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1 Categories, Functors and natural transformations

1.3 Categories and functors

Definition. (3.1) A *category* is a quadruple $\mathbf{A} = (\mathcal{O}, \hom, id, \circ)$ consisting of

1. a class $\mathcal{O}$, whose members are called $\mathbf{A}$-*objects,*
2. for each pair (A, B) of $\mathbf{A}$-objects a set $\hom(A, B)$, whose member are called $\mathbf{A}$-*morphisms from A to B*
3. for each $\mathbf{A}$-object A, a morphism $A \xrightarrow{id_A} A$, called the $\mathbf{A}$-*identity* on A,
4. a *composition law* associating with each $\mathbf{A}$-morphism $A \xrightarrow{f} B$ and each $\mathbf{A}$-morphism $B \xrightarrow{g} C$ an $\mathbf{A}$-morphism $A \xrightarrow{g \circ f} C$, called the *composite of f and g,*

subject to the following conditions

(a) composition is associative, i.e., $h \circ (g \circ f) = (h \circ g) \circ f$

(b) $\mathbf{A}$-identities act as identities with respect to composition; i.e., for $\mathbf{A}$-morphisms $A \xrightarrow{f} B$ we have $id_B \circ f = f$ and $f \circ id_A = f$.

(c) the sets $\hom(A, B)$ are pairwise disjoint.

Example. (3.3) **Set**, **Vec**, **Grp**, **Top**

Rel with objects all pairs (X, ρ), where X is a set and ρ is a relation on X. Morphisms $f\colon (X, \rho) \to (Y, \sigma)$ are relation-preserving maps, i.e., maps $f\colon X \to Y$ such that $x \rho x'$ implies $f(x) \sigma f(x')$.

$\mathbf{Alg}(\Omega)$ with objects all Ω-algebras and morphisms all Ω-morphisms between them. $\Omega = (n_i)_{i \in I}$, $n_i \in \mathbb{N}$, I is a set.

Met - all non-expansive maps (=contractions),
$\mathbf{Met}_u$ - all uniformly continuous maps,
$\mathbf{Met}_c$ - all continuous maps

Ban - linear contractions,
$\mathbf{Ban}_b$ - bounded linear maps (= continuous linear maps = uniformly continuous linear maps)

Mat with objects all natural numbers, and for which $\hom(m, n)$ is the set of all real $m \times n$ matrices, composition is defined by $A \circ B = BA$.

Every class X give rise to a category $C(X)$, with only identities as morphisms. = *Discrete category*

Preordered class as category. = *thin category* (For each A, B $\hom(A, B)$ has at most one member.)

Monoid as category.

$\mathbf{Set} \times \mathbf{Set}$ - pairs

The duality principle, isomorphisms, functors and examples

1.3.1 Properties of functors

Proposition. *(3.21) All functors $F\colon \mathbf{A} \to \mathbf{B}$ preserve isomorphisms. (f is an isomorphism $\Rightarrow Ff$ is isomorphism.)*

Functors need not reflect isomorphisms.
Composition of functors.

Definition. (3.24) A functor $F\colon \mathbf{A} \to \mathbf{B}$ is called an *isomorphism* provided that there is a functor $G\colon \mathbf{B} \to \mathbf{A}$ such that $G \circ F = id_{\mathbf{A}}$ and $F \circ G = id_{\mathbf{B}}$.

Definition. (3.27) Let $f\colon \mathbf{A} \to \mathbf{B}$ be a functor.
F is called an *embedding* provided that F is injective on morphisms.
F is called an *faithful* provided that all the hom-set restrictions $F\colon \hom_A(A, A') \to \hom_B(FA, FA')$ are injective.
F is called *full* provided that all hom-set restrictions are surjective.

Notice that a functor is:

1. an embedding if and only if it is faithful and injective on objects,

2. an isomorphism if and only if it is full, faithful and bijective on objects.

Proposition. *(3.30) Let $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$ be functors.*

(1) If F and G are both isomorphism (resp. embeddings, faithful, or full), then so is $G \circ F$.

(2) If $G \circ F$ is an embedding (resp. faithful), then so is F.

(3) If $G \circ F$ is full, then so is G.

Proposition. *(3.31) If $F\colon \mathbf{A} \to \mathbf{B}$ is a full, faithful functor, then for every $\mathbf{B}$-morphism $f\colon FA \to FA'$ there exists a unique $\mathbf{A}$-morphism $g\colon A \to A'$ with $Fg = f$.*

Corollary. *(3.32) Functors $F\colon \mathbf{A} \to \mathbf{B}$ that are full and faithful reflect isomorphisms; i.e., whenever g is an $\mathbf{A}$-morphism such that Fg is a $\mathbf{B}$-isomorphism, then g is an $\mathbf{A}$-isomorphism.*

Definition. (3.33) A functor $F\colon \mathbf{A} \to \mathbf{B}$ is called *equivalence* provided that it is full, faithful and *isomorphism-dense* in the sense that for any $\mathbf{B}$-object B there exists $A \in \mathbf{A}$ such that $F(A)$ is isomorphic to B.
Categories $\mathbf{A}$ and $\mathbf{B}$ are called *equivalent* provided that there is an equivalence from $\mathbf{A}$ to $\mathbf{B}$.

1.3.2 Categories of categories

Definition. (3.44) A category $\mathbf{A}$ is said to be *small* provided that its class of objects, $\mathrm{Ob}(\mathbf{A})$, is a set. Otherwise it is called *large.*

Definition. (3.47) The category **Cat** of small categories has as objects all small categories, as morphisms from $\mathbf{A}$ to $\mathbf{B}$ all functors from $\mathbf{A}$ to $\mathbf{B}$, as identities the identity functors, and as composition the usual composition of functors.

Definition. (3.49) A *quasicategory* is a quadruple $\mathbf{A} = (\mathcal{O}, \hom, id, \circ)$ defined in the same way except that the restrictions that $\mathcal{O}$ be a class and that each conglomerate $\hom(A, B)$ be a set are removed.

Definition. (3.50) The quasicategory **CAT** of all categories.

1.3.3 Object-free definition of categories

skipped

1.4 Subcategories

Proposition. *(4.5)*

(1) A functor $F\colon \mathbf{A} \to \mathbf{B}$ is a (full) embedding if and only if there exists a (full) subcategory $\mathbf{C}$ of $\mathbf{B}$ with inclusion functor $E\colon \mathbf{C} \to \mathbf{B}$ and an isomorphism $G\colon \mathbf{A} \to \mathbf{C}$ with $F = E \circ G$.

(2) A functor $F\colon \mathbf{A} \to \mathbf{B}$ is faithful if and only if there exist embeddings $E_1\colon \mathbf{D} \to \mathbf{B}$ and $E_2\colon \mathbf{A} \to \mathbf{C}$ and an equivalence $G\colon \mathbf{C} \to \mathbf{D}$ such that the diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "E_2"'] & \mathbf{B} \\ \mathbf{C} \arrow[r, "G"'] & \mathbf{D} \arrow[u, "E_1"'] \end{tikzcd}$$

commutes.

Definition. (4.6) A category $\mathbf{A}$ is said to be *fully embeddable* into $\mathbf{B}$ provided that there exists a full embedding $\mathbf{A} \to \mathbf{B}$, or, equivalently, provided that $\mathbf{A}$ is isomorphic to a full subcategory of $\mathbf{B}$.

Example. (4.7) Although it is far from easy to prove (see Pultr and Trnková), each category of the form $\mathbf{Alg}(\Omega)$ is fully embeddable into each of the following constructs: **Sgr**, **Rel**, $\mathbf{Alg}(1,1)$ (i.e., the construct of unary algebras on two operations). Under an additional set-theoretical hypothesis (the non-existence of measurable cardinals), *every* construct is fully embeddable into **Sgr** (or **Rel** or $\mathbf{Alg}(1,1)$).

Definition. (4.9) A full subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is called

(i) *isomorphism-closed* provided that every $\mathbf{B}$-object that is isomorphic to some $\mathbf{A}$-object is itself an $\mathbf{A}$-object,

(ii) *isomorphism-dense* provided that every $\mathbf{B}$-object is isomorphic to some $\mathbf{A}$-object.

If $\mathbf{A}$ is a full subcategory of $\mathbf{B}$, then the following conditions are equivalent:

(1) $\mathbf{A}$ is an isomorphism-dense subcategory of $\mathbf{B}$,

(2) the inclusion functor $\mathbf{A} \hookrightarrow \mathbf{B}$ is isomorphism-dense,

(3) the inclusion functor $\mathbf{A} \hookrightarrow \mathbf{B}$ is an equivalence.

Definition. (4.12) A *skeleton* of a category is a full, isomorphism-dense subcategory in which no two distinct objects are isomorphic.

Proposition. *(4.14)*

(1) Every category has a skeleton.

(2) Any two skeletons of a category are isomorphic.

(3) Any skeleton of a category $\mathbf{C}$ is equivalent to $\mathbf{C}$.

Corollary. *(4.15) Two categories are equivalent if and only if they have isomorphic skeletons.*

1.4.1 Reflective and coreflective subcategories

Definition. (4.16) Let $\mathbf{A}$ be a subcategory of $\mathbf{B}$, and let B be a $\mathbf{B}$-object.

(1) An $\mathbf{A}$-*reflection* (or $\mathbf{A}$-*reflection arrow*) for $\mathbf{B}$ is a morphism $B \xrightarrow{r} A$ from B to an $\mathbf{A}$-object A with the following universal property: for any morphism $B \xrightarrow{f} A'$ from B into some $\mathbf{A}$-object A', there exists a unique $\mathbf{A}$-morphism $f' \colon A \to A'$ such that the triangle

$$\begin{tikzcd} B \arrow[r, "r"] \arrow[dr, "f"'] & A \arrow[d, "f'"] \\ & A' \end{tikzcd}$$

commutes.

(2) $\mathbf{A}$ is called a *reflective subcategory* of $\mathbf{B}$ provided that each $\mathbf{B}$-object has an $\mathbf{A}$-reflection.

Proposition. *(4.20) If* $\mathbf{A}$ *is a reflective subcategory of* $\mathbf{B}$*, then the following conditions are equivalent:*

(1) $\mathbf{A}$ *is a full subcategory of* $\mathbf{B}$.

(2) For each $\mathbf{A}$*-object* A*,* $A \xrightarrow{id_A} A$ *is an* $\mathbf{A}$*-reflection.*

(3) For each $\mathbf{A}$*-object* A*,* $\mathbf{A}$*-reflection arrows* $A \xrightarrow{r_A} A^*$ *are* $\mathbf{A}$*-isomorphisms.*

(4) For each $\mathbf{A}$*-object* A*,* $\mathbf{A}$*-reflection arrows* $A \xrightarrow{r_A} A^*$ *are* $\mathbf{A}$*-morphisms.*

Existence of *reflector.*
Dual notions: *coreflective subcategory*, $\mathbf{A}$-*coreflection*, *coreflector.*

1.5 Concrete categories and concrete functors

Definition. (5.1) Let $\mathbf{X}$ be a category. A *concrete category* over $\mathbf{X}$ is a pair $(\mathbf{A}, U)$, where $\mathbf{A}$ is a category and $U \colon \mathbf{A} \to \mathbf{X}$ is a faithful functor. Sometimes U is called the *forgetful* (or *underlying*) functor of the concrete category $\mathbf{X}$ and $\mathbf{X}$ is called the *base category*.

A concrete category over $\mathbf{Set}$ is called a *construct.*

1.5.1 Fibres in concrete categories

Definition. (5.4) Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$.

The *fibre* of an $\mathbf{X}$-object X is the preordered class consisting of all $\mathbf{A}$-objects A with $U(A) = X$ ordered by: $A \leq B$ if and only if $id_X : UA \to UB$ is an $\mathbf{A}$-morphism.

$\mathbf{A}$-objects A and B are said to be *equivalent* provided that $A \leq B$ and $B \leq A$.

$(\mathbf{A}, U)$ is said to be *amnestic* provided that its fibres are partially ordered classes, i.e., no two different $\mathbf{A}$-objects are equivalent.

$(\mathbf{A}, U)$ is said to be *fibre-small* provided that each of its fibres is small, i.e., a preordered set.

