A}_\sigma$, is an inverse system and $\varprojlim \mathbf{S}_A = \overline{A} \subset X$.*

Corollary. *(2.5.7) Any closed subspace A of the limit X of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is the limit of the inverse system $\mathbf{S}_A = \{\overline{A}_\sigma, \tilde{\pi}^\sigma_\varrho, \Sigma\}$ of closed subspaces $\overline{A}_\sigma$ of the spaces X_σ.*

Theorem. *(2.5.8) Let $\mathcal{P}$ be a topological property that is hereditary with respect to closed subsets and finitely multiplicative. A topological space X is homeomorphic to the limit of an inverse system of T_2-spaces with the property $\mathcal{P}$ if and only if X is homeomorphic to a closed subspace of a Cartesian product of T_2-spaces with the property $\mathcal{P}$.*

Suppose we are given two inverse systems $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ and $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\varrho'}, \Sigma'\}$; a *mapping of the system* $\mathbf{S}$ *to the system* $\mathbf{S}'$ is a family $\{\varphi, f_{\sigma'}\}$ consisting of a nondecreasing function φ from Σ' to Σ such that the set $\varphi[\Sigma']$ is cofinal in Σ, and of continuous mappings $f_{\sigma'} : X_{\varphi(\sigma')} \to Y_{\sigma'}$ such that

$$\pi^{\sigma'}_{\varrho'} f_{\sigma'} = f_{\varrho'} \pi^{\varphi(\sigma')}_{\varphi(\varrho')},$$

i.e., such that the diagram

$$\begin{tikzcd}
X_{\varphi(\sigma')} \arrow[r, "f'_\sigma"] \arrow[d, "\pi^{\varphi(\sigma')}_{\varphi(\varrho')}"'] & Y_{\sigma'} \arrow[d, "\pi^{\sigma'}_{\varrho'}"] \\
X_{\varphi(\varrho')} \arrow[r, "f_{\varrho'}"'] & Y'_\varrho
\end{tikzcd}$$

is commutative for any $\sigma', \varrho' \in \Sigma'$ satisfying $\varrho' \leq \sigma'$.

Any mapping of an inverse system $\mathbf{S}$ to an inverse system $\mathbf{S}'$ induces a continuous mapping of $\varprojlim \mathbf{S}$ to $\varprojlim \mathbf{S}'$. This mapping is called the *limit mapping induced by* $\{\varphi, f_{\sigma'}\}$ and is denoted by $\varprojlim\{\varphi, f_{\sigma'}\}$.

Lemma. *(2.5.9) Let $\{\varphi, f_{\sigma'}\}$ be a mapping of an inverse system $\mathbf{S}$ to an inverse system $\mathbf{S}'$. If all mappings $f_{\sigma'}$ are one-to-one, the limit mapping $f = \varprojlim\{\varphi, f_{\sigma'}\}$ also is one-to-one. If, moreover, all mappings $f_{\sigma'}$ are onto, f also is a mapping onto.*

Proposition. *(2.5.10) Let $\{\varphi, f_{\sigma'}\}$ be a mapping of an inverse system $\mathbf{S}$ to an inverse system $\mathbf{S}'$. If all mappings $f_{\sigma'}$ are homeomorphisms, the limit mapping $f = \varprojlim\{\varphi, f_{\sigma'}\}$ also is a homeomorphism.*

Corollary. *(2.5.11) Let $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ be an inverse system and Σ' a subset cofinal in Σ. The mapping consisting in restricting all threads from $X = \varprojlim \mathbf{S}$ to Σ' is a homeomorphism of X onto the space $X' = \varprojlim \mathbf{S}'$, where $\mathbf{S}' = \{X'_\sigma, \pi'^{\sigma'}_\varrho, \Sigma'\}$.*

Corollary. *(2.5.12) Let $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ be an inverse system; if in the directed set Σ there exists and element σ_0 such that $\sigma \leq \sigma_0$ for every $\sigma \in \Sigma$, then the limit of $\mathbf{S}$ is homeomorphic to the space X_{σ_0}.*

Theorem. *(2.5.13) For every mapping $\{\varphi, f'_\sigma\}$ of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ to an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\varrho'}, \Sigma'\}$ there exists a homeomorphic embedding $h\colon \varprojlim S \to \prod_{\sigma' \in \Sigma'} Z_{\sigma'}$, where $Z_{\sigma'} = X_{\varphi(\sigma')}$, such that $\varprojlim\{\varphi, f_{\sigma'}\} = (\prod_{\sigma' \in \Sigma'} f_{\sigma'})h$. If all $X_{\varphi(\sigma')}$ are Hausdorff spaces, then $f[\varprojlim \mathbf{S}]$ is a closed subset of $\prod_{\sigma' \in \Sigma'} Z_{\sigma'}$.*

Theorem. *(2.5.14) For every inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ and any $\sigma_0 \in \Sigma$ there exist an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\varrho'}, \Sigma'\}$, where $Y_{\sigma'} = X_{\sigma_0}$ for all $\sigma' \in \Sigma'$, a homeomorphism $h\colon \varprojlim \mathbf{S}' \to X_{\sigma_0}$, and a mapping $\{\varphi, f_{\sigma'}\}$ of $\mathbf{S}$ to $\mathbf{S}'$, where $f_{\sigma'}$ are bonding mappings of $\mathbf{S}$, such that $\pi_{\sigma_0} = h \varprojlim\{\varphi, f_{\sigma'}\}$.*

2.6 Function spaces I

Y^X = the set of all continuous mappings from X to Y
topology of uniform convergence

Proposition. *(2.6.2) For every topological space X the set I^X is closed in the space R^X with the topology of uniform convergence.*

Now let X and Y be arbitrary topological spaces; for $A \subseteq X$ and $B \subseteq Y$ define

$$M(A, B) = \{f \in Y^X; f[A] \subseteq B\}. \tag{1}$$

Denote by $\mathcal{F}$ the family of all finite subsets of X and let $\mathcal{O}$ be the topology of Y. The family $\mathcal{B}$ of all sets $\bigcap_{i=1}^{k} M(A_i, U_i))$, where $A_i \in \mathcal{F}$ and $U_i \in \mathcal{O}$ for $i = 1, 2, \ldots, k$, generates a topology on Y^X; this topology is called the *topology of pointwise convergence* on Y^X. The family $\mathcal{B}$ is a base for the space Y^X with the topology of pointwise convergence.

Proposition. *(2.6.3) The topology of pointwise convergence on Y^X coincides with the topology of a subspace of the cartesian product $\prod_{x \in X} Y_x$, where $Y_x = Y$ for every $x \in X$.*

Theorem. *(2.6.4) If Y is a T_i-space, then the space Y^X with the topology of pointwise convergence also is a T_i-space for $i \leq 3\frac{1}{2}$.*

Proposition. *(2.6.5) A net $\{f_\sigma; \sigma \in \Sigma\}$ in the space Y^X with the topology of pointwise convergence converges to $f \in Y^X$ if and only if the net $\{f_\sigma(x), \sigma \in \Sigma\}$ converges to $f(x)$ for every $x \in X$.*

Proposition. *(2.6.6) For every topological space X the topology of uniform convergence on R^X is finer than the topology of pointwise convergence.*

Proposition. *(2.6.9) For every family $\{X_s\}_{s \in S}$ of non-empty topological spaces and a topological space Y, the combination $\nabla \colon \prod_{s \in S} (Y^{X_s}) \to Y^{(\bigoplus_{s \in S} X_s)}$ is a homeomorphism with respect to the topology of pointwise convergence on function spaces.*

Proposition. *(2.6.10) For every topological space X and a family $\{Y_s\}_{s \in S}$ of topological spaces, the diagonal $\triangle \colon \prod_{s \in S} (Y_s^X) \to (\prod_{s \in S} Y_s)^X$ is a homeomorphism with respect to the topology of pointwise convergence on function spaces.*

Let us observe that any mappings $g \colon Y \to Z$ and $h \colon T \to X$ induce mapping Φ_g of Y^X to Z^X and Ψ_h of Y^X to Y^T defined by letting

$$\Phi_g(f) = gf \text{ for } f \in Y^X \qquad \text{and} \qquad \Psi_h(f) = fh \text{ for } f \in Y^X. \tag{10}$$

Since

$$\Phi_g^{-1}(M(A, B)) = M(A, g^{-1}(B)) \text{ and } \Psi_h^{-1}(M(A, B)) = M(h[A], B), \tag{11}$$

both Φ_g and Ψ_h are continuous with respect to the topology of pointwise convergence on function spaces.

The mappings Φ_g and Ψ_h are connected with the operation Σ of composition of mappings; in fact from (10) it follows immediately that

$$\Phi_g(f) = \Sigma(g, f) \qquad \text{and} \qquad \Psi_h(f) = \Sigma(f, h).$$

The mapping Ω of $Y^X \times X$ to Y defined by $\Omega(f, x) = f(x)$ is called the *evaluation mapping* of Y^X. It is also connected with the operation Σ; namely, Ω is the composition of mappings

$$Y^X \times X \xrightarrow{id_{Y^X} \times i_X} Y^X \times X^{\{p\}} \xrightarrow{\Sigma} Y^{\{p\}} \xrightarrow{i_Y^{-1}} Y, \text{ i.e. } \Omega = i_Y^{-1} \Sigma(id_{Y_X} \times i_X) \tag{12}$$

One easily sees that the formula

$$\{[\Lambda(f)](z)\}(x) = f(z,x), \tag{13}$$

where f is a mapping of $Z \times X$ to Y, defines a one-to-one correspondence Λ between the set of all (not necessarily continuous) mapping of $Z \times X$ to Y and the set of all mapping of Z to the set of all mapping of X to Y; this correspondence is called the *exponential mapping.*

We say that a topology on Y^X is *proper* if for every space Z and any $f \in Y^{(Z\times X)}$ the mapping $\Lambda(f)$ belongs to $(Y^X)^Z$. Similarly, we say that a topology on Y^X is *admissible* if for every space Z and any $g \in (Y^X)^Z$ the mapping $\Lambda^{-1}(g)$ belongs to $Y^{(Z\times X)}$. A topology on Y^X that is both proper and admissible is called an *acceptable* topology.

Proposition. *(2.6.11) A topology on Y^X is admissible if and only if the evaluation mapping of Y^X is continuous, i.e., if $\Omega\colon Y^X \times X \to Y$.*

Proposition. *(2.6.12) For every pair X, Y of topological spaces and any two topologies $\mathcal{O}$, $\mathcal{O}'$ on the function space Y^X we have:*

- *(i) If the topology $\mathcal{O}$ is proper and $\mathcal{O}' \subset \mathcal{O}$, the topology $\mathcal{O}'$ is proper.*
- *(ii) If the topology $\mathcal{O}$ is admissible and $\mathcal{O} \subset \mathcal{O}'$, then the topology $\mathcal{O}$ is admissible.*
- *(iii) If the topology $\mathcal{O}$ is proper and the topology $\mathcal{O}'$ is admissible, then $\mathcal{O} \subset \mathcal{O}'$.*
- *(iv) On Y^X there exists at most one acceptable topology.*

The topology of pointwise convergence is proper.

The topology of pointwise convergence is generally not admissible; indeed for this topology the fact that g is in $(Y^X)^Z$ means that for all $z_0 \in Z$ and $x_0 \in X$ the mapping $[g(z_0)](x)$ and $[g(z)](x_0)$ are continuous, while the fact that $\Lambda^{-1}(g)$ is in $Y^{(Z\times X)}$ means that g is continuous with respect to both coordinates.

The topology of uniform convergence is admissible. On the other hand, the topology of uniform convergence is generally not proper.

2.7 Problems

2.7.1 Cardinal functions II

f is cardinal function $\to hf$ is supremum over all subspaces. Hereditary density, hereditary Souslin number etc.

$hw(X) = w(X)$, $h\chi(X) = \chi(X)$, $h\tau(X) = \tau(X)$, $hc(X) = he(X)$

$hd(X) \geq \tau(X)$

If A is a dense subspace of X, then $c(A) = c(X)$, but not necessarily $d(A) \leq d(X)$.

$\mathbb{R}$ with the topology generated by the base $(a,b) \setminus A$, where $|A| \leq \aleph_0$, is a Hausdorff space such that $hd(X) > hc(X)$. The existence of such regular space is connected with Souslin's problem.

Souslin's problem - the question whether there exists a linearly ordered space X such that $c(X) = \aleph_0$ and $d(X) > \aleph_0$ (a *Souslin space*). If X is a Souslin space then $c(X \times X) > \aleph_0$.

2.7.2 Spaces of closed subsets I

2.7.20(a) For any topological space X we denote by 2^X the family of all non-empty closed subsets of X. The family $\mathcal{B}$ of all the sets of the form $\mathcal{V}(U_1, \ldots, U_k) = \{B \in 2^X : B \subset \bigcup_{i=1}^{k} U_i$ and $B \cap U_i \neq \emptyset$ for $i = 1, 2, \ldots, k\}$, where $U_1, \ldots, U_k$ is a sequence of open subsets of X generates a topology on 2^X; this topology is called the *Vietoris topology* on 2^X and the set 2^X with the Vietoris topology is called the *exponential space* of X.

3 Compact spaces

3.1 Compact spaces

Let us recall that a cover of a set X is a family $\{A_s\}_{s\in S}$ of subsets of X such that $\bigcup_{s\in S} A_s = X$, and that - if X is a topological space- $\{A_s\}_{s\in S}$ is an open (a closed) cover of X if all sets A_s are open (closed). We say that a cover $\mathcal{B} = \{B_t\}_{t\in T}$ is a *refinement* of another cover $\mathcal{A} = \{A_s\}_{s\in S}$ of the same set X if for every $t \in T$ there exists an $s(t) \in S$ such that $B_t \subset A_{s(t)}$; in this situation we say also that $\mathcal{B}$ *refines* $\mathcal{A}$. A cover $\mathcal{A}' = \{A'_s\}_{s\in S'}$ of X is a *subcover* of another cover $\mathcal{A} = \{A_s\}_{s\in S}$ of X if $S' \subset S$ and $A'_s = A_s$ for every $s \in S'$. In particular, any subcover is a refinement.

A topological space X is called a *compact space* if X is a Hausdorff space and every open cover of X has a finite subcover, i.e., if for every open cover $\{U_s\}_{s\in S}$ of the space X there exists a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that $X = U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_k}$.

Theorem. *(3.1.1) A Hausdorff space X is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection.*

Theorem. *(3.1.2) Every closed subspace of a compact space is compact.*

Theorem. *(3.1.3) If a subspace A of a topological space X is compact, then for every family $\{U_s\}_{s\in S}$ of open subsets of X such that $A \subset \bigcup_{s\in S} U_s$ there exists a finite set $\{s_1, \ldots, s_k\} \subset S$ such that $A \subset \bigcup_{i=1}^{k} U_{s_i}$.*

Corollary. *(3.1.4) Let X be a Hausdorff space and $\{F_1, \ldots, F_k\}$ a family of closed subsets of X. The subspace $F = \bigcup_{i=1}^{k} F_i$ of X is compact if and only if all subspaces F_i are compact.*

Corollary. *(3.1.5) Let U be an open subset of a topological space X. If a family $\{F_s\}_{s\in S}$ of closed subsets of X contains at least one compact set - in particular, if X is compact - and if $\bigcap_{s\in S} F_s \subset U$, then there exists a finite set $\{s_1, \ldots, s_k\} \subset S$ such that $\bigcap_{i=1}^{k} F_{s_i} \subset U$.*

Theorem. *(3.1.6) If A is a compact subspace of a regular space X, then for every closed subset $B \subset X \setminus A$ there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

If, moreover, B is a compact subspace of X, then it suffices to assume that X is a Hausdorff space.

Theorem. *(3.1.7) If A is a compact subspace of a Tychonoff space X, then for every closed set $B \subset X \setminus A$ there exists a continuous function $f\colon X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

Theorem. *(3.1.8) Every compact subspace of a Hausdorff space X is a closed subset of X.*

Theorem. *(3.1.9) Every compact space is normal.*

Theorem. *(3.1.10) If there exists a continuous mapping $f\colon X \to Y$ of a compact space X onto a Hausdorff space Y, then Y is a compact space.*

In other words, a continuous image of a compact space is compact, provided it is a Hausdorff space.

Corollary. *(3.1.11) If $f\colon X \to Y$ is a continuous mapping of a compact space X to a Hausdorff space Y, then $\overline{f[A]} = f[\overline{A}]$ for every $A \subset X$.*

Theorem. *(3.1.12) Every continuous mapping of a compact space to a Hausdorff space is closed.*

Theorem. *(3.1.13) Every continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism.*

Corollary. *(3.1.14) Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies defined on a set X and let $\mathcal{O}_1$ be finer than $\mathcal{O}_2$. If the space $(X, \mathcal{O}_1)$ is compact and $(X, \mathcal{O}_2)$ is a Hausdorff space, then $\mathcal{O}_1 = \mathcal{O}_2$.*

In other words, among all Hausdorff topologies, compact topologies are minimal.

Lemma. *(3.1.15) If A is a compact subspace of a space X and y a point of a space Y, then for every open set $W \subset X \times Y$ containing $A \times \{y\}$ there exist open sets $U \subset X$ and $V \subset Y$ such that $A \times \{y\} \subset U \times V \subset W$.*

Theorem (The Kuratowski theorem)**.** *(3.1.16) For a Hausdorff space X the following conditions are equivalent:*

(i) The space X is compact.

(ii) For every topological space Y the projection $p\colon X \times Y \to Y$ is closed.

(iii) For every normal space Y the projection $p\colon X \times Y \to Y$ is closed.

A family $\mathcal{N} = \{M_s\}_{s \in S}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighborhood U if x there exists an $s \in S$ such that $x \in M_s \subset U$. Clearly, any base for X is a network for X: it is a network of a special kind, one whose members all are open. The *network weight* of a space X is defined as the smallest cardinal number of the form $|\mathcal{N}|$, where $\mathcal{N}$ is a network for X. Clearly, for every topological space X we have $nw(X) \leq w(X)$, $nw(X) \leq |X|$ and $d(X) \leq nw(X)$. For every T_0-space we have $|X| \leq \exp nw(X)$.

Lemma. *(3.1.18) For every Hausdorff space X there exists a continuous one-to-one mapping of X onto a Hausdorff space Y such that $w(Y) \leq nw(X)$.*

Theorem. *(3.1.19) For every compact space X we have $nw(X) = w(X)$.*

Corollary. *(3.1.20) If a compact space X has a cover $\{A_s\}_{s\in S}$ such that $w(A_s) \leq m \geq \aleph_0$ for $s \in S$ and $|S| \leq m$, then $w(X) \leq m$.*

Theorem. *(3.1.21) For every compact space X we have $w(X) \leq |X|$.*

Theorem. *(3.1.22) If a compact space Y is a continuous image of a space X, then $w(Y) \leq w(X)$.*

Theorem. *(3.1.23) A Hausdorff space X is compact if and only if every net in X has a cluster point.*

The filter counterpart of the above theorem reads as follows:

Theorem. *(3.1.24) A Hausdorff space X is compact if and only if every filter in X has a cluster point.*

Example. (3.1.26) $X = C_1 \cup C_2$ - two concentric circle, the projection of C_1 onto C_2 from the point $(0,0)$ will be denoted by p. On the set X we shall generate a topology by defining a neighbourhood system $\{\mathcal{B}(z)\}_{z\in X}$; namely let $\mathcal{B}(z) = \{z\}$ for $z \in C_2$ and for $z \in C_1$ let $\mathcal{B}(z) = \{U_j(z)\}_{j=1}^{\infty}$, where $U_j = V_j \cup p[V_j \setminus \{z\}]$ and V_j is the arc of C_1 with centre at z and of length $1/j$.