Definition. (5.7) A concrete category is called *fibre-complete* provided that its fibres are (possibly large) complete lattices. (A partially ordered class $(X, \leq)$ is called a large complete lattice provided that every subclass of X has join and meet.)

A concrete category is called *fibre-discrete* provided that its fibres are ordered by equality.

Proposition. *(5.8) A concrete category* $(\mathbf{A}, U)$ *over* $\mathbf{X}$ *is fibre-discrete if and only if* U reflects identities *(i.e., if* $U(k)$ *is an* $\mathbf{X}$*-identity, then* k *must be an* $\mathbf{A}$*-identity.)*

1.5.2 Concrete functors

Definition. (5.9) If $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ are concrete categories over $\mathbf{X}$, then a *concrete functor from* $(\mathbf{A}, U)$ *to* $(\mathbf{B}, V)$ is a functor $F\colon \mathbf{A} \to \mathbf{B}$ with $U = V \circ F$. We denote such a functor $F\colon (\mathbf{A}, U) \to (\mathbf{B}, V)$.

Proposition. *(5.10) Every concrete functor is faithful.*
Every concrete functor is completely determined by its values on objects.
Objects that are identified by a full concrete functor are equivalent.
Every full concrete functor with amnestic domain is an embedding.

concrete isomorphism=concrete functor + isomorphism

Proposition. *(5.14) The identity functor on a concrete category is a concrete isomorphism. Any composite of concrete functors over* $\mathbf{X}$ *is a concrete functor over* $\mathbf{X}$*.*

Definition. (5.15) The quasicategory that has as objects all concrete categories over $\mathbf{X}$ and as morphisms all concrete functors between them is denoted by $\mathbf{CAT}(\mathbf{X})$. In particular, $\mathbf{CONST} = \mathbf{Cat}(\mathbf{Set})$ is the quasicategory of all constructs.

Definition. (5.18) If F and G are both concrete functors from $(\mathbf{A}, U)$ to $(\mathbf{B}, V)$, then F is *finer then* G (or G is *coarser* than F), denoted by $F \leq G$, provided that $F(A) \leq G(A)$ for each $\mathbf{A}$-object A.

1.5.3 Concrete subcategories

If $(\mathbf{B}, U)$ is a concrete category over $\mathbf{X}$ and $\mathbf{A}$ is a subcategory of $\mathbf{B}$ with inclusion $E\colon \mathbf{A} \hookrightarrow \mathbf{B}$, then $\mathbf{A}$ will often be regarded (via the functor $U \circ E$) as a concrete category over $\mathbf{X}$ - *concrete subcategory.* If $\mathbf{X} = \mathbf{Set}$ - *subconstruct.*

Definition. (5.22) A concrete subcategory $(\mathbf{A}, U)$ of $(\mathbf{B}, V)$ is called *concretely reflective* in $(\mathbf{B}, V)$ (or a *reflective modification* of $(\mathbf{B}, V)$) provided that for each $\mathbf{B}$-object there exists an identity-carried $\mathbf{A}$-reflection arrow.

Reflectors induced by identity-carried reflection arrows are called *concrete reflectors.*

Proposition. *(5.24) Every concretely reflective subcategory of an amnestic concrete category is a full subcategory.*

Proposition. *(5.26) For a concrete full subcategory* $(\mathbf{A}, U)$ *of a concrete category* $(\mathbf{B}, V)$ *over* $\mathbf{X}$*, with inclusion functor* $E\colon (\mathbf{A}, U) \hookrightarrow (\mathbf{B}, V)$*, the following are equivalent:*

(1) $(\mathbf{A}, U)$ *is concretely reflective in* $(\mathbf{B}, V)$*,*

(2) there exists a concrete functor $R\colon (\mathbf{B}, V) \to (\mathbf{A}, U)$ *that is a reflector with* $R \circ E = id_{\mathbf{A}}$ *and* $id_{\mathbf{B}} \leq E \circ R$*,*[1]

(3) there exists a concrete functor $R\colon (\mathbf{B}, V) \to (\mathbf{A}, U)$ *with* $R \circ E \leq id_{\mathbf{A}}$ *and* $id_{\mathbf{B}} \leq E \circ R$*.*

[1] Observe that $R \circ E = id_{\mathbf{A}}$ just means that $RA = A$ for each $\mathbf{A}$-object A and that $id_{\mathbf{B}} \leq E \circ R$ just means that $B \leq RB$ for each $\mathbf{B}$-object B.

1.5.4 Transportability

Definition. (5.28) A concrete category $(\mathbf{A}, U)$ is said to be *(uniquely) transportable* provided that for every $\mathbf{A}$-object A and every $\mathbf{X}$-isomorphism $UA \xrightarrow{k} X$ there exists a (unique) $\mathbf{A}$-object B with $UB = X$ such that $A \xrightarrow{k} B$ is an $\mathbf{A}$-isomorphism.

Proposition. *(5.29) A concrete category is uniquely transportable if and only if it is transportable and amnestic.*

Proposition. *(5.31) If $(\mathbf{A}, U)$ is an isomorphism-closed full concrete subcategory of a transportable concrete category $(\mathbf{B}, V)$ over $\mathbf{X}$, then the following are equivalent:*

(1) $(\mathbf{A}, U)$ is concretely reflective in $(\mathbf{B}, V)$,

(2) there exists a reflector $R\colon \mathbf{B} \to \mathbf{A}$ that is concrete from $(\mathbf{B}, V)$ to $(\mathbf{A}, U)$.

Proposition. *(5.33) For every concrete category $(\mathbf{A}, U)$ over $\mathbf{X}$, there exists an amnestic concrete category $(\mathbf{B}, V)$ over $\mathbf{X}$ that is uniquely determined up to a concrete isomorphism by each of the following properties:*

(1) there exists an (injective) concrete equivalence $E\colon (\mathbf{B}, V) \to (\mathbf{A}, U)$,

(2) there exists a surjective concrete equivalence $P\colon (\mathbf{A}, U) \to (\mathbf{B}, V)$.

Moreover, if $(\mathbf{A}, U)$ is transportable, then so is $(\mathbf{B}, V)$.

Proposition. *(5.36) For every concrete category $(\mathbf{A}, U)$ over $\mathbf{X}$ there exists a uniquely transportable concrete category $(\mathbf{B}, V)$ over $\mathbf{X}$ and a concrete equivalence: $E\colon (\mathbf{A}, U) \to (\mathbf{B}, V)$ that is uniquely determined up to concrete isomorphism.*

1.5.5 Functors inducing concrete categories

Definition. (5.37) Let $T\colon \mathbf{X} \to \mathbf{X}$ be a functor. $\mathbf{Alg}(T)$ is the concrete category over $\mathbf{X}$, the objects of which (called T-*algebras*) are pairs (X, h) with X an $\mathbf{X}$-object and $h\colon T(X) \to X$ an $\mathbf{X}$-morphism. Morphisms $f\colon (X, h) \to (X', h')$ (called T-*homomorphisms*) are $\mathbf{X}$-morphisms such that the diagram

$$\begin{tikzcd} T(X) \arrow[r, "h"] \arrow[d, "T(f)"'] & X \arrow[d, "f"] \\ T(X') \arrow[r, "h'"'] & X' \end{tikzcd}$$

commutes. The underlying functor to $\mathbf{X}$ is given by: $|(X, h) \xrightarrow{f} (X'h')| = X \xrightarrow{f} X'$.

Proposition. *(5.39) Each concrete category of the form $\mathbf{Alg}(T)$ is fibre-discrete.*

Definition. (5.40) Let $T\colon \mathbf{X} \to \mathbf{Set}$ be a functor. $\mathbf{Spa}(T)$ is the concrete category over $\mathbf{X}$, the objects of which (called T-*spaces*) are pairs (X, α) with $\alpha \subseteq T(X)$. Morphisms $(X, \alpha) \xrightarrow{f} (Y, \beta)$ (called T-*maps*) are $\mathbf{X}$-morphisms $f\colon X \to Y$ such that $T(f)[\alpha] \subseteq \beta$. The underlying functor to $\mathbf{X}$ is given by: $|(X, \alpha) \xrightarrow{f} (Y, \beta)| = X \xrightarrow{f} Y$. Concrete categories of the form $\mathbf{Spa}(T)$ are called *functor-structured categories*.

Proposition. *(5.42) Each concrete category of the form $\mathbf{Spa}(T)$ is fibre-complete.*

1.6 Natural transformations

Definition. (6.1) Let $F, G\colon \mathbf{A} \to \mathbf{B}$ be functors. A *natural transformation τ from F to G* (denoted by $\tau\colon F \to G$ or $F \xrightarrow{\tau} G$) is a function that assigns to each $\mathbf{A}$-object A a $\mathbf{B}$-morphism $\tau_A\colon FA \to GA$ in such a way that the following *naturality condition* hold: for each $\mathbf{A}$-morphism $A \xrightarrow{f} A'$, the square

$$\begin{tikzcd} FA \arrow[r, "\tau_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FA' \arrow[r, "\tau_{A'}"'] & GA' \end{tikzcd}$$

commutes.

Definition. (6.3) If $G, G'\colon \mathbf{A} \to \mathbf{B}$ are functors and $G \xrightarrow{\tau} G'$ is a natural transformation, then
the natural transformation $G'^{op} \xrightarrow{\tau^{op}} G^{op}$ is defined by $\tau^{op}_A = \tau_A$,
for each functor $F\colon \mathbf{C} \to \mathbf{A}$, the natural transformation $\tau F\colon G \circ F \to G' \circ F$ is defined by $(\tau F)_C = \tau_{FC}$,
for each functor $H\colon \mathbf{B} \to \mathbf{D}$, the natural transformation $H\tau\colon H \circ G \to H \circ G'$ is defined by $(H\tau)_A = H(\tau_A)$.

1.6.1 Natural isomorphisms

Definition. (6.5) Let $F, G\colon \mathbf{A} \to \mathbf{B}$ be functors, let $F \xrightarrow{\tau} G$ be natural transformation, and let M be a class of $\mathbf{B}$-morphisms.
τ is called an *M-transformation* provided every τ_A belongs to M.
Iso-transformations are called *natural isomorphisms.*
F and G are called *naturally isomorphic* (denoted by $F \cong G$) provided that there exists a natural isomorphism from F to G.

Proposition. *(6.7) If $\mathbf{A}$ is a reflective subcategory of $\mathbf{B}$, then any two reflectors for $\mathbf{A}$ are naturally isomorphic.*

Proposition. *(6.8) A functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ is an equivalence if and only if there exists a functor $\mathbf{B} \xrightarrow{G} \mathbf{A}$ such that $id_{\mathbf{A}} \cong G \circ F$ and $F \circ G \cong id_{\mathbf{B}}$.*

Definition. (6.9) A functor $F\colon \mathbf{A} \to \mathbf{Set}$ is called *representable* (by an $\mathbf{A}$-object A) provided that F is naturally isomorphic to the hom-functor $\hom(A, -)\colon \mathbf{A} \to \mathbf{Set}$.

Proposition. *(6.10) Objects that represent the same functor are isomorphic.*

1.6.2 Functor categories

Definition. (6.13) *composition of natural transformations* $(\tau \circ \sigma)_A = \tau_A \circ \sigma_A$

quasicategory $[\mathbf{A}, \mathbf{B}]$ = functors from $\mathbf{A}$ to $\mathbf{B}$ and all natural transformations

Proposition. *(6.18) For any functor $F\colon \mathbf{A} \to \mathbf{Set}$, any $\mathbf{A}$-objects A and any element $a \in F(A)$, there exists a unique natural transformation $\tau\colon \hom(A, -) \to F$ with $\tau_A(id_A) = a$.*

Corollary. *(6.19) If $F\colon \mathbf{A} \to \mathbf{Set}$ is a functor, and A is $\mathbf{A}$-object, then there exists a bijective function*

$$Y\colon [\hom(A,-),F] \to F(A) \text{ defined by } Y(\sigma) = \sigma_A(id_A),$$

where $\hom(A,-),F]$ *is the set of all natural transformations from* $\hom(A,-)$ *to* F.

Theorem. *(6.20) For any category $\mathbf{A}$, the functor $E\colon \mathbf{A} \to [\mathbf{A}^{op}, \mathbf{Set}]$, defined by*

$$E(A \xrightarrow{f} B) = \hom(-,A) \xrightarrow{\sigma_f} \hom(-,B),$$

where $\sigma_f(g) = f \circ g$, is full and faithful; i.e., is equivalent to a full embedding.

1.6.3 Concrete natural transformations and Galois correspondences

Definition. (6.23) If $\mathbf{A}$ and $\mathbf{B}$ are concrete categories over $\mathbf{X}$ and $F, G\colon \mathbf{A} \to \mathbf{B}$ are concrete functors, then a natural transformations $\tau\colon F \to G$ is called *concrete* (or *identity carried*) provided that $|\tau_A| = id_{|A|}$ for each $\mathbf{A}$-object A.

Proposition. *(6.24) If $F, G\colon \mathbf{A} \to \mathbf{B}$ are concrete functors, then the following are equivalent:*

(1) $F \leq G$,

(2) there exists a (necessarily unique) concrete natural transformation $\tau\colon F \to G$.

Definition. (6.25) Let $\mathbf{A}$ and $\mathbf{B}$ be concrete categories over $\mathbf{X}$.

(1) If $G\colon \mathbf{A} \to \mathbf{B}$ and $F\colon \mathbf{B} \to \mathbf{A}$ are concrete functors over $\mathbf{X}$, then the pair (F, G) is called a *Galois correspondence* (between $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{X}$) provided that $F \circ G \leq id_{\mathbf{A}}$ and $id_{\mathbf{B}} \leq G \circ F$.

(2) A concrete functor $G\colon \mathbf{A} \to \mathbf{B}$ over $\mathbf{X}$ is called a *residual functor* (or *Galois adjoint*) provided that there is a concrete functor $F\colon \mathbf{B} \to \mathbf{A}$ such that (F, G) is a Galois correspondence over $\mathbf{X}$. Dual notion: *residuated functor*

(1) *Galois isomorphism:* If $K\colon \mathbf{A} \to \mathbf{B}$ is a concrete isomorphism, then (K^{-1}, K) is a Galois correspondence, called a *Galois isomorphism.*

(2) *Galois reflections and coreflections:*

  (a) If $E\colon \mathbf{A} \to \mathbf{B}$ is a concrete embedding and $R\colon \mathbf{B} \to \mathbf{A}$ is concrete reflector, then (R, E) is a Galois correspondence, called a *Galois reflection.*

  (b) If $E\colon \mathbf{A} \to \mathbf{B}$ is a concrete embedding and $C\colon \mathbf{B} \to \mathbf{A}$ is concrete coreflector, then (E, C) is a Galois correspondence, called a *Galois coreflection.*

(3) Galois correspondences for constructs

(4) Galois connections

Proposition. *(6.27)*

(1) If (F, G) is a Galois correspondence between $\mathbf{A}$ and $\mathbf{B}$ and $(\hat{F}, \hat{G})$ is a Galois correspondence between $\mathbf{B}$ and $\mathbf{C}$, then $(F \circ \hat{F}, G \circ \hat{G})$ is a Galois correspondence between $\mathbf{A}$ and $\mathbf{C}$.

(2) If (F, G) is a Galois correspondence between $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{X}$, then (G^{op}, F^{op}) is a Galois correspondence between $\mathbf{B}^{op}$ and $\mathbf{A}^{op}$ over $\mathbf{X}^{op}$.

Proposition. *(6.28) Let* $G\colon \mathbf{A} \to \mathbf{B}$ *and* $F\colon \mathbf{B} \to \mathbf{A}$ *be concrete functors over* $\mathbf{X}$*. Then the following are equivalent:*

(1) (F, G) *is a Galois correspondence,*

(2) an $\mathbf{X}$*-morphism* $|F(B)| \xrightarrow{f} |A|$ *is an* $\mathbf{A}$*-morphism if and only if* $|B| \xrightarrow{f} |G(A)|$ *is a* $\mathbf{B}$*-morphism.*

Proposition. *(6.29) The functors in a Galois correspondence between amnestic concrete categories determine each other uniquely; in particular, if* (F, G) *and* (F', G) *are such Galois correspondences, then* $F = F'$.

Proposition. *(6.30) If* (F, G) *is a Galois correspondence between amnestic concrete categories, then* $G \circ F \circ G = G$ *and* $F \circ G \circ F = F$.

Corollary. *(6.31) If* (F, G) *is a Galois correspondence between amnestic concrete categories, then* $(G \circ F) \circ (G \circ F) = G \circ F$ *and* $(F \circ G) \circ (F \circ G) = F \circ G$.

Corollary. *(6.32) Let* $G\colon \mathbf{A} \to \mathbf{B}$ *and* $F\colon \mathbf{B} \to \mathbf{A}$ *be concrete functors between amnestic concrete categories such that* (F, G) *is a Galois correspondence, and let* $\mathbf{A}^*$ *be the full subcategory of* $\mathbf{A}$ *with objects* $\{F(B) | B \in \mathrm{Ob}(\mathbf{B})\}$ *and* $\mathbf{B}^*$ *be the full subcategory of* $\mathbf{B}$ *with objects* $\{G(A) | A \in \mathrm{Ob}(\mathbf{A})\}$*. Then*

(1) $\mathbf{A}^*$ *is coreflective in* $\mathbf{A}$*, and* $A \in \mathrm{Ob}(\mathbf{A}^*)$ *if and only if* $A = (F \circ G)(A)$.

(2) $\mathbf{B}^*$ *is reflective in* $\mathbf{B}$*, and* $B \in \mathrm{Ob}(\mathbf{B}^*)$ *if and only if* $B = (G \circ F)(B)$.

(3) The restrictions of G *and* F *to* $\mathbf{A}^*$ *and* $\mathbf{B}^*$ *are concrete isomorphisms* $G^*\colon \mathbf{A}^* \to \mathbf{B}^*$ *and* $F^*\colon \mathbf{B}^* \to \mathbf{A}^*$*, that are inverse to each other.*

Proposition. *(6.34) Let* $G\colon \mathbf{A} \to \mathbf{B}$ *and* $F\colon \mathbf{B} \to \mathbf{A}$ *be concrete functors between amnestic concrete categories such that* (F, G) *is a Galois correspondence. Then the following are equivalent:*

(1) G *is a full embedding,*

(2) G *is full,*

(3) G *is injective on objects,*

(4) F *is surjective on objects,*

(5) $F \circ G = id_{\mathbf{A}}$,

(6) up to Galois isomorphism, (F, G) *is a Galois reflection; i.e. there exists a Galois reflection* (R, E) *and a Galois isomorphism* (K^{-1}, K) *with* $(F, G) = (R, E) \circ (K^{-1}, K)$.

Theorem (Decomposition theorem for Galois correspondences). *(6.35) Every Galois correspondence* (F, G) *between amnestic concrete categories is a composite* $(F, G) = (R, E_{\mathbf{B}}) \circ (K^{-1}, K) \circ (E_{\mathbf{A}}, C)$ *of*

(1) a Galois coreflection, $(E_{\mathbf{A}}, C)$,

(2) a Galois isomorphism, (K^{-1}, K) *and*

(3) a Galois reflection, $(R, E_{\mathbf{B}})$.

6A. Composition of Natural Transformations Let $F, F' \colon \mathbf{A} \to \mathbf{B}$ and $G, G' \colon \mathbf{B} \to \mathbf{C}$ be functors and let $F \xrightarrow{\tau} F'$ and $G \xrightarrow{\sigma} G'$ be natural transformations. Show that:

(a) $\sigma F' \circ G\tau = G'\tau \circ \sigma F$. [This natural transformation is called the *star product* of τ and σ and is denoted by $G \circ F \xrightarrow{\sigma * \tau} G' \circ F'$.]

(b) $\sigma F = \sigma * id_F$ and $G\tau = id_G * \tau$.

(c) $id_G \circ id_F = id_{G \circ F}$.

(d) If $H, H' \colon \mathbf{C} \to \mathbf{D}$ are functors and $H \xrightarrow{\delta} H'$ is a natural transformation, then $\delta * (\sigma * \tau) = (\delta * \sigma) * \tau$.

(e) If $F'' \colon \mathbf{A} \to \mathbf{B}$ and $G'' \colon \mathbf{B} \to \mathbf{C}$ are functors and $F' \xrightarrow{\tau'} F''$ and $G' \xrightarrow{\sigma'} G''$ are natural transformations, then

$$(\sigma' \circ \sigma) * (\tau' \circ \tau) = (\sigma' * \tau') \circ (\sigma * \tau).$$

(f) If $H \colon \mathbf{C} \to \mathbf{D}$ is a functor, then $(H\sigma)F = H(\sigma F)$.

(g) If $H \colon \mathbf{C} \to \mathbf{D}$ is a functor, then $(H \circ G)\tau = H(G\tau)$.

(h) If $K \colon \mathbf{D} \to \mathbf{A}$ is a functor, then $\sigma(F \circ K) = (\sigma F)K$.

(i) If $G'' \colon \mathbf{B} \to \mathbf{C}$ and $H \colon \mathbf{C} \to \mathbf{D}$ are functors and $G' \xrightarrow{\sigma'} G''$ is a natural transformation, then $H(\sigma' \circ \sigma)F = (H\sigma' F) \circ (H\sigma F)$.

2 Objects and morphisms

2.7 Objects and morphisms in abstract categories

Definition. (7.1) An object A is said to be an *initial object* provided that for each object B there is exactly one morphism from A to B.

Proposition. *(7.3) Initial objects are essentially unique, i.e.*

(1) if A and B are initial objects, then A and B are isomorphic,

(2) if A is an initial object, then so is every object that is isomorphic to A.

Definition. (7.4) An object A is called a *terminal object* provided that for each object B there is exactly one morphism from B to A.

Terminal objects are essentially unique.

Definition. (7.7) An object A is called *zero object* provided that it is both an initial object and terminal object.

Definition. (7.10) An object S is called a *separator* provided that whenever $A \overset{f}{\underset{g}{\rightrightarrows}} B$ are distinct morphisms, there exists a morphism $S \xrightarrow{h} A$ such that

$$S \xrightarrow{h} A \xrightarrow{f} B \neq S \xrightarrow{h} A \xrightarrow{g} B.$$

In **Set** the separators are precisely the nonempty sets. In **Top** (resp. **Pos**) the separators are precisely the nonempty spaces (resp. nonempty posets). In **Vec** the separators are precisely the nonzero vector spaces.

The group of integers $\mathbb{Z}$ under addition is separator for **Grp** and for **Ab**. The monoid of natural numbers $\mathbb{N}$ is separator for **Mon**.

(X, ϱ) is a separator in **Rel** if and only if $X \neq \emptyset = \varrho$.

Proposition. *(7.12) An object S of a category $\mathbf{A}$ is a separator if and only if* $\hom(S, -)\colon \mathbf{A} \to$ **Set** *is a faithful functor.*

Proposition. *(7.22) Every functor preserves sections.*

Proposition. *(7.28) Every functor preserves retractions.*

Proposition. *(7.29) Every full, faithful functor reflects retractions.*

Proposition. *(7.37)*

(1) Every representable functor preserves monomorphisms.

(2) Every faithful functor reflect monomorphisms.

Corollary. *(7.38) In any construct all morphisms with injective underlying functions are monomorphisms. When the underlying functor is representable, the monomorphisms are precisely the morphisms with injective underlying functions.*

In **Haus** the epimorphisms are precisely the continuous functions with dense images. Also in $\mathbf{Ban_b}$, and **Ban** (with either of the two natural forgetful functors) the epimorphisms are precisely the morphisms with dense images. For Hausdorff topological groups it is not yet known whether or not epimorphisms must have dense images.