The space X is called the *Alexandroff double circle.*

X is a compact space

Example. (3.1.27) $W = \omega_1 + 1$, base $(y, x\rangle$ and $\{0\}$. W is a compact space. $W_0 = W \setminus \{\omega_1\}$ - subspace. Every continuous function $f\colon W_0 \to I$ is extendable over W (every such a function is eventually constant). W_0 is not perfectly normal. W is hereditarily normal but not perfectly normal. W_0 is first countable. W is not a sequential space and it has no countable base at ω_1.

Example. (3.1.28) Cantor set $D^{\aleph_0}$ is homeomorphic to a subspace of the real line. $C =$ sets of all numbers of the form $\sum\limits_{i=1}^{\infty} \frac{2x_i}{3^i}$, where $x_i \in \{0,1\}$ for $i = 1, 2, \ldots$ We put $f(x) = \{x_i\}$, f is a homeomorphism.

Theorem. *(3.1.29) For every infinite compact space X we have $|X| \leq \exp \chi(X)$.*

Corollary. *(3.1.30) Every first countable compact space has cardinality $\leq c$.*

A topological space X is called a *quasi-compact space* if every open cover of X has a finite subcover. The reader can easily verify that Theorems 3.1.1-3.1.3, Corollaries 3.1.4-3.1.5, Theorem 3.1.10, 3.1.16, 3.1.23 and 3.1.24 of this section, as well as Theorems 3.2.3, 3.2.4, and 3.2.10 of the next section, remain valid, along with their proofs, when one replaces "compact" by "quasi-compact" and "Hausdorff space" by "topological space".

3.1.F: The *pseudocharacter of a point* x in a T_1-space X is defined as the smallest cardinal number of the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\bigcap \mathcal{U} = \{x\}$; this cardinal number is denoted by $\psi(x, X)$. The *pseudocharacter of a T_1-space* X is defined as the supremum of all numbers $\psi(x, X)$ for all $x \in X$; this cardinal number is denoted by $\psi(X)$.

For every T_1-space X we have $\psi(x, X) \leq \chi(x, X)$ and $\psi(X) \leq \chi(X)$. If X is a compact space then $\psi(x, X) = \chi(x, X)$ and $\psi(X) = \chi(X)$. For every Hausdorff space X we have $\psi(X) \leq \exp d(X)$. For every regular space X we have $|X| \leq \exp[d(X)\psi(X)]$.

3.2 Operations on compact spaces

Theorem. *(3.2.1) Let A be a dense subspace of a topological space X and f a continuous mapping of A to a compact space Y. The mapping f has a continuous extension over X if and only if for every pair B_1, B_2 of disjoint closed subsets of Y the inverse images $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have disjoint closures in the space X.*

Theorem. *(3.2.2) Every compact space of weight $m \geq \aleph_0$ is a continuous image of a closed subspace of the Cantor cube $D^{\mathfrak{m}}$.*

Theorem. *(3.2.3) The sum $\bigoplus_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is compact if and only if all spaces X_s are compact and the set S is finite.*

Theorem (The Tychonoff theorem). *(3.2.4) The Cartesian product $\prod_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is compact if and only if all spaces X_s are compact.*

Theorem. *(3.2.5) The Tychonoff cube $I^{\mathfrak{m}}$ is universal for all compact spaces of weight $\mathfrak{m} \geq \aleph_0$.*

Theorem. *(3.2.6) A space X is a Tychonoff space if and only if it is embeddable in a compact space.*

Theorem. *(3.2.8) A subspace A of Euclidean n-space R^n is compact if and only if the set A is closed and bounded.*

Corollary. *(3.2.9) Every continuous real-valued function defined on a compact space is bounded and attains its bounds.*

Theorem (The Wallace theorem). *(3.2.10) If A_s is a compact subspace of a topological space X_s for $s \in S$, then for every open subset W of the Cartesian product $\prod_{s \in S} X_s$ which contains the set $\prod_{s \in S} A_s$ there exist open sets $U_s \subset X_s$ such that $U_s \neq X_s$ for only finitely many $s \in S$ and $\prod_{s \in S} A_s \subset \prod_{s \in S} U_s \subset W$.*

Theorem (The Alexandroff theorem). *(3.2.11) For every closed equivalence relation E on a compact space X there exists exactly one (up to a homeomorphism) Hausdorff space Y and a continuous mapping $f\colon X \to Y$ of X onto Y such that $E = E(f)$, viz. the quotient space X/E and the natural quotient mapping $q\colon X \to X/E$; moreover Y is a compact space.*

Conversely, for every continuous mapping $f\colon X \to Y$ of a compact space X onto a Hausdorff space Y the equivalence relation $E(f)$ is closed.

Theorem. *(3.2.13) The limit of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ of non-empty compact spaces is compact and non-empty.*

Theorem. *(3.2.14) Let $\{\varphi, f_{\sigma'}\}$ be a mapping of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ of compact spaces to an inverse system $\mathbf{S}' = \{Y_{\sigma'}, \pi^{\sigma'}_{\varrho'}, \Sigma'\}$ of T_1-spaces. If all mappings $f_{\sigma'}$ are onto, the limit mapping $f = \varprojlim\{\varphi, f_{\sigma'}\}$ also is a mapping onto.*

Corollary. *(3.2.15) If in an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ of compact spaces all bonding mappings π^σ_ϱ are onto, then the projections $\pi_\sigma\colon \varprojlim \mathbf{S} \to X_\sigma$ also are mappings onto.*

Corollary. *(3.2.16) If $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$, where $\Sigma \neq \emptyset$, is an inverse system of T_1-spaces, X is a compact space, and $\{f_\sigma\}_{\sigma\in\Sigma}$ where $f_\sigma\colon X \to X_\sigma$, is a family of mappings onto such that $\pi^\sigma_\varrho f_\sigma = f_\varrho$ for any $\sigma, \varrho \in \Sigma$ satisfying $\varrho \leq \sigma$, then the limit mapping $\varprojlim f_\sigma$ also is a mapping onto.*

Corollary. *(3.2.17) If $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$, where $\Sigma \neq \emptyset$, is an inverse system of compact spaces, X is a T_1-space, and $\{f_\sigma\}_{\sigma\in\Sigma}$ where $f_\sigma\colon X_\sigma \to X$, is a family of mappings onto such that $f_\varrho \pi^\sigma_\varrho = f_\varrho$ for any $\sigma, \varrho \in \Sigma$ satisfying $\varrho \leq \sigma$, then the limit mapping $\varprojlim f_\sigma$ also is a mapping onto.*

Lemma (The Dini theorem)**.** *(3.2.18) Let X be a compact space and $\{f_i\}$ a sequence of continuous real-valued functions defined on X and satisfying $f_i(x) \leq f_{i+1}(x)$ for all $x \in X$ and $i = 1, 2, \ldots$ If there exists a function $f \in R^X$ such that $f(x) = \lim f_i(x)$ for every $x \in X$, then $f = \lim f_i$, i.e. the sequence $\{f_i\}$ is uniformly convergent to f.*

Lemma. *(3.2.19) There exists a sequence $\{w_i\}$ of polynomials which is uniformly convergent to the function $\sqrt{t}$ on the closed interval I.*

Lemma. *(3.2.20) Let P be a ring of continuous and bounded real-valued functions defined on a topological space X. If the ring P contains all constant functions and is closed with respect to uniform convergence, then for every $f, g \in P$ the functions $\max(f, g)$ and $\min(f, g)$ belong to P.*

Theorem (The Stone-Weierstrass theorem)**.** *(3.2.21) If a ring P of continuous real-valued functions defined on a compact space X contains all constant functions, separates points and is closed with respect to uniform convergence (i.e., is a closed subset of the space R^X with the topology of uniform convergence), then P coincides with the ring of all continuous real-valued functions on X.*

3.3 Locally compact spaces and k-spaces

A topological space X is called a *locally compact space* if for every $x \in X$ there exists a neighbourhood U of the point x such that $\overline{U}$ is a compact subspace of X.

Theorem. *(3.3.1) Every locally compact space is a Tychonoff space.*

Theorem. *(3.3.2) For every compact subspace A of a locally compact space X and every open set $V \subset X$ that contains A there exists an open set $U \subset X$ such that $A \subset U \subset \overline{U} \subset V$ and $\overline{U}$ is compact.*

Corollary. *(3.3.3) For every compact subspace A of a locally compact space X and every open set V that contains A there exists a continuous function $f\colon X \to I$ such that $f(x) = 0$ for $x \in A$, $f(x) = 1$ for $x \in X \setminus V$ and the set $f^{-1}(\langle 0, a\rangle)$ is compact for every $a < 1$.*

Theorem. *(3.3.4) The character of a point x in a locally compact space X is equal to the smallest cardinal number of the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\bigcap \mathcal{U} = \{x\}$.*

Theorem. *(3.3.5) For every locally compact space X we have $nw(X) = w(X)$.*

Corollary. *(3.3.6) For every locally compact space X we have $w(X) \leq |X|$.*

Corollary. *(3.3.7) If a locally compact space Y is a continuous image of a space X, then $w(Y) \leq w(X)$.*

Theorem. *(3.3.8) If X is a locally compact space, then every subspace of X that can be represented in the form $F \cap V$, where F is closed in X and V is open in X, also is locally compact.*

Theorem. *(3.3.9) A locally compact subspace M of a Hausdorff space X is an open subset of the closure $\overline{M}$ of the set M in the space X, i.e., it can be represented in the form $F \cap V$, where F is closed in X and V is open in X.*

Corollary. *(3.3.10) A subspace M of a locally compact space X is locally compact if and only if it can be represented in the form $F \cap V$, where F is closed in X and V is open in X.*

Corollary. *(3.3.11) A topological space is locally compact if and only if it is homeomorphic to an open subspace of a compact space.*

Theorem. *(3.3.12) The sum $\bigoplus_{s \in S} X_s$ is locally compact if and only if all spaces X_s are locally compact.*

Theorem. *(3.3.13) The Cartesian product $\prod_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is locally compact if and only if all spaces X_s are locally compact and there exists a finite set $S_0 \subset S$ such that X_s is a compact for $s \in S \setminus S_0$.*

Theorem. *(3.3.15) If there exists an open mapping $f\colon X \to Y$ of a locally compact space X onto a Hausdorff space Y, then Y is a locally compact space.*

Theorem (The Whitehead theorem)**.** *(3.3.17) For every locally compact space X and any quotient mapping $g\colon Y \to Z$, the Cartesian product $f = id_X \times g\colon X \times Y \to X \times Z$ is a quotient mapping.*

A topological space X is called a *k-space* if X is a Hausdorff space and if X is an image of a locally compact space under a quotient mapping.

Theorem. *(3.3.18) A Hausdorff space X is a k-space if and only if for each $A \subset X$, the set A is closed in X provided that the intersection of A with any compact subspace Z of the space X is closed in Z.*

Corollary. *(3.3.19) A Hausdorff space X is a k-space if and only if for each $A \subset X$, the set A is open in X provided that the intersection of A with any compact subspace Z of the space X is open in Z.*

Theorem. *(3.3.20) Every sequential Hausdorff space - and, in particular, every first-countable Hausdorff space - is a k-space.*

Theorem. *(3.3.21) A mapping f of a k-space X to a topological space Y is continuous if and only if for every compact subspace $Z \subset X$ the restriction $f|Z\colon Z \to Y$ is continuous.*

Theorem. *(3.3.22) A continuous mapping $f\colon X \to Y$ of a topological space X to a k-space Y is closed (open, quotient) if and only if for every compact subspace $Z \subset Y$ the restriction $f_Z\colon f^{-1}(Z) \to Z$ is closed (open, quotient).*

Theorem. *(3.3.23) If there exists a quotient mapping $f\colon X \to Y$ of a k-space X onto a Hausdorff space Y, then Y is a k-space.*

Theorem. *(3.3.25) The property of being a k-space is hereditary both with respect to closed subsets and with respect to open subsets.*

Theorem. *(3.3.26) The sum $\bigoplus_{s\in S} X_s$ is a k-space if and only if all spaces X_s are k-spaces.*

Theorem. *(3.3.27) The Cartesian product $X \times Y$ of a locally compact space X and a k-space Y is a k-space.*

Theorem. *(3.3.28) If $f_i\colon X_i \to Y_i$ is a quotient mapping for $i = 1, 2$ and if X_1 and $Y_1 \times Y_2$ are k-spaces, then the Cartesian product $f = f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is a quotient mapping.*

Example. (3.3.29) k-spaces are not finitely productive.

3.4 Function spaces II: The compact-open topology

The *compact-open* topology on Y^X is the topology generated by the base consisting of all sets $\bigcap_{i=1}^{k} M(C_i, U_i)$ where C_i is a compact subset of X and U_i is an open subset of Y for $i = 1, 2, \ldots, k$ and where, for any $A \subset X$ and U_i is an open subset of Y for $i = 1, 2, \ldots, k$.

Formulas (11) in section 2.6 imply that

$$\Phi_g\colon Y^X \to Z^X \text{ is continuous for every mapping } g\colon Y \to Z \tag{14}$$

and

$$\Psi_h\colon Y^X \to Y^T \text{ is continuous for every mapping } h\colon T \to X \text{ to a Hausdorff space } X \tag{15}$$

where $\Phi_g(f) = gf$ for $f \in Y^X$ and $\Psi_h(f) = fh$ for $f \in Y^X$ and the function space have the compact-open topology.

Theorem. *(3.4.1) For every pair X, Y if a topological spaces the compact-open topology on Y^X is proper.*

Theorem. *(3.4.2) For every pair X, Z of topological spaces and every locally compact space Y the composition $\Sigma\colon Z^Y \times Y^X \to Z^X$ is continuous with respect to the compact-open topology on function spaces.*

Theorem. *(3.4.3) If X is a locally compact then for every topological space Y the compact-open topology on Y^X is acceptable.*

It turns out that local compactness of X is crucial; indeed, one can prove (see Exercise 3.4.A) that if for a completely regular space X there exists an acceptable topology on the set R^X then X is locally compact.

Proposition. *(3.4.4) For every family $\{X_s\}_{s\in S}$ of non-empty topological spaces and a topological space Y, the combination $\nabla\colon \prod_{s\in S}(Y^{X_s}) \to Y^{(\bigoplus_{s\in s} X_s)}$ is a homeomorphism with respect to the compact-open topology on function spaces.*

Proposition. *(3.4.5) For every topological space X and a family $\{Y_s\}_{s\in S}$ of topological spaces, the diagonal $\triangle\colon \prod_{s\in S}(Y_s^X) \to (\prod_{s\in S} Y_s)^X$ is a homeomorphism with respect to the compact-open topology on function spaces.*

Lemma. *(3.4.6) For every pair X, Y of topological spaces and every subbase $\mathcal{P}$ for the space Y, the sets $M(C,U)$ where C is a compact subset of X and $U \in \mathcal{P}$, form a subbase for the space Y^X with the compact-open topology.*

Theorem. *(3.4.7) For every pair X, Z of Hausdorff spaces and every topological space Y, the exponential mapping $\Lambda\colon Y^{(Z\times X)} \to (Y^X)^Z$ is a homeomorphic embedding with respect to the compact-open topology on function spaces.*

Theorem. *(3.4.8) For every topological space Y, a Hausdorff space Z and a locally compact space X, the exponential mapping $\Lambda\colon Y^{(Z\times X)} \to (Y^X)^Z$ is a homeomorphism with respect to the compact-open topology on function spaces.*

Theorem. *(3.4.9) If $Z \times X$ is a k-space, then for every topological space Y the exponential mapping $\Lambda\colon Y^{(Z\times X)} \to (Y^X)^Z$ is a homeomorphism with respect to the compact-open topology on function spaces.*

Corollary. *(3.4.10) If X and Z are first-countable Hausdorff spaces, then for every topological space Y the exponential mapping $\Lambda\colon Y^{(Z\times X)} \to (Y^X)^Z$ is a homeomorphism with respect to the compact-open topology on function spaces.*

Let $\mathcal{Z}(X)$ denote the family of all non-empty compact subsets of a Hausdorff space X ordered by inclusion ($\leq=\subset$). $\mathcal{Z}(X)$ is directed by $\leq$. For any $C_1, C_2 \in \mathcal{Z}(X)$ satisfying $C_2 \leq C_1$, and for an arbitrary topological space Y, a continuous mapping $\pi^{C_1}_{C_2}\colon Y^{C_1} \to Y^{C_2}$, viz., $\pi^{C_1}_{C_2} = \Psi_i$, where $i\colon C_2 \to C_1$ is the embedding; clearly $\pi^{C_1}_{C_2}(f) = f|C_2$ for any $f \in Y^{C_1}$.

Theorem. *(3.4.11) If X is a k-space, then for every topological space Y the space Y^X with the compact-open topology (with the topology of pointwise convergence) is homeomorphic to the limit of the inverse system $\mathbf{S}(X) = \{Y^C, \pi^{C_1}_{C_2}, \mathcal{Z}(X)\}$ of the space Y^C with the compact-open topology (with the topology of pointwise convergence).*

Lemma. *(3.4.12) For every pair X, Y of topological spaces, any subset A of X and any closed subset B of Y, the set $M(A,B)$ is closed in the space Y^X with the topology of pointwise convergence and, a fortiori, in the space Y^X with the compact-open topology.*

Theorem. *(3.4.13) If Y is a regular space, the space Y^X with the compact-open topology also is a regular space.*

Lemma. *(3.4.14) Let X be a topological space and C a compact subspace of X. Assigning to any $f \in I^X$ the number $\Xi(f) = \sup_{x \in C} f(x)$ defines a function $\Xi \colon I^X \to I$ continuous with respect to the compact-open topology on I^X.*

Theorem. *(3.4.15) If Y is a Tychonoff space, then the space Y^X with the compact-open topology also is a Tychonoff space.*

Theorem. *(3.4.16) If the weight of both X and Y is not larger then $\mathfrak{m} \geq \aleph_0$ and X is locally compact, then the weight of the space Y^X with the compact-open topology is not larger then $\mathfrak{m}$.*

We say that a family F of mappings of X to Y is *evenly continuous* if for every $x \in X$, every $y \in Y$ and any neighbourhood V of y there exists a neighbourhood U of x and a neighbourhood W of y such that $\Omega\left[(F \cap M(\{x\}, W)) \times U\right] \subset V$, i.e., such that the conditions $f \in F$ and $f(x) \in W$ imply the inclusion $f[U] \subset V$. It follows directly from the definition that if a family F of mappings of X to Y is evenly continuous, then all members of F are continuous, i.e., $F \subset Y^X$.

Lemma. *(3.4.17) If Y is a regular space, then for every evenly continuous family of mappings $F \subset Y^X$ the closure $\overline{F}$ of the set F in the Cartesian product $\prod_{x \in X} Y_x$, where $Y_x = Y$ for every $x \in X$, is an evenly continuous family of mappings, and, in particular $\overline{F} \subseteq Y^X$.*

Lemma. *(3.4.18) If $F \subset Y^X$ is an evenly continuous family of mapping then the restriction $\Omega | F \times X$ of the evaluation mapping is continuous with respect to the topology of pointwise convergence on F.*

Lemma. *(3.4.19) Let Y be a regular space, X an arbitrary topological space and Y^X the space of all continuous mappings of X to Y with the topology of pointwise convergence. If a set $F \subset Y^X$ is compact and the restriction $\Omega | F \times X$ of the evaluation mapping is continuous, then F is an evenly continuous family of mappings.*

Theorem (The Ascoli theorem)**.** *(3.4.20) If X is a k-space and Y is a regular space, then a closed subset F of the space Y^X with the compact-open topology is compact if and only if F is an evenly continuous family of mappings and the set $\Omega(F \times \{x\}) = \{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.*

The following theorem is a variant of the Ascoli theorem; the symbol $F|Z$ that appears in it denotes, for $F \subset Y^X$ and $Z \subset X$, the family of restrictions $\{f|Z : f \in F\} \subset Y^X$.

Theorem. *(3.4.21) If X is a k-space and Y is a regular space, then a closed subset F of the space Y^X with the compact-open topology is compact if and only if $F|Z$ is an evenly continuous family of mappings for every compact subspace $Z \subset X$ and the set $\Omega(F \times \{x\}) = \{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.*

3.4.A: If X is a regular space and there exists an acceptable topology on R^X, then X is locally compact.

3.4.E: A Hausdorff space X is *hemicompact* if in the family of all compact subspaces of X there exists a countable cofinal subfamily.

(a) Prove that every first-countable hemicompact space is locally compact.

(b) Give an example of a countable hemicompact space which is not a k-space.

(c) Show that in the realm of second-countable spaces hemicompactness is equivalent to local compactness.

(d) Prove that if the space R^X with the compact-open topology is first-countable and X is a Tychonoff space, then X is hemicompact.

3.4.G (a) Show that $nw(Y^X) \leq w(X)w(Y)$ with respect both to the compact-open topology and to the topology of pointwise convergence on Y^X. Deduce that if X and Y are second-countable then Y^X is hereditarily separable with respect both to the compact-open topology and to the topology of pointwise convergence.

3.5 Compactifications

Let X be a topological space.

A pair (Y, c), where Y is a compact space and $c\colon X \to Y$ is a homeomorphic embedding of X in Y such that $\overline{c[X]} = Y$, is called a *compactification of the space X.*

Theorem. *(3.5.1) X has compactification $\Leftrightarrow$ X is Tychonoff.*

Theorem. *(3.5.2) Every Tychonoff space has a compactification Y such that $w(X) = w(Y)$.*

We shall say that compactifications c_1X and c_2X of a space X are *equivalent* if there exists a homeomorphism $f\colon c_1X \to c_2X$ such that the diagram

$$\begin{tikzcd} c_1X \arrow[r, "f"] & c_2X \\ X \arrow[u, "c_1"] \arrow[ur, "c_2"'] & \end{tikzcd}$$

is commutative, i.e., $fc_1(x) = c_2(x)$ for every $x \in X$.

Theorem. *(3.5.3) For every compactification Y of a space X we have $|Y| \leq \exp\exp d(X)$ and $w(Y) \leq \exp d(X)$.*

Let $c_1X \leq c_2X$ if there exists a continuous mapping $f\colon c_1X \to c_2X$ such that $fc_1 = c_2$.

Theorem. *(3.5.4) Compactifications c_1X and c_2X of a space X are equivalent if and only if $c_1X \leq c_2X$ and $c_2X \leq c_1X$.*

Theorem. *(3.5.5) Compactifications c_1X and c_2X of a space X are equivalent if and only if for every pair A, B of closed subsets of X we have*

$$\overline{c_1[A]} \cap \overline{c_1[B]} = \emptyset \text{ if and only if } \overline{c_2[A]} \cap \overline{c_2[B]} = \emptyset. \tag{16}$$

Lemma. *(3.5.6) Let A be a dense subspace of a Hausdorff space X and let $f\colon X \to Y$ be a mapping of X to an arbitrary space Y. If $f|A\colon A \to f[A] \subset Y$ is a homeomorphism, then $f[X \setminus A] \cap f[A] = \emptyset$.*

Theorem. *(3.5.7) If c_1X and c_2X are compactifications of a space X and a mapping $f\colon c_1X \to c_2X$ satisfies the condition $fc_1 = c_2$, then*

$$f(c_1(X)) = c_2(X) \text{ and } f(c_1X \setminus c_1(X)) = c_2X \setminus c_2(X).$$

Theorem. *(3.5.8) For every Tychonoff space X the following conditions are equivalent:*

(i) The space X is locally compact.

(ii) For every compactification cX of the space X the remainder $cX \setminus c[X]$ is closed in cX.

(iii) There exists a compactification cX of the space X such that the remainder $cX \setminus c[X]$ is closed in cX.

The next theorem states an important property of the family $\mathcal{C}(X)$ of all compactifications of X.

Theorem. *(3.5.9) Every non-empty subfamily $\mathcal{C}_0 \subset \mathcal{C}$ has a least upper bound with respect to the order $\leq$ in $\mathcal{C}(X)$.*

Corollary. *(3.5.10) For every Tychonoff space X there exists a largest element with respect to the order $\leq$ in $\mathcal{C}(X)$.*

The largest element in $\mathcal{C}(X)$ is called the *Čech-Stone compactification*

Theorem (The Alexandroff compactification theorem)**.** *(3.5.11) Every non-compact locally compact space X has a compactification ωX with one-point remainder. This compactification is the smallest element in $\mathcal{C}(X)$ with respect to the order $\leq$, its weight is equal to the weight of the space X.*

Theorem. *(3.5.12) If in the family $\mathcal{C}(X)$ of all compactifications of a non-compact Tychonoff space X there exists an element cX which is the smallest with respect to the order $\leq$, then X is locally compact and cX is equivalent to the Alexandroff compactification ωX of X.*

Theorem. *(3.5.13) If a compact space Y is a continuous image of the remainder $cX \setminus c[X]$ of a compactification cX of a locally compact space X, then the space X has a compactification $c'X \leq cX$ with the remainder homeomorphic to Y.*

3.5.E Maximal compactification of a Tychonoff space X can be obtained by taking the closure in $\prod_{f\in\mathcal{F}} I_f$ of the image of the space X under the mapping $\mathop{\triangle}_{f\in\mathcal{F}} f$, where $\mathcal{F}$ is the family of all continuous functions from X to I and $I_f = I$ for $f \in \mathcal{F}$.

3.6 The Čech-Stone compactification and the Wallman extension

Let us recall that the largest element in the family $\mathcal{C}(X)$ of all compactifications of a Tychonoff space X is called the Čech-Stone compactification of X and is denoted by βX.

Theorem. *(3.6.1) Every continuous mapping $f: X \to Z$ of a Tychonoff space X to a compact space Z is extendable to a mapping $F: \beta X \to Z$.*

If every continuous mapping of a Tychonoff space X to a compact space is continuously extendable over a compactification αX of X, then αX is equivalent to the Čech-Stone compactification of X.

Corollary. *(3.6.2) Every pair of completely separated subsets of a Tychonoff space X has disjoint closures in βX.*

If a compactification αX of X has the property that every pair of completely separated subsets of the space X has disjoint closures in αX, then αX is equivalent to the Čech-Stone compactification of X.

Corollary. *(3.6.3) Every continuous $f: X \to I$ (X is Tychonoff) is extendable to continuous $F: \beta X \to I$.*

If every continuous function from a Tychonoff space X to the closed interval I is continuously extendable over a compactification αX of X, then αX is equivalent to the Čech-Stone compactification of X.

Corollary. *(3.6.4) Every pair of disjoint closed subsets of a normal space X has disjoint closures in βX.*

If a compactification αX of a Tychonoff space X has the property that every pair of disjoint closed subsets of the space X has disjoint closures in αX, then αX is equivalent to the Čech-Stone compactification of X.

Corollary. *(3.6.5) For every clopen subset A of a Tychonoff space X the closure $\overline{A}$ of A in βX is clopen.*

Corollary. *(3.6.6) For every compactification αY of a Tychonoff space Y and every continuous mapping $f: X \to Y$ of a Tychonoff space X to the space Y there exists an extension $F: \beta X \to \alpha Y$ over βX and αY.*

Corollary. *(3.6.7) If a subspace M of a Tychonoff space X has the property that every continuous function $f: M \to I$ is continuously extendable over X, then the closure $\overline{M}$ of M in βX is a compactification of M equivalent to βM. If, moreover, M is dense in X, then $\beta X = \beta M$.*

Corollary. *(3.6.8) For every closed subspace M of a normal space X the closure $\overline{M}$ of M in βX is a compactification of M equivalent to βM.*

Corollary. *(3.6.9) For every Tychonoff space X and a space T such that $X \subset T \subset \beta X$ we have $\beta T = \beta X$.*

Theorem. *(3.6.11) For every $m \geq \aleph_0$ the Čech-Stone compactification of the space $D(\mathfrak{m})$ has cardinality $2^{2^{\mathfrak{m}}}$ and weight $2^{\mathfrak{m}}$.*

Corollary. *(3.6.12) The space βN has cardinality 2^c and weight c.*

Theorem. *(3.6.13) For every point $x \in \beta D(\mathfrak{m})$ and each neighbourhood V of x there exists an open-and-closed subset U of $\beta D(\mathfrak{m})$ such that $x \in U \subset V$.*

Theorem. *(3.6.14) Every infinite closed set $F \subset \beta N$ contains a subset homeomorphic to βN; in particular F has cardinality 2^c and weight c.*

Corollary. *(3.6.15) The space βN does not contain any subspace homeomorphic to $A(\aleph_0)$, i.e., in βN there are no non-trivial convergent sequences.*

Corollary. *(3.6.16) No non-discrete subspace of βN is a sequential space.*

Corollary. *(3.6.17) No space $N \cup \{x\} \subset \beta N$, where $x \in \beta N \setminus N$, is first-countable.*

Let X be a T_1-space and let $\mathcal{D}(X)$ denote the family of all closed subsets of X. The family of all ultrafilters in $\mathcal{D}(X)$ will be denoted by $F(X)$.

Properties of ultrafilters in $F(X)$:

(1) $\emptyset \notin \mathcal{F}$.

(2) If $A, B \in \mathcal{F}$, then $A \cap B$ in $\mathcal{F}$.

(3) If $B \in \mathcal{D}(X)$ and $B \cap A \neq \emptyset$ for every $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

(4) If $A \in \mathcal{F}$ and $A \subset B \in \mathcal{D}(X)$, then $B \in \mathcal{F}$.

(5) If $A, B \in \mathcal{D}(X)$ and $A \cup B \in \mathcal{F}$, then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(6) If $\mathcal{F} \neq \mathcal{F}'$, then there exist $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$ such that $A \cap A' = \emptyset$.

Ultrafilters that have an empty intersection are called *free ultrafilters*; they form a subfamily $F_0(X)$ of the family $F(X)$.

Let $wX = X \cup F_0(X)$; for every open set $U \subset X$ define

$$U^* = U \cup \{\mathcal{F} \in F_0(X) : A \subset U \text{ for some } A \in \mathcal{F}\} \subset wX.$$

$\mathcal{B} =$ the family of all sets U^* where U is an open subset of X. The set wX with the topology generated by the base $\mathcal{B}$ is called the *Wallman extension of the space X*.

Theorem. *(3.6.21) For every T_1-space X the Wallman extension wX is a quasi-compact T_1-space that contains X as a dense subspace and has the property that every continuous mapping $f\colon X \to Z$ of X to a compact space Z is extendable to a mapping $F\colon wX \to Z$.*

Theorem. *(3.6.22) The Wallman extension wX of a T_1-spaces X is a Hausdorff space if and only if X is a normal space.*

Corollary. *(3.6.23) For every normal space X the Wallman extension wX is a compactification of the space X equivalent to the Čech-Stone compactification of X.*

3.7 Perfect mappings

A continuous mapping $f\colon X \to Y$ is *perfect* if X is a Hausdorff space, f is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X. A one-to-one mapping $f\colon X \to Y$ defined on a Hausdorff space X is perfect if and only if it is a closed mapping, i.e., if f is a homeomorphic embedding and the set $f[X]$ is closed in Y.

Topological properties of Hausdorff spaces which are both invariants and inverse invariants of prefect mappings are called *perfect properties;* a class of all Hausdorff spaces that have a fixed perfect property is called a *perfect class of spaces.*

3.7.D: Let $f\colon X \to Y$ be a hereditarily quotient mapping with compact fibers defined on a Hausdorff space X. Then $w(Y) \leq w(X)$ and if X is locally compact and Y is a Hausdorff space, then Y also is locally compact.

3.8 Lindelöf spaces

We say that a topological space X is a *Lindelöf space,* or has the *Lindelöf property,* if X is regular and every open cover of X has a countable refinement.

Theorem. *(3.8.1) Every regular second-countable space is Lindelöf space.*

Theorem. *(3.8.2) Every Lindelöf space is normal.*

Theorem. *(3.8.3) A regular space X has the Lindelöf property if and only if every family of closed subsets of X which has the countable intersection property has non-empty intersection.*

Theorem. *(3.8.4) Every closed subspace of a Lindelöf space is a Lindelöf space.*

Theorem. *(3.8.5) If a subspace A of a topological space X has the Lindelöf property, then for every family $\{U_s\}_{s\in S}$ of open subsets of X such that $A \subset \bigcup_{s\in S} U_s$ there exists a countable set $\{s_1, s_2, \ldots\} \subset S$ such that $A \subset \bigcup_{i=1}^{\infty} U_{s_i}$.*

Theorem. *(3.8.6) If there exists a continuous mapping $f\colon X \to Y$ of a Lindelöf space X onto a regular space Y, then Y is a Lindelöf space.*

Every regular space which can be represented as a countable union of subspaces each of which has the Lindelöf property itself has Lindelöf property. In particular, every regular space which can be represented as a countable union of compact subspaces (Hausdorff spaces with this property are called *σ-compact spaces*) has the Lindelöf and is therefore normal. Lindelöf spaces are hereditary with respect to F_σ-sets.

Theorem. *(3.8.7) The sum $\coprod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, has the Lindelöf property if and only if all spaces X_s have the Lindelöf property and the set S is countable.*

Theorem. *(3.8.8) If $f\colon X \to Y$ is a closed mapping defined on a regular space X and all fibers $f^{-1}(y)$ have the Lindelöf property, then for every subspace $Z \subset Y$ that has the Lindelöf property the inverse image $f^{-1}(Z)$ also has the Lindelöf property.*

Theorem. *(3.8.9) The class of Lindelöf spaces is perfect.*

Corollary. *(3.8.10) The Cartesian product $X \times Y$ of a Lindelöf X and a compact space Y is a Lindelöf space.*

Theorem. *(3.8.11) Every open cover of a Lindelöf space has a locally finite open refinement.*

The smallest cardinal number $\mathfrak{m}$ such that every open cover of a space X has an open refinement of cardinality $\leq \mathfrak{m}$ is called the *Lindelöf number* of the space X and is denoted by $l(X)$.

Theorem. *(3.8.12) For every topological space X we have $l(X) \leq nw(X)$.*

Example. (3.8.13) Niemytzki plane = separable, not Lindelöf

$A(\mathfrak{m})$ for $\mathfrak{m} > \aleph_0$ = Lindelöf space, not separable

Since every countable regular space has the Lindelöf property, it follows from 3.3.24 that there exist Lindelöf spaces which are not k-spaces.

Example. (3.8.14, 15) Sorgenfrey line K is a Lindelöf space. $K \times K$ is not.

3.8.C: Observe that every hemicompact space is σ-compact but not necessarily vice versa. For a locally compact space X the following conditions are equivalent:

(1) The space X has the Lindelöf property.

(2) The space X is hemicompact.

(3) The space X is σ-compact.

(4) There exists a sequence $A_1, A_2, \ldots$ of compact subspaces of the space X such that $A_i \subset \operatorname{Int} A_{i+1}$ and $X = \bigcup\limits_{i=1}^{\infty} A_i$.

(5) The space X is compact or $\chi(\Omega, \omega X) \leq \aleph_0$.

3.8.A Observe that X is hereditary Lindelöf space if and only if all open subspaces of X have the Lindelöf property.

Show that a Lindelöf space X is a hereditarily Lindelöf space if and only if X is perfectly normal.

3.8.D: Prove that if X and Y are second-countable spaces and Y is regular, then the space Y^X is hereditarily Lindelöf with respect to both compact-open topology and the topology of pointwise convergence.

3.9 Čech-complete spaces

Theorem. *(3.9.1) For every Tychonoff space X the following conditions are equivalent:*

(i) For every compactification cX of the space X the remainder $cX \setminus c(X)$ is an F_σ-set in cX.

(ii) The remainder $\beta X \setminus \beta(X)$ is an F_σ-set in βX.

(iii) There exists a compactification cX of the space X such that the remainder $cX \setminus c(X)$ is an F_σ-set in cX.

A topological space X is *Čech complete* if X is a Tychonoff space and satisfies condition (i), and hence all the conditions, in Theorem 3.9.1.

We shall say that the diameter of a subset A of a topological space X is less than a cover $\mathcal{A} = \{A_s\}_{s \in S}$ of the space X, and we shall write $\delta(A) < \mathcal{A}$, provided that there exists an $s \in S$ such that $A \subset A_s$.

Theorem. *(3.9.2) A Tychonoff space X is Čech-complete if and only if there exists a countable family $(\mathcal{A}_i)_{i=1}^{\infty}$ of open covers of the space X with the property that any family $\mathcal{F}$ of closed subsets of X, which has the finite intersection property and contains sets of diameter less than $\mathcal{A}_i$ for $i = 1, 2, \ldots$, has non-empty intersection.*

Theorem (The Baire category theorem)**.** *(3.8.3) In a Čech-complete space X the union $A = \bigcup\limits_{i=1}^{\infty} A_i$ of a sequence $A_1, A_2, \ldots$ of nowhere dense sets is a co-dense set, i.e., the complement $X \setminus A$ is dense in X.*

Theorem. *(3.9.6) Čech-completeness is hereditary with respect to closed subsets and with respect to G_δ-subsets.*

3.10 Countably compact, pseudocompact and sequentially compact spaces

A topological space X is called a *countably compact space* if X is Hausdorff space and every countable open cover of X has a finite subcover.

Theorem. *(3.10.1) A topological space is compact if and only if it is a countably compact space with the Lindelöf property.*

Theorem. *(3.10.2) For every Hausdorff space X the following conditions are equivalent:*

(i) The space X is countably compact.

(ii) Every countable family of closed subsets of X which has the finite intersection property has non-empty intersection.

(iii) For every decreasing sequence $F_1 \supset F_2 \supset \ldots$ of non-empty closed subsets of X the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty.

Theorem. *(3.10.3) For every Hausdorff space X the following conditions are equivalent:*

(i) The space X is countably compact.

(ii) Every locally finite family of non-empty subsets of X is finite.

(iii) Every locally finite family of one-point subsets of X is finite.

(iv) Every infinite subset of X has an accumulation point.

(v) Every countably infinite subset of X has an accumulation point.

Theorem. *(3.10.4) Every closed subspace of a countably compact space is countably compact.*

Theorem. *(3.10.5) If there exists a continuous mapping $f\colon X \to Y$ onto Hausdorff space Y, then Y is a countably compact space.*

Theorem. *(3.10.6) Every continuous real-valued function defined on a countably compact space is bounded and attains its bounds.*

Theorem. *(3.10.7) If X is a countably compact space and Y is a sequential space, in particular, a first-countable space, then the projection $p\colon X \times Y \to Y$ is closed.*

Theorem. *(3.10.8) The sum $\coprod_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is countably compact if and only if all spaces X_s are countably compact and the set S is finite.*

Theorem. *(3.10.9) If $f\colon X \to Y$ is a closed mapping defined on a Hausdorff space X and all fibers $f^{-1}(y)$ are countably compact, then for every countable compact subspace $Z \subset Y$ the inverse image $f^{-1}(Z)$ is countably compact.*

Theorem. *(3.10.10) The class of countably compact spaces is perfect.*

Theorem. *(3.10.13) The Cartesian product $X \times Y$ of a countably compact space X and a countably compact k-space Y is countably compact.*

Corollary. *(3.10.14) The Cartesian product $X \times Y$ of a countably compact space X and a compact space Y is countably compact.*

Corollary. *(3.10.15) The Cartesian product $X \times Y$ of a countably compact space X and a countably compact sequential space Y is countably compact.*

A topological space X is called *pseudocompact* if X is a Tychonoff space and every real-valued continuous function defined on X is bounded. One can readily check that the last condition is equivalent to the condition that every continuous real-valued function on X attains its bounds.