Proposition. *(7.44) Every faithful functor reflect epimorphisms.*

Corollary. *(7.45) In any construct all morphisms with surjective underlying functions are epimorphisms.*

Although faithful functors reflect epimorphisms and monomorphisms, they need not preserve them (as the above examples show). In fact, even full embeddings may fail to do so. For example the full embedding $E\colon \mathbf{Haus} \hookrightarrow \mathbf{Top}$ doesn't preserve epimorphisms and so the full embedding E^{op} doesn't preserve monomorphisms. However, if such functors are also isomorphism-dense, then they preserve monomorphisms and epimorphisms, as the following shows:

Proposition. *(7.47) Every equivalence functor preserves and reflect each of the following: monomorphisms, epimorphisms, sections, retractions and isomorphisms.*

Definition. (7.49)

(1) A morphism is called a *bimorphism* provided that it is simultaneously a monomorphism and epimorphism.

(2) A category is called *balanced* provided that each of its bimorphisms is an isomorphism.

Set, **Vec**, **Grp**, **Ab** and **HComp** are balanced categories.
Rel, **Pos**, **Top**, **Mon**, **Sgr**, **Rng**, **Cat**, **Ban** and $\mathbf{Ban_b}$ are not balanced categories.

Proposition. *(7.54) If $E \xrightarrow{e} A$ is an equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$, then the following are equivalent:*

(1) $f = g$,

(2) e is an epimorphism,

(3) e is an isomorphism,

(4) id_A is an equalizer of f and g.

Proposition. *(7.59)*

(1) Every section is a regular monomorphism.

(2) Every regular monomorphism is a monomorphism.

Neither implication of the previous proposition can be reversed.

Definition. (7.61) A monomorphism m is called *extremal* provided that it satisfies the following *extremal condition:* If $m = f \circ e$, where e is an epimorphism, then e must be an isomorphism.

Proposition. *(7.62) Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be morphisms.*

(1) If f is an extremal monomorphism and g is a regular monomorphism, then $g \circ f$ is a an extremal monomorphism.

(2) If $g \circ f$ is an extremal monomorphism, then f is an extremal monomorphism.

(3) If $g \circ f$ is a regular monomorphism and g is a regular monomorphism, then f is a regular monomorphism.

Corollary. *(7.63) Every regular monomorphism is extremal.*

A composite of extremal monomorphisms may fail to be extremal.

Proposition. *(7.67) For any category $\mathbf{A}$, the following are equivalent:*

(1) $\mathbf{A}$ is balanced,

(2) in $\mathbf{A}$ each monomorphism is extremal.

In most of the familiar categories the regular epimorphisms and the extremal epimorphisms coincide.

In general "between" regular epimorphism and extremal epimorphism there are several other commonly used types of epimorphisms (strong, swell, strict).

Definition. (7.87) Let E be a class of epimorphisms of category $\mathbf{A}$.

(1) $\mathbf{A}$ is called *E-co-wellpowered* provided that no $\mathbf{A}$-object has a proper class of pairwise non-isomorphic E-quotient objects.

(2) In case E is the class of all (regular, extremal) epimorphisms, then E-co-wellpowered is called *(regular, extremally) co-wellpowered.*

Theorem. *(7.88) Every construct is regular wellpowered and regular co-wellpowered.*

2.8 Objects and morphisms in concrete categories

Forgetful functor will be denoted by $|\ |$.

Definition. (8.1, 8.3) An object A in a concrete category $\mathbf{A}$ over $\mathbf{X}$ is called *discrete* whenever, for each object B, every $\mathbf{X}$-morphism $|A| \to |B|$ is an $\mathbf{A}$-morphism.

An object A in a concrete category $\mathbf{A}$ over $\mathbf{X}$ is called *indiscrete* whenever, for each object B, every $\mathbf{X}$-morphism $|B| \to |A|$ is an $\mathbf{A}$-morphism.

Definition. (8.6) Let $\mathbf{A}$ be a concrete category over $\mathbf{X}$.

(1) An $\mathbf{A}$-morphism $A \xrightarrow{f} B$ is called *initial* provided that for any $\mathbf{A}$-object C and $\mathbf{X}$-morphism $|C| \xrightarrow{g} |A|$ is an $\mathbf{A}$-morphism whenever $|C| \xrightarrow{g \circ f} |A|$ is an $\mathbf{A}$-morphism.

(2) An initial morphism $A \xrightarrow{f} B$ that has a monomorphic underlying $\mathbf{X}$-morphism $|A| \xrightarrow{f} |B|$ is called an *embedding.*

(3) If $A \xrightarrow{f} B$ is an embedding, then (f, B) is called an *extension* of A and (A, f) is called an *initial subobject* of B.

Proposition. *(8.7) For any concrete category the following hold:*

(1) Each embedding is a monomorphism.

(2) Each section (and in particular each isomorphism) is an embedding.

(3) If the forgetful functor preserves regular monomorphisms, then each regular monomorphism is an embedding.

Proposition. *(8.9)*

(1) If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are initial morphisms (resp. embeddings), then $A \xrightarrow{g \circ f} C$ is an initial morphism (resp. an embedding).

(2) If $A \xrightarrow{g \circ f} C$ is an initial morphism (resp. an embedding), then f is initial (resp. an embedding).

Definition. (8.10) Let $\mathbf{A}$ be a concrete category over $\mathbf{X}$.

(1) An $\mathbf{A}$-morphism $A \xrightarrow{f} B$ is called *final* provided that for any $\mathbf{A}$-object C, an $\mathbf{X}$-morphism $|B| \xrightarrow{g} |C|$ is an $\mathbf{A}$-morphism whenever $|A| \xrightarrow{g \circ f} |C|$ is an $\mathbf{A}$-morphism.

(2) A final morphism $A \xrightarrow{f} B$ with epimorphic underlying $\mathbf{X}$-morphism $|A| \xrightarrow{f} |B|$ is called a *quotient morphism.*

(3) If $A \xrightarrow{f} B$ is a quotient morphism, then (f, B) is called a *final quotient object* of A.

Proposition. *(8.12) For any concrete category the following hold:*

(1) Each quotient morphism is an epimorphism.

(2) Each retraction (and in particular each isomorphism) is a quotient morphism.

(3) If the forgetful functor preserves regular epimorphisms, then each regular epimorphism is a quotient morphism.

Proposition. *(8.13)*

(1) If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are final morphisms (resp. quotient morphisms), then $A \xrightarrow{g \circ f} C$ is final (resp. a quotient morphism).

(2) If $A \xrightarrow{g \circ f} C$ is a final morphism (resp. a quotient morphism), then g is final (resp. a quotient morphism).

Proposition. *(8.14) In a concrete category* $\mathbf{A}$ *over* $\mathbf{X}$ *the following conditions are equivalent for each* $\mathbf{A}$*-morphism f:*

(1) f is an $\mathbf{A}$*-isomorphism.*

(2) f is an initial morphism and $\mathbf{X}$*-isomorphism.*

(3) f is a final morphism and $\mathbf{X}$*-isomorphism.*

2.8.1 Structured arrows

Definition. (8.15) Let $\mathbf{A}$ be a concrete category over $\mathbf{X}$.

(1) A *structured arrow with domain* $\mathbf{X}$ is a pair (f, A) consisting of an $\mathbf{A}$-object A and $\mathbf{X}$-morphism $X \xrightarrow{f} |A|$.

Such a structured arrow will be often denoted by $X \xrightarrow{f} |A|$.

(2) A structured arrow (f, A) is said to be *generating* provided that for any pair of $\mathbf{A}$-morphisms $r, s\colon A \to B$ the equality $r \circ f = s \circ f$ implies that $r = s$.

(3) A generating arrow (f, A) is called *extremally generating* (resp. concretely generating) provided that each $\mathbf{A}$-monomorphism (resp. $\mathbf{A}$-embedding) $m\colon A' \to A$, through which f factors (i.e., $f = m \circ g$ for some $\mathbf{X}$-morphism g), is an $\mathbf{A}$-isomorphism.

(4) In a construct, an object $\mathbf{A}$ is (*extremally* resp. *concretely*) *generated by a subset X of* $|A|$ provided that the inclusion map $X \hookrightarrow |A|$ is (extremally resp. concretely) generating.

Proposition. *(8.16) In a concrete category* $\mathbf{A}$ *over* $\mathbf{X}$ *the following hold for each structured arrow $f\colon X \to |A|$:*

(1) If (f, A) is extremally generating, then (f, A) is concretely generating.

(2) If (f, A) is concretely generating, then (f, A) is generating.

(3) If $X \xrightarrow{f} |A|$ is an $\mathbf{X}$*-epimorphism, then (f, A) is generating.*

(4) If $X \xrightarrow{f} |A|$ is an extremal epimorphism in $\mathbf{X}$*, and if $|\ |$ preserves monomorphisms, then (f, A) is extremally generating.*

Definition. (8.19) Let $\mathbf{A}$ be a concrete category over $\mathbf{X}$.

(1) Structured arrows (f, A) and (g, B) with the same domain are said to be *isomorphic* provided that there exists an $\mathbf{A}$-isomorphism $k\colon A \to B$ with $k \circ f = g$.

(2) $\mathbf{A}$ is said to be *concretely co-wellpowered* provided that for each $\mathbf{X}$-object X any class of pairwise non-isomorphic concretely generating arrows with domain X is a set.

Proposition. *(8.21) Each concretely co-wellpowered concrete category is extremally co-wellpowered.*

2.8.2 Universal arrows and free objects

Definition. (8.22) In a concrete category **A** over **X**

(1) a *universal arrow* over an **X**-object X is a structured arrow $X \xrightarrow{u} |A|$ with domain X that has the following universal property: for each structured arrow $f \xrightarrow{X} |B|$ with domain X there exists a unique **A**-morphism $\hat{f}\colon A \to B$ such that the triangle

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "f"'] & {|A|} \arrow[d, "\hat{f}"] \\ & {|B|} \end{tikzcd}$$

commutes.

(2) a *free object* over **X**-object X is an **A**-object A such that there exists a universal arrow (u, A) over X.

In a construct, an object A is a free object

(1) over the empty set if and only if A is an initial object.

(2) over a singleton set if and only if A represents the forgetful functor (6.9).

In the construct **Vec** each object is a free object over any basis for it.
In the construct **Top** and **Pos** the free objects are precisely the discrete ones.

Proposition. *(8.24) Every universal arrow is concretely generating.*

Proposition. *(8.25) For any* **X***-object X, universal arrows over X are essentially unique; i.e., any two universal arrows with domain X are isomorphic, and conversely, if $u \xrightarrow{X} |A|$ is a universal arrow and $A \xrightarrow{k} A'$ is an* **A***-isomorphism, then $k \circ u \xrightarrow{X} |A'|$ is also universal.*

Definition. (8.26) A concrete category over **X** is said to *have free objects* provided that for each **X**-object X there exists a universal arrow over X.

Proposition. *(8.28) If a concrete category* **A** *over* **X** *has free objects, then an* **A***-morphism is an* **A***-monomorphism if and only if it is an* **X***-monomorphism.*

Proposition. *(8.29) If a construct* **A** *has a free object over a singleton set, then the monomorphisms in* **A** *are precisely those morphisms that are injective functions.*

2.8.3 Objects and morphisms with respect to a functor

Definition. (8.30) Let $G\colon \mathbf{A} \to \mathbf{B}$ be a functor, and let B be a **B**-object.

(1) A *G-structured arrow with domain B* is a pair (f, A) consisting of an **A**-object A and a **B**-morphism $f\colon B \to GA$.

(2) A G-structured arrow (f, A) with domain B is called

   (a) *generating* provided that for any pair of **A**-morphisms $A \overset{r}{\underset{s}{\rightrightarrows}} \hat{A}$, the equality $Gr \circ f = Gs \circ f$ implies that $r = s$.

(b) *extremally generating* provided that it is generating and whenever $A' \xrightarrow{m} A$ is an $\mathbf{A}$-monomorphism and (g, A') is a G-structured arrow with $f = G(m) \circ g$ then m is $\mathbf{A}$-isomorphism.

(c) *G-universal for B* provided that for each G-structured arrow (f', A') with domain B there exists a unique $\mathbf{A}$-morphism $A \xrightarrow{\hat{f}} A'$ with $f' = G(\hat{f}) \circ f$, i.e. such that the triangle $\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "f'"'] & GA \arrow[d, "G\hat{f}"] \\ & GA' \end{tikzcd}$ commutes.