Theorem. *(3.10.20) Every countably compact Tychonoff space is pseudocompact.*

Theorem. *(3.10.21) Every pseudocompact normal space is countably compact.*

Theorem. *(3.10.22) For every Tychonoff space X the following conditions are equivalent:*

(i) The space X is pseudocompact.

(ii) Every locally finite family of non-empty open subsets of X is finite.

(iii) Every locally finite open cover of X consisting of non-empty sets is finite.

(iv) Every locally finite cover of X has a finite subcover.

Theorem. *(3.10.23) For every Tychonoff space the following conditions are equivalent:*

(i) The space X is pseudocompact.

(ii) For every decreasing sequence $W_1 \supset W_2 \supset \dots$ of non-empty subsets of X the intersection $\bigcap_{i=1}^{\infty} \overline{W_i}$ if non-empty.

(iii) For every countable family $\{V_i\}_{i=1}^{\infty}$ of open subsets of X which has finite intersection property the intersection $\bigcap_{i=1}^{\infty} \overline{V_i}$ is non-empty.

Theorem. *(3.10.24) If there exists a continuous mapping $f\colon X \to Y$ of a pseudocompact space X onto a Tychonoff space Y, then Y is a pseudocompact space.*

Theorem. *(3.10.25) The sum $\coprod_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$ is pseudocompact if and only if all spaces X_s are pseudocompact and the set S is finite.*

Theorem. *(3.10.26) The cartesian product $X \times Y$ of a pseudocompact space X and a pseudocompact k-space Y is pseudocompact.*

Corollary. *(3.10.27) The cartesian product $X \times Y$ of a pseudocompact space X and compact space Y is pseudocompact.*

Corollary. *(3.10.28) The cartesian product $X \times Y$ of a pseudocompact space X and a pseudocompact sequential Y is pseudocompact.*

A topological space X is called *sequentially compact* if X is a Hausdorff space and every sequence of points of X has a convergent subsequence.

Theorem. *(3.10.30) Every sequentially compact space is countably compact.*

The reverse implication does not hold; there exist even compact spaces which are not sequentially compact – by virtue of Corollary 3.6.15, the Čech-Stone compactification $\beta\mathbb{N}$ is such a space.

Theorem. *(3.10.31) Sequential compactness and countable compactness are equivalent in the class of sequential spaces and, in particular, in the class of first-countable spaces.*

Theorem. *(3.10.32) If there exists a continuous mapping $f : X \to Y$ of a sequentially compact space X onto a Hausdorff space Y, then Y is a sequentially compact space.*

Theorem. *(3.10.33) Every closed subspace of a sequentially compact space is sequentially compact.*

Theorem. *(3.10.34) Sum of sequentially compact spaces is sequentially compact if and only if it is finite sum and every space is sequentially compact.*

Theorem. *(3.10.35) The Cartesian product of countably many sequentially compact spaces is sequentially compact.*

Theorem. *(3.10.36) The Cartesian product $X \times Y$ of a countably compact space X and a sequentially compact space Y is countably compact.*

Theorem. *(3.10.37) The Cartesian product $X \times Y$ of a pseudocompact space X and a sequentially compact space Y is countably compact.*

	ssp	cl.ssp	op.ssp	fin.sum	sum	fin.prod	c.prod	prod	cont.im
pseudocompact	-	-	-	+	-	-	-	-	$+_{3\frac{1}{2}}$
countably compact	-	+	-	+	-	-	-	-	$+_2$
sequentially compact	-	+	-	+	-	+	+	-	$+_2$

$$\begin{tikzcd}
\text{compact metrizable} \arrow[rr] \arrow[dr] & & \text{compact} \arrow[rr] \arrow[dr] & & \text{pseudocompact} \arrow[dl, shift right, "T_4"'] \\
& \text{sequentially compact} \arrow[rr] & & \text{countably compact} \arrow[ur, shift right, "T_{3\frac{1}{2}}"'] &
\end{tikzcd}$$

3.10.C: *two arrows space*: $X = C_0 \cup C_1$, where $C_0 = \langle 0,1\rangle$ and $C_1 = \langle 0,1) \times \{1\}$. The subspaces $C_{0,1}$ are homeomorphic to Sorgenfrey line.

The space X is hereditarily separable, hereditarily Lindelöf, perfectly normal, compact.

X^2 is not hereditarily normal.

Sorgenfrey line is not Čech-complete.

3.11 Realcompact spaces

A topological space X is called a *realcompact space* if X is a Tychonoff space and there is no Tychonoff space $\tilde{X}$ which satisfies the following two conditions:

(BC1) There exists a homeomorphic embedding $r\colon X \to \tilde{X}$ such that $r(X) \neq \overline{r(X)} = \tilde{X}$.

(BC2) For every continuous real-valued function $f\colon \tilde{X} \to R$ there exists a continuous function $\tilde{f}\colon \tilde{X} \to R$ such that $\tilde{f}r = f$.

Theorem. *(3.11.1) A topological space is compact if and only if it is a pseudocompact realcompact space.*

Theorem. *(3.11.3) A topological space is realcompact if and only if it is homeomorphic to a closed subspace of a Cartesian product of copies of real line.*

Theorem. *(3.11.4) Every closed subspace of a realcompact space is realcompact.*

Theorem. *(3.11.5) The Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is realcompact if and only if all spaces X_s are realcompact.*

Corollary. *(3.11.6) The limit of an inverse system of realcompact spaces is realcompact.*

Corollary. *(3.11.7) Let X be a topological space and $\{A_s\}_{s\in S}$ a family of subspaces of X; if all A_s's are realcompact, then the intersection $\bigcap_{s\in S} A_s$ also is realcompact.*

Corollary. *(3.11.8) If $f\colon X \to Y$ is a continuous mapping of a realcompact space X to a Hausdorff space Y, then for every realcompact subspace B of Y the inverse image $f^{-1}(B) \subset X$ is realcompact.*

Lemma. *(3.11.9) Let X be a topological space and A a subspace of X. If every function $g\colon A \to R$ such that $g(x) \geq 1$ for all $x \in A$ is extendable over X, then any function $f\colon A \to R$ is extendable over X.*

Theorem. *(3.11.10) A Tychonoff space X is realcompact if and only if for every point $x_0 \in \beta X \setminus X$ there exists a function $h\colon \beta X \to I$ such that $h(x_0) = 0$ and $h(x) > 0$ for any $x \in X$.*

$\mathcal{D}_0(X)$ = family of all functionally closed subsets of a Tychonoff space X.

Theorem. *(3.11.11) A Tychonoff space X is realcompact if and only if every ultrafilter in $\mathcal{D}_0(X)$ which has the countable intersection property has non-empty intersection.*

Theorem. *(3.11.12) Every Lindelöf space is realcompact.*

Example. (3.11.13) Let $\mathcal{A}$ be a family of sets closed with respect ti countable unions; by a *countably additive two-valued measure* defined on $\mathcal{A}$ we understand any function μ from $\mathcal{A}$ to $\{0,1\}$ satisfying the condition

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

whenever $A_i \in \mathcal{A}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. A cardinal number $\mathfrak{m}$ is called *non-measurable* provided that the only countably additive two-valued measure defined on the family of all subsets of a set X of cardinality $\mathfrak{m}$ which vanishes on all one-point sets is the trivial measure, identically equal to zero.

$\aleph_0$ is non-measurable.

If $\mathfrak{m} \in \mathcal{N}$ (class of all non-measurable cardinals), then every cardinal number less than $\mathfrak{m}$, the sum of any family $\{\mathfrak{m}_s\}_{s\in S}$ of cardinal numbers from $\mathcal{N}$ such that $|S| \leq \mathfrak{m}$ and the cardinal number $2^{\mathfrak{m}}$ also belong to $\mathcal{N}$. One can also prove that the smallest cardinal number that cannot be obtained from $\aleph_0$ by means of the three operations mentioned above (called the *first strongly inaccessible aleph*) also belongs to $\mathcal{N}$.

The assumption that all cardinal numbers are non-measurable is consistent with the axioms of set theory; on the other hand, it is not known whether the assumption of the existence of measurable cardinals is also consistent with the axioms of set theory.

$D(\mathfrak{m})$ is realcompact if and only if $\mathfrak{m}$ is a non-measurable cardinal.

3.12 Problems

3.12.1 Further characterization of compactness: complete accumulation points and the Alexander subbase theorem

3.12.1: A point x in a topological space X is called *complete accumulation point of a set* $A \subset X$ if $|A \cap U| = |A|$ for every neighborhood U of x.

For a Hausdorff space X the following conditions are equivalent:

(i) The space X is compact.

(ii) Every infinite subset of X has a complete accumulation point.

(iii) For every decreasing transfinite sequence $F_0 \supseteq F_1 \supseteq \ldots \supseteq F_\xi \supseteq \ldots$, $\xi < \alpha$, of non-empty closed subsets of X the intersection $\bigcap_{\xi<\alpha} F_\xi$ is non-empty.

3.12.2: *Alexander subbase theorem* Let X be a Hausdorff space and $\mathcal{P}$ a subbase for X; show that the space X is compact if and only if every covering of X by members of $\mathcal{P}$ has a finite subcovering.

3.12.2 Cardinal functions III

If there exists a quotient mapping $f\colon X \to Y$, then $\tau(X) \geq \tau(Y)$.

If $f\colon X \to Y$ is a closed mapping of a regular space X to a topological space Y and if for $x \in X$ the inequalities $\tau(f(x), Y) \leq \mathfrak{m}$ and $\tau(x, f^{-1}f(x)) \leq \mathfrak{m}$ hold, then $\tau(x, X) \leq \mathfrak{m}$. Note that if f is a perfect mapping, then the assumption of regularity can be omitted.

Show that if X is a locally compact space, then for every Hausdorff space Y we have $\tau(X \times Y) \leq \max(\tau(X), \tau(Y))$. Prove that if a family $\{X_s\}_{s\in S}$ of topological spaces has the property $\tau(\prod_{s\in S_0} X_s) \leq \mathfrak{m}$ for every finite $S_0 \subset S$ and if $|S| \leq \mathfrak{m}$ we have $\tau(\prod_{s\in S} X_s) \leq \mathfrak{m}$.

Show that for every family $\{X_s\}_{s\in S}$ of Hausdorff spaces such that $\tau(X_s) \leq \mathfrak{m}$ and $h(X_s) \leq \mathfrak{m}$ for $s \in S$ and $|S| \in \mathfrak{m}$ we have $\tau(\prod_{s\in S} X_s) \leq \mathfrak{m}$.

3.12.10: For every Hausdorff space X we have $|X| \leq \exp[l(X)\chi(X)]$. (Arhangelskii)

X Hausdorff $\Rightarrow |X| \leq \exp[c(X)\chi(X)]$

X is T_1 $\Rightarrow |X| \leq \exp[hc(X)\psi(X)]$

3.12.3 Rings of continuous functions and compactifications

For a Tychonoff space X the symbol $C(X)$ (the symbol $C^*(X)$) denotes the ring of all continuous real-valued (all bounded continuous real-valued) functions defined on X.

Every ideal is contained in a maximal ideal.

Tychonoff space X is compact if and only if for every maximal ideal Δ in the ring $C(X)$, or - equivalently - for every maximal ideal Δ in the ring $C^*(X)$ there exists a point $x \in X$ such that the conditions $f \in \Delta$ and $f(x) = 0$ are equivalent.

In the set $\mathcal{M}$ of all maximal ideals in the ring $C(X)$ (in the ring $C^*(X)$) generate a topology by the base consisting of all sets of the form $U_f = \{\Delta; f \notin \Delta\}$ and show that $\mathcal{M}$ is a compact space. Prove that $\mathcal{M}$ is the Čech-Stone compactification of X.

Compact spaces X and Y are homeomorphic if and only if the rings $C(X)$ and $C(Y)$ are isomorphic.

Verify that if Δ is a maximal ideal in $C(X)$, then $\mathcal{F}(\Delta) = \{f^{-1}(0) : f \in \Delta\}$ is an ultrafilter in the family $\mathcal{D}_0(X)$ of all functionally closed subsets of X and that $\Delta(\mathcal{F})$ is a maximal ideal in $C(X)$. $\Rightarrow$ one-to-one correspondence

4 Metric and metrizable spaces

4.1 Metric and metrizable spaces

A *metric space* is a pair (X, ϱ) consisting of a set X and a function ϱ defined on the set $X \times X$, assuming non-negative real values, and satisfying the following conditions:

(M1) $\varrho(x, y) = 0$ if and only if $x = y$.

(M2) $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in X$.

(M3) $\varrho(x, y) + \varrho(y, z) \geq \varrho(x, z)$ for all $x, y, z \in X$.

metrizable space, metrics,

Two metrics ϱ_1 and ϱ_2 are called *equivalent* if they induce the same topology on X.

Proposition. *(4.1.1) A point x belongs to the closure $\overline{A}$ of a set $A \subset X$ with respect to the topology induced by a metric ϱ if and only if there exists a sequence of the points of A that converges to x.*

Theorem. *(4.1.2) Two metrics ϱ_1 and ϱ_2 on X are equivalent if and only if they induce the same convergence.*

Theorem. *(4.1.3) For every metric space (X, ϱ) there exists a metric ϱ_1 on the set X which is equivalent to ϱ and bounded by 1.*

Example. (4.1.5) Let S be a set of cardinality $\mathfrak{m} \geq \aleph_0$ and let $I_s = I \times \{s\}$ for every $s \in S$. By letting

$$(x, s_1)E(y, s_2) \text{ if and only if } x = 0 = y \text{ or } x = y \text{ and } s_1 = s_2$$

we define an equivalence relation E on the set $\bigcup_{s \in S} I_s$. The formula

$$\varrho([(x, s_1)], [(y, s_2)]) = \begin{cases} |x - y|, & \text{if } s_1 = s_2 \\ x + y, & \text{if } s_1 \neq s_2 \end{cases}$$

defines a metric on the set of equivalence classes of E. This space will be called the *hedgehog space of spininess* $\mathfrak{m}$ and will be denoted by $J(\mathfrak{m})$.

Example. (4.1.7) $H=$ the set of all infinite sequences $\{x_i\}$ of real numbers satisfying $\sum x_i^2 < \infty$. $\varrho(x, y) = \sqrt{\sum(x_i - y_i)^2}$ – *Hilbert space.*

Proposition. *(4.1.8) A mapping f of a space X with the topology induced by a metric ϱ to a space Y with the topology induced by a metric σ s continuous if and only if for every $x \in X$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sigma(f(x), f(x')) < \varepsilon$ whenever $\varrho(x, x') < \delta$.*

uniformly continuous mapping: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X$ we have $\sigma(f(x), f(x')) < \varepsilon$ whenever $\varrho(x, x') < \delta$.
isometry $\varrho(x, y) = \sigma(f(x), f(y))$

Proposition. *(4.1.9) For a pair of points x, y and a set A in a metric space (X, ϱ) we have*

$$|\varrho(x, A) - \varrho(y, A)| \leq \varrho(x, y).$$

Theorem. *(4.1.10) For a fixed set $A \subset X$, assigning to every point $x \in X$ the distance $\varrho(x, A)$ defines a continuous function on X.*

Corollary. *(4.1.11) For every set A we have*

$$\overline{A} = \{x : \varrho(x, A) = 0\}.$$

Corollary. *(4.1.12) Every closed subset of a metrizable space is functional closed and, in particular, is a G_δ-set.*

Corollary. *(4.1.13) Every metrizable space is perfectly normal.*

Corollary. *(4.1.14) In a metric space (X, ϱ) for every compact set $A \subset X$ and any open set U containing A there exists an $r > 0$ such that $B(A, r) \subset U$.*

Theorem. *(4.1.15) For every cardinal number $\mathfrak{m}$ and any metrizable space X the following conditions are equivalent:*

(i) The space X has a base of cardinality $\leq \mathfrak{m}$.

(ii) The space X has a network of cardinality $\leq \mathfrak{m}$.

(iii) Every open cover of the space X has a subcover $\leq \mathfrak{m}$.

(iv) Every closed discrete subspace of X has cardinality $\leq \mathfrak{m}$.

(v) Every discrete subspace of the space X has cardinality $\leq \mathfrak{m}$.

(vi) Every family of pairwise disjoint non-empty open subsets of the space X has cardinality $\leq \mathfrak{m}$.

(vii) The space X has a dense subset of cardinality $\leq \mathfrak{m}$.

Corollary. *(4.1.16) For every metrizable space X the following conditions are equivalent:*

(i) The space X is second-countable.

(ii) The space X has the Lindelöf property.

(iii) The space X is separable.

(iv) Every family of pairwise disjoint non-empty open subsets of the space X is countable.

Theorem. *(4.1.17) For every metrizable space the following conditions are equivalent:*

(i) The space X is compact.

(ii) The space X is countably compact.

(iii) The space X is sequentially compact.

Theorem. *(4.1.18) Every compact metrizable space is separable.*

Hilbert space, described in example 4.1.7 is – as shown by Anderson in [1966] – homeomorphic to $\aleph_0$ copies of the real line; this is a difficult and deep result.

4.2 Operations on metrizable spaces

Theorem. *(4.2.1) The sum $\bigoplus_{s\in S} X_s$ is metrizable if and only if all spaces X_s are metrizable.*

$$\varrho(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i}\varrho_i(x_i,y_i) \tag{6}$$

Theorem. *(4.2.2) Let $X_1, X_2, \ldots$ be a sequence of metrizable spaces and let ϱ_i be a metric on the space X bounded by 1 for $i = 1, 2, \ldots$ The topology induced on the set $X = \prod_{i=1}^{\infty} X_i$ by the metric ϱ defined in (6) coincides with the topology of the Cartesian product of the spaces $\{X_i\}_{i=1}^{\infty}$.*

Corollary. *(4.2.3) The Hilbert space $I^{\aleph_0}$ is metrizable.*

Corollary. *(4.2.4) The Cartesian product $\prod_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is metrizable if and only if all spaces X_s are metrizable and there exists a countable set $S_0 \subset S$ such that X_s is a one-point space for $s \in S \setminus S_0$.*

Corollary. *(4.2.5) The limit of an inverse sequence of metrizable spaces is metrizable.*

Corollary. *(4.2.6) Metric is continuous $X \times X \to R$*

Corollary. *(4.2.7) A sequence $\{x_i^1\}, \{x_i^2\}, \ldots$ in the Cartesian product $\prod_{i=1}^{\infty} X_i$ of metrizable spaces converges to $x = \{x_i\} \in \prod_{i=1}^{\infty} X_i$ if and only if the sequence $x_i^1, x_i^2, \ldots$, converges to x_i for $i = 1, 2, \ldots$.*

Theorem. *(4.2.8) A compact space is metrizable if and only if it is a second-countable space.*

Theorem. *(4.2.9) A second-countable space is metrizable if and only if it is a regular space.*

Theorem. *(4.2.10) The Hilbert cube $I^{\aleph_0}$ is universal for all compact metrizable spaces and for all separable metrizable spaces.*

One can readily verify that the formula

$$\varrho(\{x_i\}, \{y_i\}) = \begin{cases} \frac{1}{k}, & \text{if } x_k \neq y_k \text{ and } x_i = y_i \text{ for } i < k, \\ 0, & \text{if } x_i = y_i, \text{ for all } i, \end{cases} \tag{6}$$

defines a metric on the set $\prod_{i=1}^{\infty} X_i$, $|X_i| = \mathfrak{m}$. $B(\mathfrak{m})$=*Baire space of weight* $\mathfrak{m}$

Theorem. *(4.2.13) If E is a closed equivalence relation on a separable metrizable space X and the equivalence classes of E are compact, then the quotient space X/E is metrizable.*

On the set of all bounded continuous mappings of X to Y one can define a metric $\hat{\varrho}$ by letting

$$\hat{\varrho}(f, g) = \sup_{x \in X} \varrho(f(x), g(x)) \tag{7}$$

Example. (4.2.14) Metrics on $\mathbb{R}$: $\varrho_1(x, y) = \min(1, |x - y|)$ and $\varrho_2(x, y) = \varrho(h(x), h(y))$, where $h\colon R \to S^1 \setminus \{0, 1\} \subset R^2$ is a homeomorphism and ϱ is the natural metric on R^2. Clearly, the two metrics ϱ_1 and ϱ_2 are equivalent but $\hat{\varrho}_1$ and $\hat{\varrho}_2$ are not equivalent.

Theorem. *(4.2.15) For every topological space X and any metric space (Y, ϱ), where ϱ is bounded, the topology on Y^X induced by $\hat{\varrho}$ is admissible.*

Corollary. *(4.2.16) For every topological space X and any metric space (Y, ϱ), where ϱ is bounded, the topology on Y^X induced by $\hat{\varrho}$ is finer that the compact-open topology.*

Theorem. *(4.2.17) For every compact space X, a metrizable space Y and any metric ϱ on the space Y, the topology on Y^X induced by $\hat{\varrho}$ coincides with the compact-open topology and is independent of the choice of the metric ϱ.*

Corollary. *(4.2.18) For every compact metrizable space X and any separable metric space (Y, ϱ), the space $(Y^X, \hat{\varrho})$ is separable.*

Theorem. *(4.2.19) Let X be a topological space, (Y, ϱ) a metric space and $\{f_i\}$ a sequence of continuous mappings from X to Y. If the sequence $\{f_i\}$ is uniformly convergent to a mapping f, then f is a continuous mapping from X to Y. If all f_i's are bounded, then f is also bounded.*

Theorem. *(4.2.20) For every topological space X the function space R^X with the topology of uniform convergence is metrizable.*

More exactly, the topology of uniform convergence on R^X is induced by the metric $\hat{\varrho}$, where ϱ is the metric on the real line defined by letting $\varrho(x, y) = \min(1, |x - y|)$.

4.2.D (c) Sequential spaces can be characterized as the images of metrizable spaces under quotient mappings and Fréchet spaces can be characterized as the images of metrizable spaces under hereditarily quotient mappings.

4.2.G Let X be a Tychonoff space, Y a metrizable space that contains a subspace homeomorphic to R and let ϱ be a bounded metric on the space Y. Show that if the metric $\hat{\varrho}$ induces the compact-open topology on Y^X, then X is a compact space.

4.2.H Prove that if X is a hemicompact space, then for every metrizable space Y the space Y^X with the compact-open topology is metrizable.

4.3 Totally bounded and complete metric spaces

Let (X, ϱ) be a metric space and A a subset of X; we say that A is *ε-dense* in (X, ϱ) if for every $x \in X$ there exists an $x' \in A$ such that $\varrho(x, x') < \varepsilon$.

A metric space (X, ϱ) is *totally bounded* if for every $\varepsilon > 0$ there exists a finite set $A \subset X$ which is ε-dense in (X, ϱ); a metric ϱ on a set X is *totally bounded* if the space (X, ϱ) is totally bounded.

Theorem. *(4.3.2) If (X, ϱ) is a totally bounded space, then for every subset $M \subset X$ the space (M, ϱ) is totally bounded.*

If (X, ϱ) is an arbitrary metric space and for a subset $M \subset X$ the space (M, ϱ) is totally bounded, then the space $(\overline{M}, \varrho)$ also is totally bounded.

Theorem. *(4.3.3) Let $\{(X_i, \varrho_i)\}_{i=1}^{\infty}$ be a family of non-empty metric spaces such that the metric space ϱ_i is bounded by 1 for $i = 1, 2, \ldots$ The Cartesian product $\prod_{i=1}^{\infty} X_i$ with the metric ϱ defined by formula (6) is totally bounded if and only if all spaces (X_i, ϱ_i) are totally bounded.*

Corollary. *(4.3.4) The Hilbert cube $I^{\aleph_0}$ with the metric ϱ defined by letting*

$$\varrho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i - y_i|, \qquad \text{where } x = \{x_i\} \text{ and } y = \{y_i\},$$

is a totally bounded space.

Theorem. *(4.3.5) A metrizable space is metrizable by a totally bounded metric if and only if it is a separable space.*

Corollary. *(4.3.6) A topological space is metrizable by a totally bounded metric if and only if it is a regular second-countable space.*

A topological space X is *completely metrizable* if there exists a complete metric on the space X.

Theorem (The Cantor Theorem). *(4.3.8) A metric space (X, ϱ) is complete if and only if for every decreasing sequence $F_1 \supset F_2 \supset \ldots$ of non-empty closed subsets of X, such that $\lim \delta(F_i) = 0$, the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty.*

Corollary. *(4.3.9) If (X, ϱ) is an arbitrary metric space and M is a subset of X such that the space (M, ϱ) is complete, then M is closed in X.*

Theorem. *(4.3.10) A metric space (X, ϱ) is complete if and only if every family of closed subsets of X which has the finite intersection property and which for every $\varepsilon > 0$ contains a set of diameter less than ε has non-empty intersection.*

Theorem. *(4.3.11) If (X, ϱ) is a complete space, then for a subset $M \subset X$ the space (M, ϱ) is complete if and only if M is closed in X.*

Theorem. *(4.3.12) Let $\{(X_i, \varrho_i)\}$ be a family of non-empty metric spaces such that the metric ϱ_i is bounded by 1 for $i = 1, 2, \ldots$ The Cartesian product $\prod_{i=1}^{\infty} X_i$ with the metric defined by formula (6) is complete if and only if all spaces (X_i, ϱ_i) are complete.*

Theorem. *(4.3.13) For every topological space X and any complete metric space (Y, ϱ) the space of all bounded continuous mapping for X to Y with the metric $\hat{\varrho}$ defined by formula (7) is complete.*

Theorem. *(4.3.14) Every metric space is isometric to a subspace of a complete metric space.*

Corollary. *(4.3.15) Every metrizable space is embeddable in a completely metrizable space.*

Let X be a topological space, (Y, ϱ) a metric space and $f\colon A \to Y$ a continuous mapping defined on a dense subset A of the space X; we say that the *oscillation of the mapping f at a point $x \in X$ is equal to zero* if for every $\varepsilon > 0$ there exists a neighbourhood U of the point x such that $\delta(f[A \cap U]) < \varepsilon$. The set of all points at which the oscillation of f is equal to zero is a G_δ-set containing A.

Lemma. *(4.3.16) If X is a topological space, (Y, ϱ) a complete metric space and $f\colon A \to Y$ a continuous mapping defined on a dense subset A of the space X, then the mapping f is extendable to a continuous mapping $F\colon B \to Y$ defined on the set B consisting of all points of X at which the oscillation of f is equal to zero.*

Theorem. *(4.3.17) If (X, ϱ) is a metric space and (Y, σ) is a complete metric space, then every mapping $f\colon A \to Y$ from a dense subset A of the space X to the space Y which is uniformly continuous with respect to ϱ and σ is extendable to a mapping $F\colon X \to Y$ uniformly continuous with respect to ϱ and σ.*

Corollary. *(4.3.18) If (X, ϱ) and (Y, σ) are complete metric spaces then every isometry of (A, ϱ_A) onto (B, ϱ_B), where A and B are dense subsets of X and Y respectively, is extendable to an isometry of (X, ϱ) onto (Y, σ).*

Theorem. *(4.3.19) For every metric space (X, ϱ) there exists exactly one (up to an isometry) complete metric space $(\tilde{X}, \tilde{\varrho})$ such that $\tilde{X}$ contains a dense subspace isometric to (X, ϱ). Moreover, we have $w(\tilde{X}) = w(X)$, and if (X, ϱ) is a totally bounded space, then $(\tilde{X}, \tilde{\varrho})$ also is totally bounded. (*completion of the metric space (X, ϱ)*)*

The space $(\tilde{X}, \tilde{\varrho})$ satisfying the conditions in Theorem 4.3.19 is called the *completion of metric space* (X, ϱ).

Theorem. *(4.3.20) If Y is a completely metrizable space, then every continuous mapping $f\colon A \to Y$ from a dense subset A of a topological space X to the space Y is extendable to a continuous mapping $F\colon B \to Y$ defined on a G_δ-set $B \subset X$ containing the set A.*

Theorem (The Lavrentieff theorem)**.** *(4.3.21) Let X and Y be completely metrizable spaces and let $A \subset X$ and $C \subset Y$ be arbitrary subspaces. Every homeomorphism $f\colon A \to C$ is extendable to a homeomorphism $F\colon B \to D$, where $A \subset B \subset X$, $C \subset D \subset Y$ and B and D are G_δ sets in X and Y respectively.*

Lemma. *(4.3.22) Every G_δ-set in a metrizable space X is homeomorphic to a closed subspace of the Cartesian product $X \times R^{\aleph_0}$.*

Theorem. *(4.3.23) Complete metrizability is hereditary with respect to G_δ-sets.*

Theorem. *(4.3.24) If a subspace M of a metrizable space X is completely metrizable, then M is a G_δ set in X.*

Corollary. *(4.3.25) A separable metrizable space is completely metrizable if and only if it is embeddable in $R^{\aleph_0}$ as a closed subspace.*

Theorem. *(4.3.26) A topological space is completely metrizable if and only if it is a Čech-complete metrizable space.*

Theorem. *(4.3.27) Every metric on a compact space is totally bounded.*

Theorem. *(4.3.28) Every metric on a compact space is complete.*

Theorem. *(4.3.29) A metrizable space X is compact if and only if on the space X there exists a metric ϱ which is both totally bounded and complete.*

Corollary. *(4.3.30) The completion of a metric space (X, ϱ) is compact if and only if (X, ϱ) is a totally bounded space.*

Theorem (The Lebesgue covering theorem)**.** *(4.3.31) For every open cover $\mathcal{A}$ of a compact metric space X there exists an $\varepsilon > 0$ such that the cover $\{B(x, \varepsilon)\}_{x \in X}$ is a refinement of $\mathcal{A}$.*

Theorem. *(4.3.32) Every continuous mapping $f\colon X \to Y$ of a compact metrizable space X to a metrizable space Y is uniformly continuous with respect to any metrics ϱ and σ on the spaces X and Y respectively.*

4.3.F If X is a locally compact Lindelöf space, then for every completely metrizable space Y the space Y^X with the compact-open topology is completely metrizable. Give an example of a hemicompact space X such that the space I^X with the compact-open topology is not completely metrizable.

4.3.G The space P of all irrational numbers (with the topology of a subspace of real line) is homeomorphic to the Baire space $B(\aleph_0) = \mathbb{N}_0^{\aleph}$.

4.4 Metrization theorems I

Theorem (The Stone theorem)**.** *(4.4.1) Every open cover of a metrizable space has an open refinement which is both locally finite and σ-discrete.*

Theorem. *(4.4.3) Every metrizable space has a σ-discrete base.*

Corollary. *(4.4.4) Every metrizable space has a σ-locally finite base.*

Lemma. *(4.4.5) Every regular space which has a σ-locally finite base is normal.*

Lemma. *(4.4.6) Let X be a T_0-space and $\{\varrho_i\}_{i=1}^{\infty}$ a countable family of pseudometrics on the set X which all are bounded by 1 and satisfy the following two conditions:*

(i) $\varrho_i \colon X \times X \to R$ is a continuous function for $i = 1, 2, \ldots$

(ii) For every $x \in X$ and every non-empty closes set $A \subset X$ such that $x \notin A$ there exists an i such that $\varrho_i(x, A) = \inf_{a \in A} \varrho_i(x, a) > 0$.

Then the space X is metrizable and the function ϱ defined by letting

$$\varrho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varrho_i(x, y)$$

is a metric on the space X.

Theorem (The Nagata-Smirnov metrization theorem)**.** *(4.4.7) A topological space is metrizable if and only if it is regular and has σ-locally finite base.*

Theorem (The Bing metrization theorem)**.** *(4.4.8) A topological space is metrizable if and only if it is regular and has σ-discrete base.*

Theorem. *(4.4.9) The Cartesian product $[J(\mathfrak{m})]^{\aleph_0}$ of $\aleph_0$ copies of the hedgehog $J(\mathfrak{m})$ is universal for all metrizable spaces of weight $\mathfrak{m} \geq \aleph_0$.*

Lemma. *(4.4.12) If every open cover of a topological space X has a locally finite closed refinement, then every open cover of X has also a locally finite open refinement.*

Lemma. *(4.4.13) If there exists a perfect mapping $f \colon X \to Y$ of a metrizable space X onto Y, then every open cover of the space Y has an open locally finite refinement.*

Theorem. *(4.4.15) Metrizability is an invariant of perfect mappings.*

Theorem (Vaǐnšteǐns lemma)**.** *(4.4.16) If $f \colon X \to Y$ is a closed mapping of a metrizable space X onto a space Y, then for every $y \in Y$ such that $\chi(y, Y) \leq \aleph_0$ the set $\operatorname{Fr} f^{-1}(y)$ is compact.*

Theorem (The Hanai-Morita-Stone theorem)**.** *(4.4.17) For every closed mapping $f \colon X \to Y$ of a metrizable space X onto a space Y the following conditions are equivalent:*

(i) The space Y is metrizable.

(ii) The space Y is first-countable.

(iii) For every $y \in Y$ the set $\operatorname{Fr} f^{-1}(y)$ is compact.

Theorem. *(4.4.18) Metrizability is an invariant of closed-and-open mappings.*

Theorem. *(4.4.19) If a topological space X has a locally finite closed cover consisting of metrizable subspaces, then X is itself metrizable.*

5 Paracompact spaces

5.1 Paracompact spaces

A topological space X is called a *paracompact space* if X is a Hausdorff space and every open cover of X has a locally finite open refinement.

Let us observe that, in contrast to the definition of compactness, in the definition of paracompactness the term "refinement" cannot be replaced by the term "subcover". Every discrete space is paracompact.

Theorem. *(5.1.1) Every compact space is paracompact.*

Theorem. *(5.1.2) Every Lindelöf space is paracompact.*

Theorem. *(5.1.3) Every metrizable space is paracompact.*

Lemma. *(5.1.4) Let X be a paracompact space and A, B a pair of closed subsets of X. If for every $x \in B$ there exists open sets U_x, V_x such that $A \subset U_x$, $x \in V_x$ and $U_x \cap V_x = \emptyset$, then there also exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

Theorem. *(5.1.5) Every paracompact space is normal.*

A family $\{f_s\}_{s\in S}$ of continuous functions from a space X to the closed unit interval I is called a *partition of unity* on the space X if $\sum_{s\in S} f_s(x) = 1$ for every $x \in X$. (For a fixed $x_0 \in X$ only countable many functions f_s do not vanish at x_0.)

We say that a partition of unity $\{f_s\}$ on a space X is *locally finite* if the cover $\{f_s^{-1}((0,1\rangle)\}_{s\in S}$ of the space X is locally finite.

A partition of unity $\{f_s\}_{s\in S}$ is *subordinated to a cover* $\mathcal{A}$ of X if the cover $\{f_s^{-1}(\langle 0,1\rangle)\}_{s\in S}$ of the space X is a refinement of $\mathcal{A}$.

Lemma. *(5.1.6) If every open cover of a regular space X has a locally finite refinement (consisting of arbitrary sets), then for every open cover $\{U_s\}_{s\in S}$ of the space X there exists a closed locally finite cover $\{F_s\}_{s\in S}$ of X such that $F_s \subset U_s$ for every $s \in S$.*

Lemma. *(5.1.8) If for an open cover $\mathcal{U}$ of a space X there exists a partition of unity $\{f_s\}_{s\in S}$ subordinated to it, then $\mathcal{U}$ has an open locally finite refinement.*

Theorem. *(5.1.9) For every T_1-space X the following conditions are equivalent:*

(i) The space X is paracompact.

(ii) Every open cover of the space X has a locally finite partition of unity subordinated to it.

(iii) Every open cover of the space X has a partition of unity subordinated to it.

Lemma. *(5.1.10) Every open σ-locally finite cover $\mathcal{V}$ of a topological space X has a locally finite refinement (consisting of arbitrary sets).*

Theorem. *(5.1.11) For every regular space X the following condition are equivalent:*

(i) The space X is paracompact.

(ii) Every open cover of the space X has an open σ-locally finite refinement.

(iii) Every open cover of the space X has a locally finite refinement (consisting of arbitrary sets).

(iv) Every open cover of the space X has a closed locally finite refinement.

Let $\mathcal{A} = \{A_s\}_{s \in S}$ be a cover of a set X; the *star of a set $M \subset X$ with respect to $\mathcal{A}$* is the set $\mathrm{St}(M, \mathcal{A}) = \bigcup\{A_s : M \cap A_s \neq \emptyset\}$. The star of a one-point set $\{x\}$ with respect to a cover $\mathcal{A}$ is called the *star of the point x with respect to $\mathcal{A}$* and denoted by $\mathrm{St}(x, \mathcal{A})$. We say that a cover $\mathcal{B} = \{B_t\}_{t \in T}$ of a set X is a *star refinement* of another cover $\mathcal{A} = \{A_s\}_{s \in S}$ of the same set X if for every $t \in T$ there exists an $s(t) \in S$ such that $\mathrm{St}(B_t, \mathcal{B}) \subset A_{s(t)}$. If for every $x \in X$ there exists an $s(x) \in S$ such that $\mathrm{St}(x, \mathcal{B}) \subset A_{s(x)}$, then we say that $\mathcal{B}$ is a *barycentric refinement of $\mathcal{A}$*. Clearly, every star refinement is a barycentric refinement and every barycentric refinement is a refinement.

Theorem. *(5.1.12) For every T_1-space X the following conditions are equivalent:*

(i) The space X is paracompact.

(ii) Every open cover of the space X has an open barycentric refinement.

(iii) Every open cover of the space X has an open star refinement.

(iv) The space X is regular and every open cover of X has an open σ-discrete refinement.

Lemma. *(5.1.13) If an open cover $\mathcal{U}$ of a topological space X has a closed locally finite refinement, then $\mathcal{U}$ has also an open barycentric refinement.*

Lemma. *(5.1.15) If a cover $\mathcal{A} = \{A_s\}_{s \in S}$ of a set X is a barycentric refinement of a cover $\mathcal{B} = \{B_t\}$ of X, and $\mathcal{B}$ is a barycentric refinement of a cover $\mathcal{C} = \{C_z\}$ of the same set, then $\mathcal{A}$ is a star refinement of $\mathcal{C}$.*

Lemma. *(5.1.16) If every open cover of a topological space X has an open star refinement, then every open cover of X has also an open σ-discrete refinement.*

A topological space X is called *collectionwise normal* if X is a T_1-space and for every discrete family $\{F_s\}_{s \in S}$ of closed subsets of X there exists a discrete family $\{V_s\}_{s \in S}$ of open subsets of X such that $F_s \subset V_s$ for every $s \in S$. Clearly, every collectionwise normal space is normal.

Theorem. *(5.1.17) A T_1-space X is collectionwise normal if and only if for every discrete family $\{F_s\}_{s\in S}$ of closed subsets of X there exists a family $\{U_s\}_{s\in S}$ of open subsets of X such that $F_s \subset U_s$ for every $s \in S$ and $U_s \cap U_{s'} = \emptyset$ whenever $s \neq s'$.*

Theorem. *(5.1.18) Every paracompact space is collectionwise normal.*

Theorem. *(5.1.20) Every countably compact paracompact spaces is compact.*

Example. (5.1.21) The space W_0 of all countable ordinal numbers is not paracompact. Since W_0 is countably compact and normal, it is collectionwise normal.

Examples 5.1.22,23 skipped.