Proposition. *(8.32) If $G\colon \mathbf{A} \to \mathbf{B}$ is a functor, then the following are equivalent:*

(1) G is faithful,

(2) each $\mathbf{A}$-epimorphism, considered as a G-structured arrow, is generating,

(3) each $\mathbf{A}$-identity, considered as a G-structured arrow, is generating.

Proposition. *(8.33) Every G-universal arrow is extremally generating.*

Definition. (8.34) G-structured arrows (f, A) and (f', A') with the same domain are said to be *isomorphic* provided there exists an $\mathbf{A}$-isomorphism $k\colon A \to A'$ with $G(k) \circ f = f'$.

Proposition. *(8.35) For any functor $G\colon \mathbf{A} \to \mathbf{B}$ and any $\mathbf{B}$-object B, G-universal arrows for B are essentially unique; i.e., any two G-universal arrows with domain B are isomorphic, and conversely, if $B \xrightarrow{u} GA$ is a G-universal arrow and $A \xrightarrow{k} A'$ is an isomorphism, then $B \xrightarrow{k \circ u} GA'$ is also G-universal.*

Proposition. *(8.36) Let $G\colon \mathbf{A} \to \mathbf{B}$ be a functor. If the triangle*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "f"'] & GA \arrow[d, "G\hat{f}"] \\ & GB \end{tikzcd}$$

commutes, where (u, A) is a G-universal arrow and $A \xrightarrow{\hat{f}} B$ is an $\mathbf{A}$-morphism, then the following hold:

(1) (f, B) is generating if and only if $\hat{f}$ is an epimorphism.

(2) (f, B) is extremally generating if and only if $\hat{f}$ is an extremal epimorphism.

Definition. (8.37) A functor $G\colon \mathbf{A} \to \mathbf{B}$ is called *(extremally) co-wellpowered* provided that for any $\mathbf{B}$-object B, any class of pairwise non-isomorphic (extremally) generating G-structured arrows with domain B is a set.

A faithful functor $G\colon \mathbf{A} \to \mathbf{B}$ is called *concretely co-wellpowered* provided that the concrete category $(\mathbf{A}, G)$ is concretely co-wellpowered.

Proposition. *(8.38) If a functor $G\colon \mathbf{A} \to \mathbf{B}$ is co-wellpowered, then so is $\mathbf{A}$.*

2.8.4 Constructed arrows

All of the concepts relating to G-structured arrows have duals. In particular:

Definition. (8.40) Let $G\colon \mathbf{A} \to \mathbf{B}$ be a functor and let B be a $\mathbf{B}$-object.

(1) A *G-constructed arrow with codomain* B is a pair (A, f) consisting of an $\mathbf{A}$-object A and $\mathbf{B}$-morphism $GA \xrightarrow{f} B$.

(2) A G-constructed arrow with codomain B is called *G-co-universal for* B provided that for each G-constructed arrow (A', f') with codomain B there exists a unique $\mathbf{A}$-morphism $A' \xrightarrow{f} A$ with $f' = f \circ G(f)$.

2.9 Injective objects and essential embeddings

2.9.1 Injectivity in concrete categories

Definition. (9.1) In a concrete category an object C is called *injective* provided that for any embedding $A \xrightarrow{f} B$ an any morphism $A \xrightarrow{m} C$ there exists a morphism $B \xrightarrow{g} C$ extending f, i.e., such that the triangle

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[dr, "f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

commutes.

Proposition. *(9.4) Every terminal object is injective.*

Proposition. *(9.5) Every retract of an injective object is injective.*

Definition. (9.6) In a concrete category an object C is called an *absolute retract* provided that any embedding with domain C is a section.

Proposition. *(9.7) Every injective object is an absolute retract.*

Definition. (9.9) A concrete category *has enough injectives* provided that each of its object is an initial subobject of an injective object.

Proposition. *(9.10) If a concrete category* $\mathbf{A}$ *has enough injectives, then in* $\mathbf{A}$ *injective objects are precisely the absolute retracts.*

Definition. (9.12) In a concrete category an embedding $A \xrightarrow{m} B$ is called *essential* provided that a morphism $B \xrightarrow{f} C$ is an embedding, whenever $A \xrightarrow{f \circ m} C$ is an embedding.

Proposition. *(9.14)*

(1) Every isomorphism is essential.

(2) The composition of essential embeddings is essential.

(3) If f and g are embeddings with $g \circ f$ essential, then g is essential.

(4) If f and $g \circ f$ are essential embeddings, then g is an essential embedding.

Proposition. *(9.15) Injective objects have no proper essential extensions.*

Definition. (9.16) An *injective hull* of A is an extension $A \xrightarrow{m} B$ of A such that B is injective and m is essential.

Proposition. *(9.19) Injective hulls are essentially unique, i.e.,*

(1) if (m, B) and (m, B') are injective hulls of A, then there exists an isomorphism $B \xrightarrow{k} B'$ with $m' = k \circ m$,

(2) if (m, B) is an injective hull of A, and if $B \xrightarrow{k} B'$ is an isomorphism, then $(k \circ m, B')$ is an injective hull of A.

Proposition. *(9.20) If an object A has an injective hull, then for any extension (m, B) of A the following conditions are equivalent:*

(1) (m, B) is an injective hull of A,

(2) (m, B) is a maximal essential extension of A (B has no proper essential extension) of A,

(3) (m, B) is a largest essential extension of A,

(4) (m, B) is a smallest injective extension of A.

(5) (m, B) is a minimal injective extension *of A (i.e., (m, B) is an injective extension of A and whenever $m \xrightarrow{A} B = A \xrightarrow{m'} B' \xrightarrow{\overline{m}} B$ with m' and $\overline{m}$ embeddings, and B' an injective object, then $\overline{m}$ is an isomorphism.*

2.9.2 M-injectives in abstract categories

Definition. (9.22) Let M be a class of morphisms in a category $\mathbf{A}$.

(1) An object C is called *M-injective* provided that for every morphism $A \xrightarrow{m} B$ in M and every morphism $A \xrightarrow{f} C$ there exists a morphism $B \xrightarrow{g} C$ with $f = g \circ m$.

(2) A morphism $A \xrightarrow{m} B$ in M is called *M-essential* provided that a morphism $B \xrightarrow{f} C$ belongs to M whenever $f \circ m$ does.

Proposition. *(9.25) If $\mathbf{B}$ is a reflective, isomorphism-closed, full subcategory of $\mathbf{A}$ and M is the class of all $\mathbf{B}$-reflection arrows, then*

(1) the M-injective object of $\mathbf{A}$ are precisely the $\mathbf{B}$-objects, and

(2) the M-injective hulls are precisely the $\mathbf{B}$-reflections.

2.9.3 Projectivity

concept	**dual concept**
embedding	quotient morphism
injective object	projective object
essential embedding	essential quotient object
M-injective object	M-projective object
M-essential morphism	M-coessential morphism
M-injective hull	M-projective hull

Proposition. *(9.29) If $(\mathbf{A}, U)$ is a concrete category over $\mathbf{X}$ that has free objects, and E is the class of all $\mathbf{A}$-morphisms f for which Uf is a retraction, then the following are equivalent:*

(1) A is an E-projective object,

(2) A is a retract of a free object.

Corollary. *(9.30) If in construct with free object every surjective morphisms is a quotient morphism then the projective objects are precisely the retracts of the free objects.*

Exercise 9A: Many results of this book can be expressed (in localized form) in the realm of Zermelo-Fraenkel set theory (ZF). But if the axiom of choice for sets (AC) is no assumed, several fail tu be true:

(ET) In **Set** every epimorphism is a retraction.

(PT) In **Set** every product of injective objects is injective

(BT) The injective objects in **Ab** are precisely the divisible abelian groups.

(ST) The injective objects in **Boo** are precisely the complete Boolean algebras.

(UT) In **Boo** the two-element boolean algebra is injective.

(GT) The projective objects in **HComp** are the extremally disconnected Hausdorff spaces.

(a) Show that in ZF the following implication hold:
(AC) $\Leftrightarrow$ (ET) $\Leftrightarrow$ (PT) $\Leftrightarrow$ (BT) $\Rightarrow$ (ST) $\Rightarrow$ (UT)

(b) Show that in ZF the following holds: (ST) $\Leftrightarrow$ [(GT) and (UT)]

(c) Does (ST) imply (AC)? [Unsolved.]

3 Sources and sinks

3.10 Sources and sinks

3.10.1 Sources

Definition. (10.1) A *source* is a pair $(A, (f_i)_{i \in I})$ consisting of an object A and a family of morphisms $f_i \colon A \to A_i$ with domain A, indexed by some class I. A is called the *domain of the source* and the family $(A_i)_{i \in I}$ is called the *codomain of the source*.

Definition. (10.3) If $\mathcal{S} = (A \xrightarrow{f_i} A_i)_I$ is a source and, for each $i \in I$, $\mathcal{S}_i = (A_i \xrightarrow{g_{ij}} A_{ij})$ is a source then the source

$$(\mathcal{S}_i) \circ \mathcal{S} = (A \xrightarrow{g_{ij} \circ f_i} A_{ij})_{i \in I, j \in J_i}$$

is called the *composite* of $\mathcal{S}$ and the family $(\mathcal{S}_i)_I$.

Notation $\mathcal{S} \circ f$.

The composition of morphisms can be regarded as a special case of composition of sources.

3.10.2 Mono-sources

Definition. (10.5) A source $\mathcal{S} = (A, f_i)_I$ is called a *mono-source* provided that it can be cancelled from the left, i.e., provided that for any pari $B \overset{r}{\underset{s}{\rightrightarrows}} A$ of morphisms the equation $\mathcal{S} \circ r = \mathcal{S} \circ s$ implies $r = s$.

Proposition. *(10.7)*

(1) Representable functors preserve mono-sources.

(2) Faithful functor reflect mono-sources.

Corollary. *(10.8) In a construct $(\mathbf{A}, U)$ every point-separating source is a mono-source. The converse holds whenever U is representable.*

Proposition. *(10.9) Let $\mathcal{T} = (\mathcal{S}_i) \circ \mathcal{S}$ be a composite of sources.*

(1) If $\mathcal{S}$ and all $\mathcal{S}_i$ are mono-sources, then so is $\mathcal{T}$.

(2) If $\mathcal{T}$ is a mono-source, then so is $\mathcal{S}$.

Proposition. *(10.10) Let $(A, f_i)_I$ be a source.*

(1) If $(A, f_j)_J$ is a mono-source for some $J \subseteq I$, then so is $(A, f_i)_I$.

(2) If f_j is a monomorphism for some $j \in I$, then $(A, f_i)_I$ is a mono-source.

Definition. (10.11) A mono-source $\mathcal{S}$ is called *extremal* provided that whenever $\mathcal{S} = \overline{\mathcal{S}} \circ e$ for some epimorphism e, then e must be an isomorphism.

Proposition. *(10.13)*

(1) If a composite source $(\mathcal{S}_i) \circ \mathcal{S}$ is an extremal mono-source, then so is $\mathcal{S}$.

(2) If $\mathcal{S} \circ f$ is an extremal mono-source, then f is an extremal monomorphism.

Proposition. *(10.15) Let $(A, f_i)_I$ be a source.*

(1) If $(A, f_j)_J$ is an extremal mono-source for some $J \subseteq I$, then so is $(A, f_i)_I$.

(2) If f_j is an extremal monomorphism for some $j \in I$, then $(A, f_i)_I$ is an extremal mono-source.

Definition. (10.17) An object A is called an *extremal coseparator* provided that for any object B the source $(B, \hom(B, A))$ is an extremal mono-source.

3.10.3 Products

Definition. (10.19) A source $\mathcal{P} = (P \xrightarrow{p_i} A_i)$ is called a *product* provided that…

Proposition. *(10.21) Every product is an extremal mono-source.*

Proposition. *(10.22) Products are essential unique.*

Proposition. *(10.25) Let $\mathcal{Q} = (\mathcal{P}_i) \circ \mathcal{P}$ be a composite of sources.*

(1) If $\mathcal{P}$ and all $\mathcal{P}_i$ are products, then so is $\mathcal{Q}$.

(2) If $\mathcal{Q}$ is a product and all $\mathcal{P}_i$ are mono-sources, then $\mathcal{P}$ is a product.