Lemma. *(5.1.24) Every locally finite family of non-empty subsets of a Lindelöf space is countable.*

Theorem. *(5.1.25) If a paracompact space X contains a dense subspace A which has the Lindelöf property, then X is a Lindelöf space.*

Corollary. *(5.1.26) Every separable paracompact space is a Lindelöf space.*

Theorem. *(5.1.27) Every locally compact paracompact space X can be represented as the union of a family of disjoint closed-and-open subspaces of X each of which has the Lindelöf property.*

Theorem. *(5.1.28) Paracompactness is hereditary with respect to F_σ-sets.*

Corollary. *(5.1.29) Every closed subspace of a paracompact space is paracompact.*

Theorem. *(5.1.30) The sum $\bigoplus_{s\in S} X_s$ is paracompact if and only if all spaces X_s are paracompact.*

Example. (5.1.31) Sorgenfrey line K is a paracompact space. Since the Cartesian product $K \times K$ is not normal, the Cartesian product of two paracompact spaces is not necessarily paracompact.

Example 5.1.32 skipped

Theorem (The Michael theorem)**.** *(5.1.33) Paracompactness is an invariant of closed mappings.*

Theorem. *(5.1.34) If a topological space X has a locally finite closed cover consisting of paracompact subspaces, then X is itself paracompact.*

Theorem. *(5.1.35) Paracompactness is an inverse invariant of perfect mappings.*

Theorem. *(5.1.36) The cartesian product $X \times Y$ of a paracompact space X and a compact space Y is paracompact.*

Theorem. *(5.1.37) The class of paracompact spaces is perfect.*

Theorem (The Tamano theorem)**.** *(5.1.38) For every Tychonoff space X the following conditions are equivalent:*

(i) The space X is paracompact.

(ii) For every compactification cX of the space X the Cartesian product $X \times cX$ is normal.

(iii) The Cartesian product $X \times \beta X$ is normal.

(iv) There exists a compactification cX of the space X such that the Cartesian product $X \times cX$ is normal.

Theorem. *(5.1.39) A topological space X is paracompact if and only if for every compact space Y the Cartesian product $X \times Y$ is normal.*

Example 5.1.40 skipped.

5.2 Countably paracompact spaces

5.3 Weakly and strongly paracompact spaces

A topological space X is called *weakly paracompact*[3] if X is a Hausdorff space and every open cover of X has a point-finite open refinement. Every paracompact space is weakly paracompact, but not vice-versa.

5.4 Metrization theorems II

A sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of covers of a topological space X is called a *development for the space* X if all covers $\mathcal{W}_i$ are open, and for every point $x \in X$ and any neighbourhood U of x there exists a natural number i such that $\mathrm{St}(x, \mathcal{W}_i) \subset U$. One easily observes that a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of a topological space X is a development for X if and only if for every $x \in X$ any family $\{W_i\}_{i=1}^{\infty}$ such that $x \in W_i \in \mathcal{W}_i$ for $i = 1, 2, \ldots$ is a base for X at the point x.

Theorem (Bing's metrization criterion)**.** *(5.4.1) A topological space is metrizable if and only if it is collectionwise normal and has a development.*

A sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of covers of a topological space X is called a *strong development for the space* X if all covers $\mathcal{W}_i$ are open and for every point $x \in X$ and any neighbourhood U of x there exist a neighbourhood V of the point x and a natural number i such that $\mathrm{St}(V, \mathcal{W}_i) \subset U$. Clearly, every strong development is a development.

Theorem (The Moore metrization theorem)**.** *(5.4.2) A topological space is metrizable if and only if it is a T_0-space and has a strong development.*

We say that a base $\mathcal{B}$ for a topological space X is *point-regular* if for every point $x \in X$ and any neighbourhood U of x the set of all members of $\mathcal{B}$ that contain x and meet $X \setminus U$ is finite. One easily observes that a base $\mathcal{B}$ for a space X is point-regular if and only if for every $x \in X$ any family consisting of $\aleph_0$ members of $\mathcal{B}$ which all contain x is a base for X at the point x.[4]

We say that a base $\mathcal{B}$ for a topological space X is *regular* if for every point $x \in X$ and any neighbourhood U of x there exists a neighbourhood $V \subset U$ of

[3] The terms *metacompact* and *pointwise paracompact* are also used.

[4] The reader should be warned that point-regular bases are also called *uniform* bases.

the point x such that the set of all members of $\mathcal{B}$ that meet both V and $X \setminus U$ is finite. Clearly, every regular base is point-regular.

To simplify the statement of the next lemmas, for a family $\mathcal{A}$ of sets we shall denote by $\mathcal{A}^{\mathrm{m}}$ the subfamily of $\mathcal{A}$ consisting of all maximal elements, and for a topological space X we shall denote by $\mathcal{J}(X)$ the family of all open one-point subsets of X.

Lemma. *(5.4.3) If $\mathcal{B}$ is a point-regular (regular) base for a space X, then the family $\mathcal{B}^m \subset \mathcal{B}$ is a point-finite (locally finite) cover of X.*

Lemma. *(5.4.4) If $\mathcal{B}$ is a base for a T_1-space, then for every point-finite cover $\mathcal{B}' \subset \mathcal{B}$ the family $\mathcal{B}'' = (\mathcal{B} \setminus \mathcal{B}') \cup \mathcal{J}(X)$ is a base for X. Moreover, if the base $\mathcal{B}$ is point-regular (regular), then the base $\mathcal{B}''$ also is point-regular (regular).*

Theorem (The Arhangelskii metrization theorem)**.** *(5.4.6) A topological space is metrizable if and only if it is a T_1-space and has a regular base.*

Lemma. *(5.4.7) For every Hausdorff space X the following conditions are equivalent:*

(i) The space X has a point-regular base.

(ii) The space X is weakly paracompact and has a development.

(iii) The space X has a development consisting of point-finite covers.

Theorem (Alexandroff's metrization criterion)**.** *(5.4.8) A topological space is metrizable if and only if it is collectionwise normal and has a point-regular base.*

Theorem (The Alexandroff-Urysohn metrization theorem)**.** *(5.4.9) A topological space is metrizable if and only if it is a T_0-space and has a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ such that for every natural number i and any two sets $W_1, W_2 \in \mathcal{W}_{i+1}$ with non-empty intersection there exists a set $W \in \mathcal{W}_i$ such that $W_1 \cup W_2 \subset W$.*

Corollary. *(5.4.10) A topological space is metrizable if and only if it is a T_0-space and has a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ such that $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$*

6 Connected spaces

6.1 Connected spaces

We say that a topological space X is *connected* if X cannot be represented in the form $X_1 \oplus X_2$, where X_1 and X_2 are non-empty subspaces of X.

Theorem. *(6.1.1) For every topological space X the following conditions are equivalent:*

(i) The space X is connected.

(ii) The empty set and the whole space are the only closed-and-open subsets of the space X.

(iii) If $X = X_1 \cup X_2$ and the sets X_1 and X_2 are separated, then one of them is empty.

(iv) Every continuous mapping $f\colon X \to D$ of the space X to the two-point discrete space $D = \{0,1\}$ is constant, i.e., either $f[X] \subset \{0\}$ or $f[X] \subset \{1\}$.

Corollary. *(6.1.2) A space X is connected if and only if it cannot be represented as the union $X_1 \cup X_2$ of two closed (open), non-empty and disjoint subsets.*

Corollary. *(6.1.3) Every connected Tychonoff space containing at least two points has cardinality not less than $\mathfrak{c}$.*

Theorem. *(6.1.4) Connectedness is an invariant of continuous mappings.*

Theorem. *(6.1.7) A subspace C of a topological space X is connected if and only if for every pair X_1, X_2 of separated subsets of X such that $C = X_1 \cup X_2$ we have either $X_1 = \emptyset$ or $X_2 = \emptyset$.*

Corollary. *(6.1.8) If a subspace C of a topological space X is connected then for every pair X_1, X_2 of separated subsets of X such that $C \subset X_1 \cup X_2$ we have either $C \subset X_1$ or $C \subset X_2$.*

Theorem. *(6.1.9) Let $\{C_s\}_{s\in S}$ be a family of connected subspaces of a topological space X. If there exists an $s_0 \in S$ such that the set C_{s_0} is not separated from any of the sets C_s, then the union $\bigcup\limits_{s\in S} C_s$ is connected.*

Corollary. *(6.1.10) If the family $\{C_s\}_{s\in S}$ of connected subspaces of a topological space has non-empty intersection, then the union $\bigcup\limits_{s\in S} C_s$ is connected.*

Corollary. *(6.1.11) If a subspace C of X is connected, then every subspace A of X which satisfies $C \subset A \subset \overline{C}$ also is connected.*

Corollary. *(6.1.12) If a topological space X contains a connected dense subspace, then X is itself connected.*

Corollary. *(6.1.13) If any two points of a topological space X can be joined by a connected subspace of X, then the space X is connected.*

Theorem. *(6.1.14) The Čech-Stone compactification βX of a Tychonoff space X is connected if and only if the space X is connected.*

Theorem. *(6.1.15) The Cartesian product $\prod\limits_{s\in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is connected if and only if all spaces X_s are connected.*

Corollary. *(6.1.16) Euclidean n-space R^n, the Tychonoff cube $I^{\mathfrak{m}}$ and the Alexandroff cube $F^{\mathfrak{m}}$ are all connected.*

A topological space X is called a *continuum* if X is both connected and compact.

Theorem. *(6.1.18) The limit of an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ of continua is a continuum.*

Corollary. *(6.1.19) If a family* $\{X_s\}_{s\in S}$ *of continua is closed with respect to finite intersections, then the intersection* $\bigcap_{s\in S} X_s$ *is a continuum.*

Corollary. *(6.1.20) The intersection* $\bigcap_{s\in S} X_s$ *of a decreasing sequence* $X_1 \supset X_2 \supset \ldots$ *of continua is a continuum.*

The *component of a point* x in a topological space X is the union of all connected subspaces of X which contain the point x. *Components of the space* X constitute a decomposition of the space X into pairwise disjoint, connected, closed subsets.

Theorem. *(6.1.21) The component of a point* $x = \{x_s\}$ *in the Cartesian product* $\prod_{s\in S} X_s$ *coincides with the Cartesian product* $\prod_{s\in S} C_s$*, where* C_s *is the component of the point* x_s *in the space* X_s*.*

The *quasi-component of a point* x is the intersection of all closed-and-open subsets of X which contain the point x. Quasi-components are closed subsets of X, they constitute a decomposition of X.

Theorem. *(6.1.22) The component* C *of a point* x *in a topological space* X *is contained in the quasi-component* Q *of the point* x*.*

Theorem. *(6.1.23) In a compact space* X *the component of a point* $x \in X$ *coincides with the quasi-component of the point* x*.*

Lemma. *(6.1.25) If* A *is a closed subspace of a continuum* X *such that* $\emptyset \neq A \neq X$*, then for every component* C *of the space* A *we have* $C \cap \operatorname{Fr} A \neq \emptyset$*.*

Lemma. *(6.1.26) If a continuum* X *is covered by pairwise disjoint closed sets* $X_1, X_2, \ldots$ *of which at least two are non-empty, then for every* i *there exists a continuum* $C \subset X$ *such that* $C \cap X_i = \emptyset$ *and at least two sets in the sequence* $C \cap X_1, C \cap X_2, \ldots$ *are non-empty.*

Theorem (The Sierpiński theorem)**.** *(6.1.27) If a continuum* X *has a countable cover* $\{X_i\}_{i=1}^{\infty}$ *by pairwise disjoint closed subsets, then at most one of the sets* X_i *is non-empty.*

We say that a continuous mapping $f \colon X \to Y$ is *monotone* if all fibers $f^{-1}(y)$ are connected.

Theorem. *(6.1.28) If* $f \colon X \to Y$ *is a monotone quotient mapping, then for every connected subset* C *of the space* Y *which is either closed or open, the inverse image* $f^{-1}(C)$ *is connected.*

Theorem. *(6.1.29) If* $f \colon X \to Y$ *is a monotone mapping which is either closed or open, then for every connected subset* C *of the space* Y *the inverse image* $f^{-1}(C)$ *is connected.*

6.2 Various kinds of disconnectedness

A topological space X is called *hereditarily disconnected* if X does not contain any connected subset of cardinality larger than one. Hence, a space X is hereditarily disconnected if and only if the component of any point $x \in X$ is the point x alone.

A topological space X is called *zero-dimensional* if X is a non-empty T_1-space and has a base consisting of open-and-closed sets. Clearly, every zero-dimensional space is a Tychonoff space.

Theorem. *(6.2.1) Every zero-dimensional space is hereditarily disconnected.*

A cover of a topological space consisting of functionally open (closed) sets will be called in the sequel a *functionally open (closed) cover.*

A topological space X is called *strongly zero-dimensional* if X is a non-empty Tychonoff space and every finite functionally open cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Clearly, the refinement $\{V_i\}_{i=1}^m$ consists of open-and-closed sets and thus is a functionally open cover of X.

Lemma. *(6.2.2) For every pair A, B of completely separated subsets of a strongly zero-dimensional space X there exists an open-and-closed set $U \subset X$ such that $A \subset U \subset X \setminus B$.*

Lemma. *(6.2.3) If for every pair A, B of completely separated subsets of a topological (normal) space X there exists an open-and-closed set $U \subset X$ such that $A \subset U \subset X \setminus B$, then every finite functionally open (open) cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^k$ such that $V_i \subset U_i$ for $i = 1, 2, \ldots, k$ and $V_i \cap V_j = \emptyset$ whenever $i \neq j$.*

Theorem. *(6.2.4) A non-empty Tychonoff space X is strongly zero-dimensional if and only if for every pair A, B of completely separated subsets of the space X there exists an open-and-closed set $U \subset X$ such that $A \subset U \subset X \setminus B$.*

Theorem. *(6.2.5) A non-empty normal space X is strongly zero-dimensional if and only if every open cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$.*

Theorem. *(6.2.6) Every strongly zero-dimensional space is zero-dimensional.*

Theorem. *(6.2.7) Every zero-dimensional Lindelöf space is strongly zero-dimensional.*

Corollary. *(6.2.8) Every non-empty regular space X such that $|X| \leq \aleph_0$ is strongly zero-dimensional.*

Theorem. *(6.2.9) Hereditary disconnectedness, zero-dimensionality and strong zero-dimensionality are equivalent in the realm of non-empty locally compact paracompact spaces.*

Corollary. *(6.2.10) Hereditary disconnectedness, zero-dimensionality and strong zero-dimensionality are equivalent in the realm of non-empty compact spaces.*

Theorem. *(6.2.11) Hereditary disconnectedness is a hereditary property and zero-dimensionality is hereditary with respect to non-empty sets.*

If X is a strongly zero-dimensional space and M is a non-empty subspace of X with the property that every continuous function $f\colon M \to I$ is continuously extendable over X, then the space M also is strongly zero-dimensional.

In particular, in normal spaces strong zero-dimensionality is hereditary with respect to non-empty closed sets.

Theorem. *(6.2.12) The Čech-Stone compactification βX of a Tychonoff space X is strongly zero-dimensional if and only if the space X is strongly zero-dimensional.*

Theorem. *(6.2.13) The sum $\bigoplus_{s \in S} X_s$, where $S \neq \emptyset$ and $X_s \neq \emptyset$ for $s \in S$, is hereditarily disconnected (zero-dimensional, strongly zero-dimensional) if and only if all spaces X_s are hereditarily disconnected (zero-dimensional, strongly zero-dimensional).*

Theorem. *(6.2.14) The Cartesian product $\prod_{s \in S} X_s$, where $S \neq \emptyset$ and $X_s \neq \emptyset$ for $s \in S$, is hereditarily disconnected (zero-dimensional) if and only if all spaces X_s are hereditarily disconnected (zero-dimensional).*

Corollary. *(6.2.15) The limit of an inverse system of hereditarily disconnected (zero-dimensional) spaces is hereditarily disconnected (zero-dimensional or empty).*

Theorem. *(6.2.16) The Cantor cube $D^{\mathfrak{m}}$ is universal for all zero-dimensional spaces.*

Corollary. *(6.2.17) Every zero-dimensional space X of weight $\mathfrak{m}$ has a zero-dimensional compactification of weight $\mathfrak{m}$.*

A continuous mapping $f\colon X \to Y$ is *light (zero-dimensional)* if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty).

Lemma. *(6.2.21) For every perfect mapping $f\colon X \to Y$ the equivalence relation E on the space X, determined by the decomposition of all fibers $f^{-1}(y)$ into components, is closed.*

Theorem. *(6.2.22) Every perfect mapping $f\colon X \to Y$ can be represented as the composition $f = hg$, where $g\colon X \to Z$ is a monotone perfect mapping and $h\colon Z \to Y$ is a zero-dimensional perfect mapping.*

Theorem. *(6.2.23) If a continuous mapping $f\colon X \to Y$ is represented for $i = 1$ and 2 as the composition $h_i g_i$, where $g_i\colon X \to Z_i$ is a monotone quotient mapping and $h_i\colon Z_i \to Y$ is a light mapping, then there exists a homeomorphism $h\colon Z_1 \to Z_2$ such that the following diagram is commutative.*

$$\begin{tikzcd}
& Z_1 \arrow[dd, "h"] \arrow[dr, "h_1"] & \\
X \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & & Y \\
& Z_1 \arrow[ur, "h_2"'] &
\end{tikzcd}$$

Theorem. *(6.2.24) For every compact space X, the decomposition of X into components, or – equivalently – into quasi-components, determines a closed equivalence relation E on the space X; the quotient space X/E is compact and zero-dimensional.*

A topological space X is called *extremally disconnected* if for every open set $U \subset X$ the closure $\overline{U}$ is open in X.

Theorem. *(6.2.25) Every non-empty extremally disconnected Tychonoff space is strongly zero-dimensional.*

Theorem. *(6.2.26) A topological space X is extremally disconnected if and only if for every pair U, V of disjoint open subsets of X we have $\overline{U} \cap \overline{V} = \emptyset$.*

Theorem. *(6.2.27) The Čech-Stone compactification βX of a Tychonoff space X is extremally disconnected if and only if the space X is extremally disconnected.*

Corollary. *(6.2.28) For every $\mathfrak{m} \geq \aleph_0$ the Čech-Stone compactification $\beta D(\mathfrak{m})$ of the discrete space $D(\mathfrak{m})$ is extremally disconnected.*

Corollary. *(6.2.29) The space βN is extremally disconnected.*

Theorem. *(6.2.30) The sum $\bigoplus_{s\in S} X_s$ is extremally disconnected if and only if all spaces X_s are extremally disconnected.*

Hereditarily disconnected spaces were introduced by Hausdorff. The spaces of this class are sometimes called totally disconnected; however, at present the term *totally disconnected* is usually applied to a space X such that the quasi-component of any point $x \in X$ consists of the point x alone (this class of spaces was introduced by Sierpiński). Every zero-dimensional space is totally disconnected and every totally disconnected space is hereditarily disconnected.

6.2.A Every G_δ-set which is both dense and co-dense in a separable zero-dimensional completely metrizable space is homeomorphic to the space of irrational numbers.

Every separable zero-dimensional completely metrizable space which does not contain any non-empty compact space is homeomorphic to the space of irrational numbers.

Every dense in itself zero-dimensional compact metrizable space is homeomorphic to the Cantor set.

Every dense in itself countable metrizable space is homeomorphic to the space of irrational numbers.

7 Dimension of topological spaces

7.1 Definitions and basic properties of dimensions ind, Ind, and dim

7.2 Further properties of the dimension dim

7.3 Dimension of metrizable spaces

8 Uniform spaces and proximity spaces

8.1 Uniformities and uniform space

$-A = \{(x,y) : (y,x) \in A\}$, $A + B = \{(x,z) :$ there exists a $y \in X$ such that $(x,y) \in A$ and $(y,z) \in B\}$.