Proposition. *(10.26) Consider*

$$\begin{tikzcd} A \arrow[r, "\langle f_i \rangle"] \arrow[dr, "f_i"'] & \prod A_i \arrow[d, "\pi_j"] \\ & A_j \end{tikzcd}$$

Then

(1) $(A, f_i)_I$ is a mono-source if and only if $\langle f_i \rangle$ is a monomorphism.

(2) $(A, f_i)_I$ is an extremal mono-source if and only if $\langle f_i \rangle$ is an extremal monomorphism.

(3) $(A, f_i)_I$ is a product if and only if $\langle f_i \rangle$ is a product; i.e., an isomorphism.

Proposition. *(10.28) If $(P \xrightarrow{p_i} A_i)_I$ is a product and if $i_0 \in I$ is such that $\hom(A_{i_0}, A_i) \neq \emptyset$ for each $i \in I$, then p_{i_0} is a retraction.*

Definition. (10.29)

(1) A category *has products* provided that for every set-indexed family $(A_i)_I$ of objects there exists a product $(\prod A_i \xrightarrow{\pi_j} A_j)_I$.

(2) A category *has finite products* provided that for every finite family $(A_i)_I$ of objects there exists a product $(\prod A_i \xrightarrow{\pi_j} A_j)_I$.

Proposition. *(10.30) A category has finite products if and only if it has terminal objects and products of pairs of object.*

Theorem. *(10.32)*

(1) A category that has products for all class-indexed families must be thin.

(2) A small category has products if and only if it is equivalent to a complete lattice.

Definition. (10.34) If $(A_i \xrightarrow{f_i} B_i)_I$ is a family of morphisms… $\prod f_i \colon \prod A_i \to \prod B_i$.

Proposition. *(10.35) Let $(f_i)_I$ be a set-indexed family of morphisms with product $\prod f_i$. If each f_i has any of the following properties, then so does $\prod f_i$:*

(1) isomorphism,

(2) section,

(3) retraction

(4) monomorphism

(5) regular monomorphism.

Proposition. *(10.36) Product of equalizers = equalizer of products.*

Definition. (10.37) A^I = *Ith power of A.*

Proposition. *(10.38) In a category that has products, an object A is an (extremal) coseparator if and only if every object is an (extremal) subobject of some power A^I of A.*

Proposition. *(10.40) For any class M of morphisms, every product of M-injective objects is M-injective.*

3.10.4 Sources in concrete categories

Initial sources

Definition. (10.41) Let $\mathbf{A}$ be a concrete category over $\mathbf{X}$. A source $(A \xrightarrow{f_i} A_i)$ in $\mathbf{A}$ is called *initial* provided that an $\mathbf{X}$-morphism $f\colon |B| \to |A|$ is an $\mathbf{A}$-morphism whenever each composite $f_i \circ f\colon |B| \to |A_i|$ is an $\mathbf{A}$-morphism.

Proposition. *(10.43) If $(A \xrightarrow{f_i} A_i)$ is an initial source in $\mathbf{A}$, then $A = \max\{B \in Ob(\mathbf{A}) \mid |B| = |A|$ and all $|B| \xrightarrow{f_i} |A_i|$ are $\mathbf{A}$-morphisms$\}$.*[2]

The above property often characterizes initial sources, e.g., in such constructs as **Top** or $\mathbf{Spa}(T)$. However, in the construct $\mathbf{Top}_1$, there are non-initial sources with the above property.

Proposition. *(10.45) Let $\mathcal{T} = (\mathcal{S}_i) \circ \mathcal{S}$ be a composite of sources in a concrete category.*

(1) If $\mathcal{S}$ and all $\mathcal{S}_i$ are initial, then so is $\mathcal{T}$.

(2) If $\mathcal{T}$ is initial, then so is $\mathcal{S}$.

Proposition. *(10.46) Let $(A, f_i)_I$ be a source in a concrete category. If $(A, f_i)_J$ is initial for some $J \subseteq I$, then so is $(A, f_i)_I$.*

Definition. (10.47) A concrete functor $F\colon \mathbf{A} \to \mathbf{B}$ over $\mathbf{X}$ is said to *preserve initial sources* provided that for every initial source $\mathcal{S}$ in $\mathbf{A}$, the source $F\mathcal{S}$ is initial in $\mathbf{B}$.

Proposition. *(10.49) If (F, G) is a Galois correspondence, then G preserves initial sources.*

Corollary. *(10.50) Embeddings of concretely reflective subcategories preserve initial sources.*

Concrete products

Definition. (10.52) Let $\mathbf{A}$ be a concrete category over $\mathbf{X}$. A source $\mathcal{S}$ in $\mathbf{A}$ is called a *concrete product* in $\mathbf{A}$ if and only if $\mathcal{S}$ is a product in $\mathbf{A}$ and $|\mathcal{S}|$ is a product in $\mathbf{X}$.

Proposition. *(10.53) A source $\mathcal{S}$ in a concrete category $\mathbf{A}$ over $\mathbf{X}$ is a concrete product if and only if it is initial in $\mathbf{A}$ and $|\mathcal{S}|$ is a product in $\mathbf{X}$.*

Definition. (10.54) A concrete category $\mathbf{A}$ *has concrete products* if and only if for every set-indexed family $(A_i)_I$ of $\mathbf{A}$-objects there exists a concrete product $(P \xrightarrow{p_i} A_i)_I$ in $\mathbf{A}$, i.e., if and only if $\mathbf{A}$ has products and the forgetful functor preserves them.

Proposition. *(10.56) Let $\mathcal{Q} = (\mathcal{P}_i) \circ \mathcal{P}$ be a composite of sources in a concrete category $\mathbf{A}$.*

(1) If $\mathcal{P}$ and all $\mathcal{P}_i$ are concrete products, then so is $\mathcal{Q}$.

(2) If $\mathcal{Q}$ is a concrete product and each $|\mathcal{P}_i|$ is a mono-source, then $\mathcal{P}$ is a concrete product.

[2] Recall the order on the fibre of $|A|$ [5.4(1)]

3.10.5 G-initial sources

Definition. (10.57) Let $G\colon \mathbf{A} \to \mathbf{B}$ be a functor. A source $\mathcal{S} = (A \xrightarrow{f_i} A_i)_I$ in $\mathbf{A}$ is called *G-initial* provided that for each source $\mathcal{T} = (B \xrightarrow{g_i} A_i)_I$ in $\mathbf{A}$ with the same codomain as $\mathcal{S}$ and each $\mathbf{B}$-morphism $GB \xrightarrow{h} GA$ with $G\mathcal{T} = G\mathcal{S} \circ h$ there exists a unique $\mathbf{A}$-morphism $B \xrightarrow{\overline{h}} A$ with $\mathcal{T} = \mathcal{S} \circ \overline{h}$ and $h = G\overline{h}$.

$$\begin{tikzcd} B \arrow[d, "\overline{h}"'] \arrow[dr, "g_i"] & \\ A \arrow[r, "f_i"'] & A_i \end{tikzcd} \qquad \begin{tikzcd} GB \arrow[d, "G\overline{h}=h"'] \arrow[dr, "Gg_i"] & \\ GA \arrow[r, "Gf_i"'] & GA_i \end{tikzcd}$$

If $(\mathbf{A}, U)$ is a concrete category, then U-initial sources are precisely the initial sources in $(\mathbf{A}, U)$.

If $\mathbf{A}$ is a category and $G\colon \mathbf{A} \to \mathbf{1}$, then G-initial sources are precisely the products in $\mathbf{A}$.

Proposition. *(10.59) For a functor $G\colon \mathbf{A} \to \mathbf{B}$ the following conditions are equivalent:*

(1) G is faithful,

(2) for each $\mathbf{A}$-object A the 2-source $(A, (id_A, id_A))$ is G-initial.

(3) whenever $(A, f_i)_I$ is a source in $\mathbf{A}$ and $(A, f_j)_J$ is G-initial for some $J \subseteq I$, then $(A, f_i)_I$ is initial.

Proposition. *(10.60) If $G\colon \mathbf{A} \to \mathbf{B}$ is a functor such that each mono-source in $\mathbf{A}$ is G-initial, then the following hold:*

(1) G is faithful,

(2) G reflects products,

(3) G reflects isomorphisms.

The property that all mono-sources be initial, is not unfamiliar. As we will see in §23, it is typical for "algebraic" categories.

3.10.6 Sinks

concept	**dual concept**
source	sink
mono-source	epi-sink
extremal mono-source	extremal epi-sink
initial source	final sink
G-initial source	G-final sink
product	coproduct
projection π_j	injection μ_j
$\langle f_i \rangle$	$[f_i]$
power A^I	copower ${}^I A$
$A \times B$	$A + B$
$\prod f_i$; $f \times g$	$\coprod f_i$; $f + g$

Definition. (10.69) A full concrete subcategory $\mathbf{A}$ of a concrete category $\mathbf{B}$ is said to be *finally dense* in $\mathbf{B}$ provided that for every $\mathbf{B}$-object B there is a final sink $(A_i \xrightarrow{f_i} B)_I$ in $\mathbf{B}$ with A_i in $\mathbf{A}$ for all $i \in I$.

Proposition. *(10.71) If* $\mathbf{A}$ *is a finally dense full concrete subcategory of a concrete category* $\mathbf{B}$*, then every initial source in* $\mathbf{A}$ *is initial in* $\mathbf{B}$.

3.11 Limits and colimits

3.11.1 Limits

Definition. (11.1) A *diagram* in a category $\mathbf{A}$ is a functor $D \colon \mathbf{I} \to \mathbf{A}$ with codomain $\mathbf{A}$. The domain $\mathbf{I}$ is called the *scheme* of the diagram.

A diagram with a small (or finite) scheme is said to be *small* (or *finite*).

Definition. (11.3) Let $D \colon \mathbf{I} \to \mathbf{A}$ be a diagram.

(1) An $\mathbf{A}$-source $(A \xrightarrow{f_i} D_i)_{i \in Ob(\mathbf{I})}$ is said to be *natural* for D provided that for each $\mathbf{I}$-morphism $i \xrightarrow{d} j$, the triangle

$$\begin{tikzcd} A \arrow[d, "f_i"'] \arrow[dr, "f_j"] & \\ D_i \arrow[r, "Dd"'] & D_j \end{tikzcd}$$

commutes.

(2) A *limit* of D is...

Every limit is an extremal mono-source. Limits are essentially unique.

Equalizer=limit of diagram with scheme $\bullet \rightrightarrows \bullet$. If in the above scheme, the two arrows are replaced by an arbitrary set of arrows, then the limits of diagrams with such schemes are called *multiple equalizers.*

3.11.2 Pullbacks

Definition. A square

$$\begin{tikzcd} P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \qquad (*)$$

is called a *pullback square* provided that it commutes and that for any commuting square of the form

$$\begin{tikzcd} \hat{P} \arrow[r, "\hat{f}"] \arrow[d, "\hat{g}"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

there exists a unique morphism $\hat{P} \xrightarrow{k} P$ for which the diagram

$$\begin{tikzcd}
\hat{P} \arrow[dr, "k"] \arrow[drr, bend left, "\hat{f}"] \arrow[ddr, bend right, "\hat{g}"'] & & \\
& P \arrow[r, "\overline{f}"'] \arrow[d, "\overline{g}"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

commutes.

Proposition. *(11.10) Let*

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

commute in **A**. *Then*

(1) *if the squares are pullback squares, then so is the outer rectangle; i.e., pullbacks can be composed by "pasting" them together,*

(2) *if the outer rectangle and right-hand square are pullback squares, then so is the left-hand square.*

3.11.3 Relationship of pullbacks to other limits

Proposition (Canonical Construction of Pullbacks). *(11.11) Mam v sk.*

Proposition. *(11.13) If T is a terminal object, then the following are equivalent:*

(1)

$$\begin{tikzcd}
P \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d] \\
B \arrow[r] & T
\end{tikzcd}$$

is a pullback square,

(2) $(P,(p_A,p_B))$ *is a product of A and B*

Proposition (Construction of Equalizers via Products and Pullbacks). *(11.14) Mam v sk*

3.11.4 Pullbacks related to special morphisms

Lemma. *(11.15) Suppose that the diagram*

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\boxed{1}"] & & \bullet \arrow[dl] \arrow[dd] \\
& \bullet \arrow[dr, "h"] & \\
\bullet \arrow[ur] \arrow[rr] & & \bullet
\end{tikzcd}$$

commutes.

(1) *If the outer square is a pullback square then so is* $\boxed{1}$.

(2) *If* $\boxed{1}$ *is a pullback square and h is a monomorphism, then the outer square is a pullback square.*

Proposition. *(11.16)* $A \xrightarrow{f} B$ *is a monomorphism if and only if*

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "id_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback square.

Definition. (11.17) A class M of morphisms in a category is called *pullback stable* (or *closed under the formation of pullbacks*) provided that for each pullback square

$$\begin{tikzcd} P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

with $f \in M$, it follows that $\overline{f} \in M$.

Proposition. *(11.18) Monomorphisms, regular monomorphisms, and retractions are pullback stable.*

3.11.5 Congruences

Definition. (11.20)

(1) If

$$\begin{tikzcd} \bullet \arrow[r, "p"] \arrow[d, "q"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

is a pullback square, then the pair (p, q) is called a *congruence relation* of f.

(2) A pair (p, q) is called a congruence relation provided that there exists some morphism f such that (p, q) is a congruence relation of f.

Lemma. *(11.21) Let* (p, q) *be a congruence relation of* $A \xrightarrow{f} B$. *Then*

(1) (p, q) *is a congruence relation of* $A \xrightarrow{m \circ f} C$ *for each monomorphism* $B \xrightarrow{m} C$,

(2) *if* $f = g \circ h$ *and* $h \circ p = h \circ q$, *then* (p, q) *is a congruence relation of* h.

Proposition. *(11.22) If* (p, q) *is a congruence relation and c is a coequalizer of p and q, then* (p, q) *is a congruence relation of c.*

If c is a regular epimorphism and (p, q) *is a congruence relation of c, then c is a coequalizer of p and q.*

3.11.6 Intersections

Definition. (11.23) Let $\mathcal{A}$ be a family of subobjects (A_i, m_i) of an object B, indexed by a class I. A subobject (A, m) of B us called an *intersection* of $\mathcal{A}$ provided that the following two conditions are satisfied:

(1) m factors through each m_i, i.e., for each i there exists an f_i with $m = m_i \circ f_i$,

(2) if a morphism $C \xrightarrow{f} B$ factors through each m_i, then it factors through m.

Definition. (11.26) A class M of monomorphisms is said to be *closed under the formation of intersections* provided that whenever (A, m) is an intersection of a family of subobjects (A_i, m_i) and each m_i belongs to M, then m belongs to M.

3.11.7 Colimits

Definition. (11.27) natural sink, colimit

Proposition. *(11.29) Colimits are essentially unique and each colimit is an extremal episink.*

Definition. (11.30) pushout

Definition. (11.32) A square, that is simultaneously a pullback square and pushout square is called a *pullation square.*

Proposition. *(11.33) Consider a commuting square*

$$\begin{tikzcd} \bullet \arrow[r, "p"] \arrow[d, "q"'] & \bullet \arrow[d, "c"] \\ \bullet \arrow[r, "c"'] & \bullet \end{tikzcd} \tag{*}$$

(1) If (*) is a pushout square, then c is a coequalizer of p and q.

(2) If (p, q) is a congruence relation of c, and c is a regular epimorphism, then (*) is a pullation square.

11L: Multiple Pullbacks

A pair (f, mcS), consisting of a morphism $A \xrightarrow{f} B$ and a source $\mathcal{S} = (A \xrightarrow{f_i} A_i)_I$, is called a *multiple pullback* of a sink $(A_i \xrightarrow{g_i} B)_I$ provided that:

(i) $f = g_i \circ f'_i$ for each $i \in I$, and

(ii) for each pair $(f', \mathcal{S}')$, with $A' \xrightarrow{f'} B$ a morphism and $\mathcal{S}' = (A' \xrightarrow{f'_i} B)_I$ a source for which $i \in I$, there exists a unique morphism $A' \xrightarrow{g} A$ with $f' = g \circ f$ and $f'_i = g \circ f_i$ for each $i \in I$.

Interpret multiple pullbacks as limits. Interpret pullbacks as multiple pullbacks of 2-sinks. Interpret intersections as multiple pullbacks. Show that each sink that consists of isomorphisms alone has a multiple pullback.

3.12 Completeness and cocompleteness

Definition. (12.1)

(1) *have (finite) products* provided that for each (finite) set-indexed family there exists a product,

(2) *have equalizers* provided that for each parallel pair of morphisms there exists an equalizer,

(3) *have pullbacks* provided that for each 2-sink there exists a pullback,

(4) *have (finite) intersections* provided that for each object A, and every (finite) family of subobjects of A, there exists an intersection.

Dual notions: have (finite) coproducts, have coequalizers, and have (finite) cointersections.

Definition. (12.2) A category $\mathbf{A}$ is said to be

(1) *finitely complete* if for each finite diagram in $\mathbf{A}$ there exists a limit,

(2) *complete* if for each small diagram in $\mathbf{A}$ there exists a limit,

(3) *strongly complete* if it is complete and has intersections.

Theorem. *(12.3) For each category $\mathbf{A}$ the following conditions are equivalent:*

(1) $\mathbf{A}$ is complete,

(2) $\mathbf{A}$ has products and equalizers,

(3) $\mathbf{A}$ has products and finite intersections.

Theorem. *(12.4) For each category $\mathbf{A}$ the following conditions are equivalent:*

(1) $\mathbf{A}$ is finitely complete,

(2) $\mathbf{A}$ has finite products and equalizers,

(3) $\mathbf{A}$ has finite products and finite intersections,

(4) $\mathbf{A}$ has pullbacks and terminal objects.

Theorem. *(12.5) Each complete and wellpowered category is strongly complete.*

3.12.1 Cocompleteness almost implies completeness

Theorem. *(12.7) A small category is complete if and only if it is cocomplete.*

Proposition (Canonical Construction of Limits via Large Colimits). *(12.8) For a small diagram $D\colon \mathbf{I} \to \mathbf{A}$, let S^D be the category whose objects are all natural sources (A, f_i) for D, whose morphisms $(A, f_i) \xrightarrow{g} (A', f'_i)$ are all those $\mathbf{A}$-morphisms $A \xrightarrow{g} A'$ with $(A, f_i) = (A', f'_i) \circ g$ and whose identity morphisms and composition law are as in $\mathbf{A}$. If $D^*\colon \mathbf{S^D} \to \mathbf{A}$ is the diagram given by:*

$$D^*((A, f_i) \xrightarrow{g} (A', f'_i)) = A \xrightarrow{g} A',$$

then for each $\mathbf{A}$-object L the following conditions are equivalent:

(1) D has limit $\mathcal{L} = (L \xrightarrow{\ell_i} D_i)_{i \in Ob(\mathbf{I})}$,

(2) D^* has a colimit $\mathcal{K} = (D^*(S) \xrightarrow{k_S} L)_{S \in Ob(S^D)}$,

(3) S^D has a terminal object $\mathcal{L} = (L \xrightarrow{\ell_i} D_i)_{i \in Ob(\mathbf{I})}$.

Proposition. *(12.9) A cocomplete category* $\mathbf{A}$ *has a terminal object if and only if it has a* weak terminal object *K; i.e., for each* $\mathbf{A}$*-object A, there exists at least one morphism from A to K.*

Definition. (12.10) A full subcategory $\mathbf{B}$ of $\mathbf{A}$ with embedding $E \colon \mathbf{B} \to \mathbf{A}$ is called *colimit-dense* in $\mathbf{A}$ provided that for every $\mathbf{A}$-object A there exists a diagram $D \colon \mathbf{I} \to \mathbf{B}$ such that the diagram $E \circ D \colon \mathbf{I} \to \mathbf{A}$ has a colimit with codomain A.

Theorem. *(12.12) Every cocomplete category with a small colimit-dense subcategory is complete.*

Theorem. *(12.13) Every co-wellpowered cocomplete category with a separator is wellpowered and complete.*

3.13 Functors and limits

Definition. (13.1)

(1) A functor $F \colon \mathbf{A} \to \mathbf{B}$ is said to *preserve* a limit $\mathcal{L} = (L \xrightarrow{\ell_i} D_i)$ of a diagram $D \colon \mathbf{I} \to \mathbf{A}$ provided that $FL = (FL \xrightarrow{F\ell_i} FD_i)$ is a limit of a diagram $F \circ D \colon \mathbf{I} \to \mathbf{B}$.

(2) F is said to *preserve limits over scheme* $\mathbf{I}$ provided that F preserves all limits of diagram $D \colon \mathbf{I} \to \mathbf{A}$ with scheme $\mathbf{I}$.

(3) F *preserves equalizers* if and only if F preserves all limits over the scheme $\bullet \rightrightarrows \bullet$; F *preserves products* if and only if F preserves all limits over small discrete schemes; F *preserves small limits* if and only if F preserves all limits over small schemes; F *preserves strong limits* if and only if F preserves all limits over small schemes and arbitrary intersection; etc.

Example. (13.2(4)) The full embedding $\mathbf{Haus} \to \mathbf{Top}$ and $\mathbf{Pos} \to \mathbf{Rel}$ preserve limits and coproducts, but not coequalizers.

Proposition. *(13.3) If $F \colon \mathbf{A} \to \mathbf{B}$ is a functor and $\mathbf{A}$ is finitely complete, then the following conditions are equivalent:*

(1) F preserves finite limits,

(2) F preserves finite products and equalizers,

(3) F preserves pullbacks and terminal objects.

Proposition. *(13.4) For a complete category $\mathbf{A}$, a functor $F \colon \mathbf{A} \to \mathbf{B}$ preserves small limits if and only if it preserves products and equalizers.*

Proposition. *(13.5)*

(1) If a functor preserves finite limits, then it preserves monomorphisms and regular monomorphism.

(2) If a functor preserves (small) limits, then it preserves (small) mono-sources.

Proposition. *(13.7) Hom-functors preserve limits.*

Proposition. *(13.8) If F and G are naturally isomorphic functors, then F preserves limits over a scheme $\mathbf{I}$ if and only if G does.*

Corollary. *(13.9) Representable functors preserve limits.*

Proposition. *(13.11) Embeddings of colimit-dense subcategories preserve limits.*

3.13.1 Concrete limits

Definition. (13.12)

(1) Let $(\mathbf{A}, U)$ be a concrete category. A limit $\mathcal{L}$ of a diagram $D\colon \mathbf{I} \to \mathbf{A}$ is called a *concrete limit* of D in $(\mathbf{A}, U)$ provided that it is preserved by U.

(2) A *concrete category* $(\mathbf{A}, U)$ *has (small) concrete limits,* resp. *concrete products,* etc., if and only if $\mathbf{A}$ has (small) limits, resp. products, etc., and U preserves them.

Proposition. *(13.14) A concrete category has small concrete limits if and only if it has concrete products and concrete equalizers.*

Proposition. *(13.15) If $(\mathbf{A}, U)$ is a concrete category and $D\colon \mathbf{I} \to \mathbf{A}$ is a diagram, then $\mathcal{L} = (L \xrightarrow{\ell_i} D_i)_{i \in Ob(\mathbf{I})}$ is a concrete limit in $(\mathbf{A}, U)$ if and only if $U(\mathcal{L})$ is a limit of $U \circ D$ and $\mathcal{L}$ is an initial source in $(\mathbf{A}, U)$.*

3.13.2 Lifting of limits

Definition. (13.17) A functor $F\colon \mathbf{A} \to \mathbf{B}$ is said to

(1) *lift limits (uniquely)* provided that for every diagram $D\colon \mathbf{I} \to \mathbf{A}$ and every limit $\mathcal{L}$ of $F \circ D$ there exists a (unique) limit $\mathcal{L}'$ of D with $F(\mathcal{L}') = \mathcal{L}$,

(2) *create limits* provided that for every diagram $D\colon \mathbf{I} \to \mathbf{A}$ and every limit $\mathcal{L}$ of $F \circ D$ there exists a unique source $\mathcal{S} = (L \xrightarrow{f_i} D_i)$ in $\mathbf{A}$ with $F(\mathcal{S}) = \mathcal{L}$, and that, moreover, $\mathcal{S}$ is a limit of D.