$1A = A$, $nA = (n-1)A + A$

$mA + nA = nA + mA = (m+n)A$

The diagonal of the Cartesian product $X \times X$ is the set $\Delta = \{(x, x) : x \in X\}$. Every set $V \subset X \times X$ that contains Δ and satisfies the condition $V = -V$ is called an *entourage of diagonal;* the family of all entourages of the diagonal $\Delta \subset X \times X$ will be denoted by $\mathcal{D}_X$. If for a pair x, y of points of X and a $V \in \mathcal{D}_X$ we have $(x, y) \in V$, we say that the *distance between x and y is less than* V and we write $|x - y| < V$; otherwise we write $|x - y| \geq V$. If for any pair of points of a set $A \subset X$ and a $V \in \mathcal{D}_X$ we have $|x - y| < V$, i.e., if $A \times A \subset V$, we say that the *diameter of A is less then* V and we write $\delta(A) < V$. One readily checks that for any $x, y, z \in X$ and any $V, V_1, V_2 \in \mathcal{D}_X$ the following conditions hold:

(1) $|x - x| < V$.

(2) $|x - y| < V$ if and only if $|y - x| < V$.

(3) If $|x - y| < V_1$ and $|y - z| < V_2$, then $|x - z| < V_1 + V_2$.

Let x_0 be a point of X and let $V \in \mathcal{D}_X$; the set $B(x_0, V) = \{x \in X : |x_0 - x| < V\}$ is called the *ball with center x_0 and radius* V or simply the *V-ball about* x_0. It follows immediately from (3) that the diameter of a V-ball is less than $2V$. For a set $A \subset X$ and $V \in \mathcal{D}_X$, by the *V-ball about* A we mean the set $B(A, V) = \bigcup_{x \in A} B(x, V)$.

A *uniformity* on a set X is a subfamily $\mathcal{U}$ of $\mathcal{D}_X$ which satisfies the following conditions:

(U1) If $V \in \mathcal{U}$ and $V \subset W \in \mathcal{D}_X$, then $W \in \mathcal{U}$.

(U2) If $V_1, V_2 \in \mathcal{U}$, then $V_1 \cap V_2 \in \mathcal{U}$.

(U3) For every $V \in \mathcal{U}$ there exists a $W \in \mathcal{U}$ such that $2W \subset V$.

(U4) $\bigcap \mathcal{U} = \Delta$.

A family $\mathcal{B} \subset \mathcal{U}$ is called a *base for the uniformity* $\mathcal{U}$ if for every $V \in \mathcal{U}$ there exists a $W \in \mathcal{B}$ such that $W \subset V$. The smallest cardinal number of the form $|\mathcal{B}|$ where $\mathcal{B}$ is a base for $\mathcal{U}$, is called the *weight of the uniformity* $\mathcal{U}$ and is denoted by $w(\mathcal{U})$.

Any base $\mathcal{B}$ for a uniformity on a set X has the following properties:

(BU1) For any $V_1, V_2 \in \mathcal{B}$ there exists a $V \in \mathcal{B}$ such that $V \subset V_1 \cap V_2$.

(BU2) For every $V \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that $2W \subset V$.

(BU3) $\bigcap \mathcal{B} = \Delta$.

Observe that every entourage of the diagonal $V \in \mathcal{D}_X$ yields a cover $\mathcal{C}(V) = \{B(x, V)\}_{x \in X}$ of the set X. Let $\mathcal{U}$ be a uniformity on a set X; any cover of the set X which has a refinement of the form $\mathcal{C}(V)$, where $V \in \mathcal{U}$, is called *uniform with respect to* $\mathcal{U}$. The collection $\mathbf{C}$ of all covers of a set X which are uniform with respect to a uniformity $\mathcal{U}$ on the set X has the following properties:

(UC1) If $\mathcal{A} \in \mathbf{C}$ and $\mathcal{A}$ is a refinement of a cover $\mathcal{B}$ of the set X, then $\mathcal{B} \in \mathbf{C}$.

(UC2) For any $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{C}$ there exists an $\mathcal{A} \in \mathbf{C}$ which is a refinement of both $\mathcal{A}_1$ and $\mathcal{A}_2$.

(UC3) For every $\mathcal{A} \in \mathbf{C}$ there exist a $\mathcal{B} \in \mathbf{C}$ which is a star refinement of $\mathcal{A}$.

(UC4) For every pair x, y of distinct points of X there exists an $\mathcal{A} \in \mathbf{C}$ such that no member of $\mathcal{A}$ contains both x and y.

A *uniform space* is a pair $(X,\mathcal{U})$ consisting of a set X and a uniformity $\mathcal{U}$ on the set X. The *weight of a uniform space* $(X,\mathcal{U})$ is defined as the weight of the uniformity $\mathcal{U}$.

Theorem. *(8.1.1) For every uniformity $\mathcal{U}$ on a set X the family*

$$\mathcal{O} = \{G \subset X : \text{ for every } x \in G \text{ there exists a } V \in \mathcal{U} \text{ such that } B(x,V) \subset G\}$$

is a topology on the set X and the topological space $(X,\mathcal{O})$ is a T_1-space.

The topology $\mathcal{O}$ is called the *topology induced by the uniformity $\mathcal{U}$.*

If X is a topological space and a uniformity $\mathcal{U}$ on the set X induces the original topology of X, then we say that $\mathcal{U}$ is *uniformity on the space X.*

Proposition. *(8.1.2) The interior of a set $A \subset X$ with respect to the topology induced by a uniformity $\mathcal{U}$ on the set X coincides with the set*

$$B = \{x \in X : \text{ there exists a } V \in \mathcal{U} \text{ such that } B(x,V) \subset A\}.$$

Corollary. *(8.1.3) If the topology of a space X is induced by a uniformity $\mathcal{U}$, then for every $x \in X$ and any $V \in \mathcal{U}$ the set $Int\, B(x,V)$ is a neighborhood of x.*

Corollary. *(8.1.4) If the topology of a space X is induced by a uniformity $\mathcal{U}$, then for every $x \in X$ and any $A \subset X$ we have*

$$x \in \overline{A} \text{ if and only if } A \cap B(x,V) \neq \emptyset \text{ for every } V \in \mathcal{U}.$$

Corollary. *(8.1.5) If the topology of a space X is induced by a uniformity $\mathcal{U}$, then for every $A \subset X$ and any $V \in \mathcal{U}$ we have*

$$\delta(\overline{A}) < 3V \qquad \text{whenever} \qquad \delta(A) < V.$$

Example. (8.1.6) $\mathcal{U} = \mathcal{D}_X$ =discrete uniformity, discrete uniform space. The one-element family $\mathcal{B} = \{\delta\}$ is a base for $\mathcal{U}$, so that $w(\mathcal{U}) = 1$.

It follows from the above example that the weight of a topological space $(X,\mathcal{O})$, where the topology $\mathcal{O}$ is induced by a uniformity $\mathcal{U}$, can be larger than the weight of $\mathcal{U}$. On the other hand, one readily verifies that the character of $(X,\mathcal{O})$ is less than or equal to the weight of $\mathcal{U}$.

Distinct uniformities may induce the same topology - example 8.1.7.

Let $\mathcal{U}$ be a uniformity on a set X; the Tychonoff topology on the Cartesian product $X \times X$, where X has the topology induced by $\mathcal{U}$, is called the *topology induced by the uniformity $\mathcal{U}$ on the set $X \times X$.*

Consider a uniform space $(X,\mathcal{U})$ and a pseudometric ϱ on the set X; we say that the pseudometric ϱ is *uniform with respect to $\mathcal{U}$* if for every $\varepsilon > 0$ there exists a $V \in \mathcal{U}$ such that $\varrho(x,y) < \varepsilon$ whenever $|x - y| < V$.

Proposition. *(8.1.9) If a pseudometric ϱ on a set X is uniform with respect to a uniformity $\mathcal{U}$ on X, then ϱ is a continuous function from the set $X \times X$ with the topology induced by the uniformity $\mathcal{U}$ to the real line.*

Theorem. *(8.1.10) For every sequence* $V_0, V_1, \ldots$ *of members of a uniformity* $\mathcal{U}$ *on a set* X*, where*

$$V_0 = X \times X \qquad and \qquad 3V_{i+1} \subset V_i \qquad for\ 1, 2, \ldots, \tag{4}$$

there exists a pseudometric ϱ *on the set* X *such that for every* $i \geq 1$

$$V_i \subset \{(x,y) : \varrho(x,y) \leq 1/2^i\} \subset V_{i-1}.$$

Corollary. *(8.1.11) For every uniformity* $\mathcal{U}$ *on a set* X *and any* $V \in \mathcal{U}$ *there exists a pseudometric* ϱ *on the set* X *which is uniform with respect to* $\mathcal{U}$ *and satisfies a condition*

$$\{(x,y) : \varrho(x,y) < 1\} \subset V.$$

Corollary. *(8.1.12) For every uniformity* $\mathcal{U}$ *on a set* X *the family of all members of* $\mathcal{U}$ *which are open with respect to the topology induced by* $\mathcal{U}$ *on* $X \times X$*, as well as the family of all members of* $\mathcal{U}$ *which are closed with respect to that topology, are both bases for* $\mathcal{U}$*.*

Corollary. *(8.1.13) For every uniformity* $\mathcal{U}$ *on a set* X*, the set* X *with the topology induced by* $\mathcal{U}$ *is a Tychonoff space.*

Let $(X,\mathcal{U})$ be a uniform space; we shall show that the family P if all pseudometrics on the set X which are uniform with respect to $\mathcal{U}$ has the following properties:

(UP1) If $\varrho_1, \varrho_2 \in P$ then $\max(\varrho_1, \varrho_2) \in P$.

(UP2) For every pair x, y of distinct points of X there exists a $\varrho \in P$ such that $\varrho(x,y) > 0$.

Proposition. *(8.1.14) Suppose we are given a set* X *and a family* $\mathcal{B} \subset \mathcal{D}_X$ *of entourages of the diagonal which has properties (BU1)-(BU3). The family* $\mathcal{U}$ *consisting of all members of* $\mathcal{D}_X$ *which contain a member of* $\mathcal{B}$ *is a uniformity on the set* X*. The family* $\mathcal{B}$ *is a base for* $\mathcal{U}$*.*

If, moreover, X *is a topological space and the family* $\mathcal{B}$ *consists of open subsets of the Cartesian product* $X \times X$*, and if for every* $x \in X$ *and any neighborhood* G *of* x *there exists a* $V \in \mathcal{B}$ *such that* $B(x,V) \subset G$*, then* $\mathcal{U}$ *is a uniformity on the space* X*.*

The uniformity $\mathcal{U}$ is called the *uniformity generated by the base* $\mathcal{B}$*.*

Proposition. *(8.1.16) Suppose we are given a set* X *and a collection* $\mathbf{C}$ *of covers of* X *which has properties (UC1)–(UC4). The family* $\mathcal{B} \subset \mathcal{D}_X$ *of all entourages of the diagonal which are of the form* $\bigcup\{H \times H : H \in \mathcal{A}\}$*, where* $\mathcal{A} \in \mathbf{C}$*, is a base for a uniformity* $\mathcal{U}$ *on the set* X*. The collection* $\mathbf{C}$ *is the collection of all covers of* X *which are uniform with respect to* $\mathcal{U}$*.*

If, moreover, X *is a topological space and the collection* $\mathbf{C}$ *consists of open covers of* X*, and if for every* $x \in X$ *and any neighborhood* G *of* x *there exists an* $\mathcal{A} \in \mathbf{C}$ *such that* $\mathrm{St}(x,\mathcal{A}) \subset G$*, then* $\mathcal{U}$ *is a uniformity on the space* X*.*

The uniformity $\mathcal{U}$ is called the *uniformity generated by the collection* $\mathbf{C}$ *of uniform covers.*

Example. (8.1.17) A *topological group* is a group G which is in the same time a T_1-space such that the following two conditions are satisfied:

(TG1) The formula $f(x,y) = xy$ defines a continuous mapping $f\colon G \times G \to G$.

(TG2) The formula $f(x) = x^{-1}$ defines a continuous mapping $f\colon G \to G$.

Let G be a group an d let A, B be subsets of G; we define $A^{-1} = \{x^{-1} : x \in A\}$ and $AB = \{xy : x \in A \text{ and } y \in B\}$. Similarly xA and Ax. If A is an open subset of a topological group G, then the set A^{-1} is also open. Similarly, the set AB is open if at least one of the sets A and B is open. In particular, for every open set $H \subset G$ the sets xH and Hx are open.

Now let G be a topological group and let $\mathcal{B} = \mathcal{B}(e)$ be a base for G at the point e. Every member H of $\mathcal{B}$ determines three covers of G:

$$\mathcal{C}_l = \{xH\}_{x\in G}, \qquad \mathcal{C}_r = \{Hx\}_{x\in G}, \qquad \text{and} \qquad \mathcal{H} = \{xHy\}_{x,y\in G}.$$

Denote by $\mathbf{C}_l$, $\mathbf{C}_r$ and $\mathbf{C}$ respectively the collection of all covers of G which have a refinement of the form $\mathcal{C}_l(H)$, $\mathcal{C}_r(H)$, or $\mathcal{C}(H)$, where $H \in \mathcal{B}$. They have properties (UC1)–(UC4) and thus generate a uniformity on the set G. Moreover it turns out that the topology induced by each of those uniformities coincides with the original topology of G.

Every topological group is a Tychonoff space.

Proposition. *(8.1.18) Suppose we are given a set X and a family P of pseudometrics on the set X which has properties* (UP1)-(UP2). *The family $\mathcal{B} \subset \mathcal{D}_X$ of all entourages of the diagonal which are of the form $\{(x,y) : \varrho(x,y) < 1/2^i\}$, where $\varrho \in P$ and $i = 1, 2, \ldots$, is a base for a uniformity $\mathcal{U}$ on the set X. Every pseudometric $\varrho \in P$ is uniform with respect to $\mathcal{U}$.*

If, moreover, X is a topological space and all pseudometrics of the family P are continuous functions from $X \times X$ to the real line, and if for every $x \in X$ and every non-empty closed set $A \subset X$ such that $x \notin A$ there exists a $\varrho \in P$ such that $\inf_{a\in A} \varrho(x,a) > 0$, then $\mathcal{U}$ is a uniformity on the space X.

The uniformity $\mathcal{U}$ is called the uniformity generated by the family P of uniform pseudometrics.

Example. (8.1.19) Family of pseudometrics on $C(X)$ and $C^*(X)$: For every finite sequence $f_1, \ldots, f_k$ of elements of $C(X)$ (resp. $C^*(X)$) the formula

$$\varrho_{f_1,\ldots,f_k} = \max\{|f_1(x) - f_1(y)|, \ldots, |f_k(x) - f_k(y)|\}$$

defines a pseudometric on the set X. The families of all such pseudometrics have properties (UP1)–(UP2). The induced topology coincides with the original topology.

Theorem. *(8.1.20) The topology of a space X can be induced by a uniformity on the set X if and only if X is a Tychonoff space.*

Let X be a set and let ϱ be a metric on the set X. Since the family $\{\varrho\}$ has properties (UP1)–(UP2), it generates a uniformity $\mathcal{U}$ on the set X. Moreover, by virtue of Corollaries 4.2.6.and 4.1.11, the topologies induced on X by the metric ϱ and by the uniformity $\mathcal{U}$ coincide. The uniformity $\mathcal{U}$ is called the *uniformity induced by the metric ϱ.* The uniform space $(X,\mathcal{U})$ is *metrizable.*

Theorem. *(8.1.21) A uniformity $\mathcal{U}$ on a set X is induced by a metric on the set X if and only if $w(\mathcal{U}) \leq \aleph_0$.*

A mapping $f\colon X \to Y$ is called *uniformly continuous with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$* if for every $V \in \mathcal{V}$ there exist a $U \in \mathcal{U}$ such that for all $x, x' \in X$ we have $|f(x) - f(x')| < V$ whenever $|x - x'| < U$. It follows immediately from the definition that f is a continuous mapping of the space X with the topology induced by $\mathcal{U}$ to the space Y with the topology induced by $\mathcal{V}$.

Proposition. *(8.1.22) Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces and f a mapping of X to Y. The following conditions are equivalent:*

- *(i) The mapping f is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$.*
- *(ii) There exist bases $\mathcal{B}$ and $\mathcal{C}$ for $\mathcal{U}$ and $\mathcal{V}$ respectively, such that for every $V \in \mathcal{C}$ there exists a $U \in \mathcal{B}$ satisfying $U \subset (f \times f)^{-1}(V)$.*
- *(iii) For every cover $\mathcal{A}$ of the set Y which is uniform with respect to $\mathcal{V}$ the cover $\{f^{-1}(A) : A \in \mathcal{A}\}$ of the set X is uniform with respect to $\mathcal{U}$.*
- *(iv) For every pseudometric ϱ on the set Y which is uniform with respect to $\mathcal{V}$ the pseudometric σ on the set X defined by letting $\sigma(x,y) = \varrho(f(x), f(y))$ is uniform with respect to $\mathcal{U}$.*

A one-to-one mapping f of a set X onto a set Y is a *uniform isomorphism with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$* on the sets X and Y respectively, if f is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$ and the inverse mapping f^{-1} is uniformly continuous with respect to $\mathcal{V}$ and $\mathcal{U}$. A uniform isomorphism is a homeomorphism of the induced topological spaces.

We say that two uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are *uniformly isomorphic* if there exists a uniform isomorphism of $(X,\mathcal{U})$ onto $(Y,\mathcal{V})$.

8.2 Operations on uniform spaces

Suppose we are given a uniform space $(X,\mathcal{U})$ and a set $M \subset X$. Let $\mathcal{U}_M = \{(M \times M) \cap V : V \in \mathcal{U}\} \subset \mathcal{D}_M$. The uniform space $(M,\mathcal{U}_M)$ is called a *subspace of the uniform space* $(X,\mathcal{U})$.

embedding of the subspace

Let $\{(X_s,\mathcal{U}_s)\}_{s\in S}$ be a family of uniform spaces. The family $\mathcal{B}$ of all entourages of the diagonal $\Delta \subset (\prod_{s\in S} X_s) \times (\prod_{s\in S} X_s)$ which are of the form

$$\{(\{x_s\}, \{y_s\}) : |x_{s_i} - y_{s_i}| < V_{s_i} \text{ for } i = 1, 2, \ldots, k\},$$

where $s_1, s_2, \ldots, s_k \in S$ and $V_{s_i} \in \mathcal{U}_{s_i}$ for $i = 1, 2, \ldots, k$. The family $\mathcal{B}$ generates a uniformity on the set $\prod_{s\in S} X_s$; this uniformity is called the *Cartesian product of the uniformities* $\{\mathcal{U}_s\}_{s\in S}$ and is denoted by $\prod_{s\in S} \mathcal{U}_s$.

Cartesian product of uniform spaces

The reader can easily verify that the topology induced on $\prod X_s$ by the uniformity $\prod \mathcal{U}_s$ coincides with the Tychonoff topology of the Cartesian product $\prod X_s$, where X_s has the topology induced by $\prod \mathcal{U}_s$.

Proposition. *(8.2.1) Let* $(X,\mathcal{U})$ *be a uniform space,* $\{(Y_s, V_s)\}_{s\in S}$ *a family of uniform spaces and* f *a mapping of the set* X *to the Cartesian product* $\prod_{s\in S} Y_s$*. The mapping* f *is uniformly continuous with respect to* $\mathcal{U}$ *and* $\prod_{s\in S} \mathcal{V}_s$ *if and only if the composition* $p_s f$ *is uniformly continuous with respect to* $\mathcal{U}$ *and* $\mathcal{V}_s$ *for every* $s \in S$*.*

Theorem. *(8.2.2) Every uniform space is uniformly isomorphic to a subspace of the Cartesian product of a family of metrizable uniform spaces.*

Remark. (8.2.4) Every uniform space of weight $\mathfrak{m}$ is uniformly isomorphic to a subspace of the Cartesian product of $\mathfrak{m}$ metrizable uniform spaces. Let us also observe that there is no universal space $(X,\mathcal{U})$ for all uniform spaces of weight $\leq \mathfrak{m}$.

In the remaining part of this section we shall discuss function spaces.

Let X be a topological space and let $(Y,\mathcal{U})$ be a uniform space. We shall denote by Y^X the set of all continuous mappings of the space X to the space Y, where Y is equipped with the topology induced by $\mathcal{U}$. For every $V \in \mathcal{U}$ denote by $\hat{V}$ the entourage of the diagonal $\Delta \subset Y^X \times Y^X$ defined by the formula

$$\hat{V} = \{(f, g) : |f(x) - g(x)| < V \text{ for every } x \in X\}.$$

From readily established formulas

$$\hat{U} \cap \hat{V} = \widehat{U \cap V} \qquad \text{and} \qquad \hat{U} + \hat{V} \subset \widehat{U + V}$$

it follows that the family $\{\hat{V} : V \in \mathcal{U}\}$ has properties (BU1)–(BU3); the uniformity on the set Y^X generated by this family will be called *uniformity of uniform convergence* induced by $\mathcal{U}$ and will be denoted by $\hat{\mathcal{U}}$.

If the uniformity $\mathcal{U}$ is induced by a bounded metric ϱ on Y, then $w(\hat{U}) \leq \aleph_0$, so that - by Theorem 8.1.21 - the uniformity $\hat{\mathcal{U}}$ is induced by a metric on Y^X. One readily verifies that the metric $\hat{\varrho}$ defined by formula (7) in Section 4.2 induces the uniformity $\hat{U}$. Hence it follows from Example 4.2.14 that two uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on Y which induce the same topology, the topologies on Y^X induced by $\hat{\mathcal{U}}_1$ and $\hat{\mathcal{U}}_2$ can be different. It turns out, however, that for a compact space X - as in the case of metric space - the topology on Y^X is independent of the choice of a particular uniformity $\mathcal{U}$ on the space Y, because the topology induced by $\hat{\mathcal{U}}$ coincides with the compact-open topology on Y^X. This fact is a corollary to Theorem 8.2.6 proved below; to formulate the theorem we have to introduce another uniformity on Y^X.

For a Hausdorff space X and a uniform space $(Y,\mathcal{U})$ we shall denote by $\hat{\mathcal{U}}|\mathcal{Z}(X)$ the uniformity on Y^X generated by the base consisting of all finite intersections of the sets of the form

$$\hat{V}|Z = \{(f, g) : |f(x) - g(x)| < V \text{ for every } x \in Z\}, \tag{3}$$

where $V \in \mathcal{U}$, $Z \in \mathcal{Z}(X)$ and $\mathcal{Z}(X)$ is the family of all compact subsets of X (the reader can easily check that the family of all finite intersection of the sets in (3) has the properties (BU1)–(BU3). The uniformity $\hat{\mathcal{U}}|\mathcal{Z}(X)$ will be called the *uniformity of uniform convergence on compacta* induced by $\mathcal{U}$.

Lemma. *(8.2.5) If the topology of a space X is induced by a uniformity $\mathcal{U}$, then for every compact set $Z \subset X$ and any open set G containing Z there exists a $V \in \mathcal{U}$ such that $B(Z,V) \subset G$.*

Theorem. *(8.2.6) For every Hausdorff space X and any uniform space $(Y,\mathcal{U})$ the topology on Y^X induced by the uniformity $\hat{\mathcal{U}}|\mathcal{Z}(X)$ of uniform convergence on compacta coincides with the compact-open topology no Y^X, where Y has the topology induced by $\mathcal{U}$.*

Corollary. *(8.2.7) For every compact space X and any uniform space $(Y,\mathcal{U})$ the topology on Y^X induced by the uniformity $\hat{\mathcal{U}}$ of the uniform convergence coincides with the compact-open topology on Y^X, and depends only on the topology induced on Y by the uniformity $\mathcal{U}$.*

We say that a family F of mapping of a topological space X to a Tychonoff space Y is *equicontinuous with respect to a uniformity $\mathcal{U}$* on the space Y if for every $x \in X$ and $V \in \mathcal{U}$ there exists a neighborhood G of the point x such that $|f(x) - f(x')| < V$ whenever $f \in F$ and $x' \in G$.

Lemma. *(8.2.8) Let X be a topological space, Y a Tychonoff space and $\mathcal{U}$ a uniformity on the space Y. If a family $F \subset Y^X$ of mappings of X to Y is equicontinuous with respect to $\mathcal{U}$, then the family F is evenly continuous.*

Lemma. *(8.2.9) Let X be a topological space, Y a Tychonoff space and $\mathcal{U}$ a uniformity on the space Y. If a family $F \subset Y^X$ of mappings of X to Y is evenly continuous and for every $x \in X$ the set $\{f(x) : f \in F\}$ has a compact closure, then the family F is equicontinuous with respect to $\mathcal{U}$.*

Theorem (The Ascoli theorem)**.** *(8.2.10) Let X be a k-space, Y a Tychonoff space and $\mathcal{U}$ a uniformity on the space Y. A closed subset F of the space Y^X with the compact-open topology is compact if and only if F is equicontinuous with respect to $\mathcal{U}$ and the set $\{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.*

Theorem. *(8.2.11) Let X be a k-space, Y a Tychonoff space and $\mathcal{U}$ a uniformity on the space Y. A closed subset F of the space Y^X with the compact-open topology is compact if and only if for every compact subspace $Z \subset X$ the family $F|Z$ is equicontinuous with respect to $\mathcal{U}$ and the set $\{f(x) : f \in F\} \subset Y$ has a compact closure for every $x \in X$.*

8.3 Totally bounded and complete uniform spaces. Compactness in uniform spaces

Let $(X,\mathcal{U})$ be a uniform space, V a member of the uniformity $\mathcal{U}$ and A a subset of X; we say that A is *V-dense* in $(X,\mathcal{U})$ if for every $x \in X$ there exists an $x' \in A$ such that $|x - x'| < V$.

A uniform space $(X,\mathcal{U})$ is *totally bounded* if for every $V \in \mathcal{U}$ there exists a finite set $A \subset X$ which is V-dense in $(X,\mathcal{U})$; a uniformity $\mathcal{U}$ on a set X is *totally bounded* if the space $(X,\mathcal{U})$ is totally bounded.

One readily verifies that if there exists a uniformly continuous mapping f of a totally bounded uniform space $(X,\mathcal{U})$ to a uniform space $(Y,\mathcal{V})$ such that $f[X] = Y$, then the space $(X,\mathcal{V})$ also is totally bounded.

Proposition. *(8.3.1) If the uniformity $\mathcal{U}$ on a set X is induced by a metric ϱ, then the uniform space $(X,\mathcal{U})$ is totally bounded if and only if the metric space (X,ϱ) is totally bounded.*

Theorem. *(8.3.2) If $(X,\mathcal{U})$ is a totally bounded uniform space, then for every subset $M \subset X$ the space $(M,\mathcal{U}_M)$ is totally bounded.*

If $(X,\mathcal{U})$ is an arbitrary uniform space and for a subset $M \subset X$ the space $(M,\mathcal{U}_M)$ is totally bounded, then the space $(\overline{M},\mathcal{U}_{\overline{M}})$ also is totally bounded.

Theorem. *(8.3.3) Let $\{(X_s,\mathcal{U}_s)\}_{s\in S}$ be a family of non-empty uniform spaces. The Cartesian product $\left(\prod_{s\in S} X_s, \prod_{s\in S} \mathcal{U}_s\right)$ is totally bounded if and only if all space $(X_s,\mathcal{U}_s)$ are totally bounded.*

Let $(X,\mathcal{U})$ be a uniform space and $\mathcal{F}$ a family of subsets of X; we say that $\mathcal{F}$ *contains arbitrarily small sets* if for every $V \in \mathcal{U}$ there exists an $F \in \mathcal{F}$ such that $\delta(F) < V$.

A uniform space $(X,\mathcal{U})$ is *complete* if every family $\mathcal{F}$ of subsets of X, closed with respect to the topology induced by $\mathcal{U}$, which has the finite intersection property and which contains arbitrarily small sets has non-empty intersection; a uniformity $\mathcal{U}$ on a set X is *complete* if the space $(X,\mathcal{U})$ is complete.

Proposition. *(8.3.5) If the uniformity $\mathcal{U}$ on a set X is induced by a metric ϱ, then the uniform space $(X,\mathcal{U})$ is complete if and only if the metric space (X,ϱ) is complete.*

Theorem. *(8.3.6) If $(X,\mathcal{U})$ is a complete uniform space, then for a subset $M \subset X$ the uniform space $(M,\mathcal{U}_M)$ is complete if and only if M is closed in X with respect to the topology induced by $\mathcal{U}$.*

Lemma. *(8.3.7) For every metrizable uniform space $(X,\mathcal{U})$ there exists a complete metrizable uniform space $(Y,\mathcal{V})$ such $M \subset Y$ the space $(X,\mathcal{U})$ is uniformly isomorphic to the space $(M,\mathcal{V}_M)$.*

Theorem. *(8.3.8) Every complete uniform space is uniformly isomorphic to a closed subspace of the Cartesian product of a family of complete metrizable uniform spaces.*

Theorem. *(8.3.9) Let $\{(X_s,\mathcal{U}_s)\}_{s\in S}$ be a family of non-empty uniform spaces. The Cartesian product $(\prod_{s\in S} X_s, \prod_{s\in S} \mathcal{U}_s)$ is complete if and only if all spaces $(X_s,\mathcal{U}_s)$ are complete.*

Theorem. *(8.3.10) If $(X,\mathcal{U})$ is a uniform space and $(Y,\mathcal{V})$ a complete uniform space, then every uniformly continuous mapping $f\colon (A,\mathcal{U}_A) \to (Y,\mathcal{V})$, where A is a subset of X dense with respect to the topology induced by $\mathcal{U}$, is extendable to a uniformly continuous mapping $F\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$.*

Corollary. *(8.3.11) If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are complete uniform spaces then every uniform isomorphism of $(A,\mathcal{U}_A)$ onto $(B,\mathcal{V}_B)$ where A and B are dense subset of X and Y respectively, is extendable to a uniform isomorphism of $(X,\mathcal{U})$ onto $(Y,\mathcal{V})$.*

Theorem. *(8.3.12) For every uniform space $(X,\mathcal{U})$ there exists exactly one (up to a uniform isomorphism) complete uniform space $(\tilde{X},\tilde{\mathcal{U}})$ such that for a dense subset A of $\tilde{X}$ the space $(X,\mathcal{U})$ is uniformly isomorphic to $(A,\tilde{\mathcal{U}}_A)$. Moreover, we have $w(\tilde{\mathcal{U}}) = w(\mathcal{U})$ and if $(X,\mathcal{U})$ is a totally bounded space, then $(\tilde{X},\tilde{\mathcal{U}})$ also is totally bounded.*

The space $(\tilde{X},\tilde{\mathcal{U}})$ is called the *completion of the uniform space* $(X,\mathcal{U})$.

Theorem. *(8.3.13) For every compact space X there exists exactly one uniformity $\mathcal{U}$ on the set X that induces the original topology of X. All entourages of the diagonal $\Delta \subset X \times X$ which are open in the Cartesian product $X \times X$ form a base for the uniformity $\mathcal{U}$.*

Theorem. *(8.3.14) Every uniformity on a countably compact space is totally bounded.*

Theorem. *(8.3.15) Every uniformity on a compact space is complete.*

A uniform space $(X,\mathcal{U})$ is called *compact* if the set X with the topology induced by $\mathcal{U}$ is a compact space.

Theorem. *(8.3.16) A uniform space $(X,\mathcal{U})$ is compact if and only if it is both totally bounded and complete.*

Corollary. *(8.3.17) The completion of a uniform space $(X,\mathcal{U})$ is compact if and only if $(X,\mathcal{U})$ is a totally bounded space.*

Let $(X,\mathcal{U})$ be a uniform space and $\{x_\sigma, \sigma \in \Sigma\}$ a net in X; we say that $\{x_\sigma, \sigma \in \Sigma\}$ is a *Cauchy net* in $(X,\mathcal{U})$ if for every $V \in \mathcal{U}$ there exists a $\sigma_0 \in \Sigma$ such that $|x_\sigma - x_{\sigma_0}| < V$ whenever $\sigma \geq \sigma_0$. Similarly, a filter $\mathcal{F}$ in the family of all subset of X is a *Cauchy filter* in $(X,\mathcal{U})$ if for every $V \in \mathcal{U}$ there exists an $F \in \mathcal{F}$ such that $\delta(F) < V$. The reader can easily verify that Cauchy nets and Cauchy filters correspond to one another under the one-to-one correspondence between nets and filters established in Section 1.6.

Theorem. *(8.3.20) A uniform space $(X,\mathcal{U})$ is complete if and only if every Cauchy net in $(X,\mathcal{U})$ is convergent to a point of X.*

Theorem. *(8.3.21) A uniform space $(X,\mathcal{U})$ is complete if and only if every Cauchy filter in $(X,\mathcal{U})$ is convergent to a point of X.*

8.4 Proximities and proximity spaces

Let X be a set and δ a relation on the family of all subsets of X. We shall write $A\overline{\delta}B$ instead of not $A\delta B$. δ is called a *proximity* on the set X if δ satisfies the following conditions:

(P1) $A\delta B$ if and only if $B\delta A$.

(P2) $A\delta(B \cup C)$ if and only if either $A\delta B$ or $A\delta C$.

(P3) $\{x\}\delta\{y\}$ if and only if $x = y$.

(P4) $\emptyset\overline{\delta}X$.

(P5) If $A\overline{\delta}B$, then there exists $C, D \subset X$ such that $A\overline{\delta}C$, $B\overline{\delta}D$ and $C \cup D = X$.

A *proximity space* is a pair (X, δ) consisting of a set X and a proximity δ on the set X. Two subsets A and B are *close* with respect to δ if $A\delta B$, otherwise they are *remote* with respect to δ.

Conditions (P1)-(P5) imply the following properties of proximities

$$\text{If } A\delta B \text{ and } B \subset C, \text{ then } A\delta C. \tag{1}$$

$$\text{If } A \cap B \neq \emptyset, \text{ then } A\delta B. \tag{2}$$

$$\emptyset\overline{\delta}A \text{ for every } A \subset X. \tag{3}$$

Every proximity δ on a set X induces a topology $\mathcal{O}$ on X. More exactly the formula

$$\overline{A} = \{x \in X : \{x\}\delta A\} \tag{4}$$

defines a closure operator on the set X.

Lemma. *(8.4.1) For every proximity δ on a set X and any sets $A, B \subset X$*

$$\textit{if } B\overline{\delta}A, \textit{ then } B\overline{\delta}\overline{A}. \tag{5}$$

Theorem. *(8.4.2) For every proximity δ on a set X formula (4) defines a closure operator which satisfies conditions (CO1)-(CO4). The space X with the topology $\mathcal{O}$ generated by that closure operator is a T_1-space.*

The topology $\mathcal{O}$ is called the topology induced by the proximity δ.

Example. (8.4.3) *discrete proximity*: $A\delta B$ if and only if $A \cap B \neq \emptyset$.

Example. (8.4.4) X - a Tychonoff space. $A\delta B$ if and only if the sets A and B are not completely separated.

A mapping f of a set X to a set Y is called *proximally continuous with respect to the proximities δ and δ'* if for any sets $A, B \subset X$ close with respect to δ, the images $f[A], f[B] \subset Y$ are close with respect to δ'.

A one-to-one mapping f of a set X onto a set Y is a *proximal isomorphism with respect to proximities δ and δ'* on the sets X and Y respectively, if f is proximally continuous with respect to δ and δ' and the inverse mapping f^{-1} is proximally continuous with respect to δ' and δ.

Theorem. *(8.4.5) Let $\mathcal{U}$ be a uniformity on a set X. Letting for $A, B \subset X$*

$$A\delta B \textit{ if and only if } V \cap (A \times B) \neq \emptyset \textit{ for every } V \in \mathcal{U}$$

we define a proximity on the set X. The topology induced by δ coincides with the topology induced by $\mathcal{U}$.

Example. (8.4.6) $A\delta B$ if and only if $\varrho(A, B) = 0$ is *proximity induced by the metric ϱ.*

Let δ be a proximity on a set X; we say that a set A is *strongly contained* in a set B with respect to δ, and we write $A \Subset B$, if $A\overline{\delta}(X \setminus B)$. Let us note that using the relation of strong inclusion we can rewrite (P5) in the following form:

$$\text{If } A\overline{\delta}B, \text{ then there exist } A_1, B_1 \subset X \text{ such that } A \Subset A_1,\ B \Subset B_1 \text{ and } A_1 \cap B_1 = \emptyset. \tag{P5'}$$

The relation $\Subset$ has the following properties (in (SI5) and (SI7) the topology induced by δ is being considered):

(SI1) If $A \Subset B$, then $X \setminus B \Subset X \setminus A$.

(SI2) If $A \Subset B$ then $A \subset B$.

(SI3) If $A_1 \subset A \Subset B \subset B_1$, then $A_1 \Subset B_1$.

(SI4) If $A_1 \Subset B_1$ and $A_2 \Subset B_2$, then $A_1 \cup A_2 \Subset B_1 \cup B_2$.

(SI5) If $A \Subset B$, then there exists an open set C such that $A \Subset C \subset \overline{C} \subset\Subset B$.

(SI6) $\emptyset \Subset \emptyset$.

(SI7) For every $x \in X$ and any neighborhood A of x we have $\{x\} \Subset A$.

Let δ be a proximity on a set X. A finite cover $\{A_i\}_{i=1}^k$ of the set X is called *δ-uniform* if there exists a cover $\{B_i\}_{i=1}^k$ of the set X such that

$$B_i \Subset A_i \qquad \text{for } i = 1, 2, \ldots, k. \tag{10}$$

Lemma. *(8.4.7) Let δ be a proximity on a set X. For $A, B \subset X$ we have $A\delta B$ if and only if every δ-uniform cover $\{A_i\}_{i=1}^k$ of the set X contains a set A_j such that $A \cap A_j \neq \emptyset \neq B \cap A_j$.*

Theorem. *(8.4.8) For every proximity δ on a set X the collection* **C** *of all covers of X which have a δ-uniform refinement has properties* (UC1)-(UC4). *The uniformity $\mathcal{U}$ on the set X generated by the collection* **C** *is totally bounded and induces the proximity δ. The topology induced by $\mathcal{U}$ coincides with the topology induced by δ.*

The uniformity $\mathcal{U}$ is called the uniformity induced by the proximity δ.

Theorem. *(8.4.9) The topology of a space X can be induced by a proximity on the set X if and only if X is a Tychonoff space.*

Theorem. *(8.4.10) For every compact space X there exists exactly one proximity δ on the set X that induces the original topology of X, viz., the proximity δ defined by letting*

$$A\delta B \text{ if and only if } \overline{A} \cap \overline{B} \neq \emptyset. \tag{17}$$

Lemma. *(8.4.11) Let X be a Tychonoff space and let cX be a compactification of X. Letting for $A, B \subset X$*

$$A\delta(c)B \text{if and only if } \overline{c(A)} \cap \overline{c(B)} \neq \emptyset,$$

where the closures are taken in cX, we define a proximity $\delta(c)$ on the space X.

For compactifications c_1X and c_2X of the space X we have $\delta(c_1) = \delta(c_2)$ if and only if the compactifications c_1X and c_2X are equivalent.

Lemma. *(8.4.12) For every proximity δ on a Tychonoff space X there exists a compactification cX of the space X such that $\delta = \delta(c)$.*

Theorem (The Smirnov theorem)**.** *(8.4.13) By assigning to any compactification cX of a Tychonoff space X the proximity $\delta(c)$ on the space X we establish a one-to-one correspondence between all compactifications of X and all proximities on the space X.*

Example. (8.4.14) The proximity defined in Example 8.4.4 on a Tychonoff space X corresponds to Čech-Stone compactification of the space X.

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