Similarly, one has lifts small limits, lifts products, creates equalizers, creates finite limits, etc.

Theorem. *(13.19) If a functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ lifts limits and $\mathbf{B}$ is (strongly) complete, then $\mathbf{A}$ is (strongly) complete and F preserves small limits (and arbitrary intersections).*

Proposition. *(13.21) For functors $F\colon \mathbf{A} \to \mathbf{B}$ the following conditions are equivalent:*

(1) F lifts limits uniquely,

(2) F lifts limits and is amnestic.

Definition. (13.22) A functor $F\colon \mathbf{A} \to \mathbf{B}$ is said to

(1) *reflect limits* provided that for each diagram $D\colon \mathbf{I} \to \mathbf{A}$ an $\mathbf{A}$-source $\mathcal{S} = (A \xrightarrow{f_i} D_i)_{i \in Ob(\mathbf{I})}$ is a limit of D whenever $F(\mathcal{S})$ is a limit of $F \circ D$,

(2) *detect limits* provided that a diagram $D\colon \mathbf{I} \to \mathbf{A}$ has a limit whenever $F \circ D$ has one.

Similarly one has reflect equalizers, detect products, etc.

Proposition. *(13.24) A functor that reflect equalizers is faithful.*

Proposition. *(13.25) For any functor $F\colon \mathbf{A} \to \mathbf{B}$ the following conditions are equivalent:*

(1) F creates limits,

(2) F lifts limits uniquely and reflects limits,

(3) F lifts limits, is faithful and amnestic, and reflects isomorphisms in the sense that whenever that Ff is a $\mathbf{B}$-isomorphism, then f is an $\mathbf{A}$-isomorphism.

Proposition. *(13.27) A full reflective subcategory $\mathbf{A}$ of $\mathbf{B}$ is limit-closed (closed under the formation of limits) in $\mathbf{B}$ if and only if $\mathbf{A}$ is isomorphism-closed in $\mathbf{B}$.*

Corollary. *(13.28) If a category has certain limits, then so does each of its isomorphism-closed full reflective subcategories.*

Isomorphism-closed full reflective subcategories $\mathbf{A}$ of $\mathbf{B}$ usually fail to be colimit-closed. However, the following proposition shows that the associated inclusion functor detect colimits.

Proposition. *(13.30) Let $\mathbf{A}$ be a full subcategory of $\mathbf{B}$ with embedding $E\colon \mathbf{A} \to \mathbf{B}$ and let $D\colon \mathbf{I} \to \mathbf{A}$ be a diagram. If $\mathcal{C} = (D_i \xrightarrow{c_i} C)$ is a colimit of $E \circ D$, and if $C \xrightarrow{r} A$ is an $\mathbf{A}$-reflection arrow for C, then $\mathcal{C}' = r \circ \mathcal{C}$ is a colimit of D.*

Example. (13.31(1)) **HComp** is a full reflective subcategory of **Top**. The construction of coproducts in **HComp** given in Example 10.67(5) is a special case of the above result.

Corollary. *(13.32) Embeddings of full reflective subcategories detect colimits.*

Proposition. *(13.34) If a functor $F\colon \mathbf{A} \to \mathbf{B}$ preserves limits, then the following conditions are equivalent:*

(1) F lift limits (uniquely),

(2) F detects limits and is (uniquely) transportable.

3.13.3 Creation and reflection of isomorphisms

Definition. (13.35) A functor $G\colon \mathbf{A} \to \mathbf{B}$ is said to

(1) *create isomorphism* provided that whenever $h\colon X \to GA$ is a G-structured $\mathbf{B}$-isomorphism, there exists precisely one $\mathbf{A}$-morphism $\hat{h}\colon B \to A$ with $G(\hat{h}) = h$, and, moreover, $\hat{h}$ is an isomorphism,

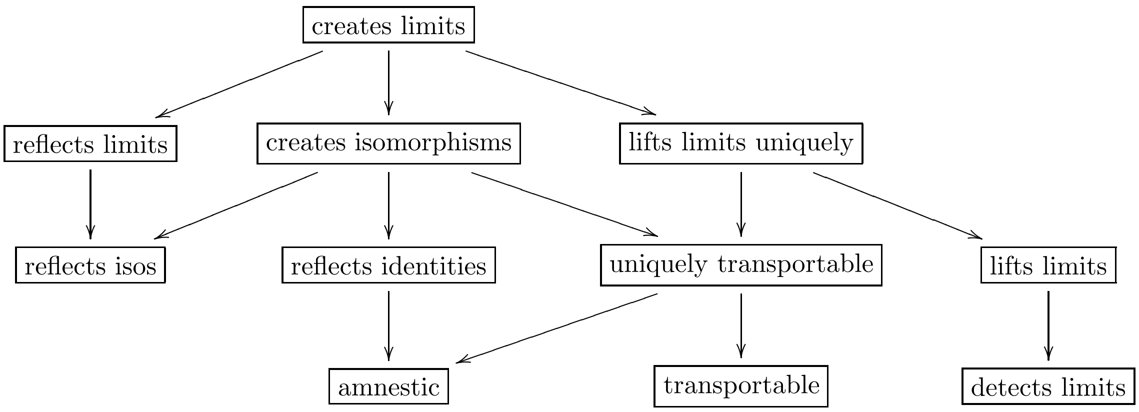
(2) *reflect isomorphism* provided that an $\mathbf{A}$-morphism f is an $\mathbf{A}$-isomorphism whenever Gf is a $\mathbf{B}$-isomorphism.

Proposition. *(13.36)*

(1) If G creates (resp. reflects) limits, then G creates (resp. reflects) isomorphisms.

(2) G creates isomorphism if and only if G reflects isomorphism and is uniquely transportable.

(3) If G creates isomorphisms, then G reflects identities.



4 Factorization structures

4.14 Factorization structures for morphisms

Definition. (14.1) Let E and M be classes of morphisms in a category $\mathbf{A}$.

(E, M) is called a *factorization structure for morphisms* in $\mathbf{A}$ and $\mathbf{A}$ is called (E, M)-*structured* provided that

(1) each of E and M is closed under composition with isomorphism,

(2) $\mathbf{A}$ has (E, M)-*factorizations (of morphisms);* i.e., each morphism f in $\mathbf{A}$ has a factorization $f = m \circ e$, with $e \in E$ and $m \in M$, and

(3) $\mathbf{A}$ has the *unique* (E, M)-*diagonalization property*; i.e., for each commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "m"'] & D \end{tikzcd} \qquad (*)$$

with $e \in E$ and $m \in M$ there exists a unique *diagonal*, i.e. a morphism d such that the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, "d"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

commutes.

Example. (14.2(5)) **Top** has a proper class (even an illegitimate conglomerate) of factorization structures for morphisms. Each of
(Epi, RegMono)=(surjection, embedding),
(RegEpi, Mono)=(quotient, injection),
(dense, closed embedding), and
(front-dense, front-closed embedding),
is a factorization structure for morphisms in **Top**, but (Epi, Mono) is not.

Proposition. *(14.3)* $\mathbf{A}$ *is* (E,M)*-structured if and only if* $\mathbf{A}^{op}$ *is* (M,E)*-structured.*

Proposition. *(14.4) If* $\mathbf{A}$ *is* (E,M)*-structured, then* (E,M)*-factorizations are essentially unique, i.e.,*

(1) if $A \xrightarrow{e_i} C_i \xrightarrow{m_i} B$ *are* (E,M)*-factorizations of* $A \xrightarrow{f} B$ *for* $i = 1,2$*, then there exists a (unique) isomorphism* h*, such that the diagram*

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[d, "e_2"'] & C_1 \arrow[d, "m_1"] \arrow[dl, "h"'] \\ C_2 \arrow[r, "m_2"'] & B \end{tikzcd}$$

commutes.

(2) If $A \xrightarrow{f} B = A \xrightarrow{e} C \xrightarrow{m} B$ *is an* (E,M)*-factorization and* $C \xrightarrow{h} D$ *is an isomorphism, then* $A \xrightarrow{f} B = A \xrightarrow{h \circ e} D \xrightarrow{m \circ h^{-1}} B$ *is also an* (E,M)*-factorization.*

Lemma. *(14.5) Let* $\mathbf{A}$ *be* (E,M)*-structured and* $e \in E$ *and* $m \in M$*. If the diagram*

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "id"'] & \bullet \arrow[d, "m"] \arrow[dl, "d"'] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

commutes, then e *is an isomorphism and* $f \in M$*.*

Proposition. *(14.6) If* $\mathbf{A}$ *is* (E,M)*-structured, then the following hold:*

(1) $E \cap M = Iso(\mathbf{A})$*,*

(2) each of E *and* M *is closed under composition,*

(3) E *and* M *determine each other via the diagonalization-property;*[3] *in particular, a morphism* M *belongs to* M *if and only if for each commutative square of the form (*) (see Definition 14.1) with* $e \in E$ *there exists a diagonal.*

Proposition. *(14.7) If* E *and* M *are classes of morphisms in* $\mathbf{A}$*, then* $\mathbf{A}$ *is* (E,M)*-structured if and only if the following conditions are satisfied:*

(1) $Iso(\mathbf{A}) \subseteq E \cap M$*,*

(2) each of E *and* M *is closed under composition,*

(3) $\mathbf{A}$ *has the* (E,M)*-factorization property, unique in the sense that for any pair of* (E,M)*-factorizations* $m_1 \circ e_1 = f = m_2 \circ e_2$ *of a morphism* f *there exists a unique isomorphism* h*, such that the diagram*

$$\begin{tikzcd} \bullet \arrow[r, "e_1"] \arrow[d, "e_2"'] & \bullet \arrow[d, "m_1"] \arrow[dl, "h"'] \\ \bullet \arrow[r, "m_2"'] & \bullet \end{tikzcd}$$

commutes.

[3]Here the diagonal needn't be required to be unique.

Proposition. *(14.9) Let* $\mathbf{A}$ *be* (E, M)*-structured and let* $f \circ g \in M$.

(1) If $f \in M$*, then* $g \in M$.

(2) If f *is a monomorphism, then* $g \in M$.

(3) If g *is a retraction, then* $f \in M$.

4.14.1 Relationship to special morphisms

Proposition. *(14.10) If* $\mathbf{A}$ *is* (E, M)*-structured, then the following hold:*

(1) $E \subseteq Epi(\mathbf{A})$ *implies that* $ExtrMono(\mathbf{A}) \subseteq M$.

If, moreover, $\mathbf{A}$ *has (Epi, Mono)-factorizations, then*

(2) $Epi(\mathbf{A}) \subseteq E$ *implies that* $M \subseteq ExtrMono(\mathbf{A})$.

(3) $Epi(\mathbf{A}) = E$ *implies that* $M = ExtrMono(\mathbf{A})$.

Proposition. *(14.11) If* $\mathbf{A}$ *is* (E, M)*-structured and has products of pairs, then the following conditions are equivalent:*

(1) $E \subseteq Epi(\mathbf{A})$,

(2) $ExtrMono(\mathbf{A}) \subseteq M$,

(3) $Sect(\mathbf{A}) \subseteq M$.

(4) for each object A*, the diagonal morphism* $\Delta_A = \langle id_A, id_A \rangle$ *belongs to* M,

(5) $f \circ g \in M$ *implies that* $g \in M$,

(6) $f \circ e \in M$ *and* $e \in E$ *imply that* $e \in Iso(\mathbf{A})$,

(7) $M = \{f \in Mor(\mathbf{A}) | f = g \circ e$ *and* $e \in E$ *imply that* $e \in Iso(\mathbf{A})\}$.

Proposition. *(14.12) If* $\mathbf{A}$ *is (E, Mono)-structured and has products of pairs, then* $E = ExtrEpi(\mathbf{A})$.

Proposition. *(14.14) If* $\mathbf{A}$ *has (RegEpi, Mono)-factorizations, then the following hold:*

(1) $\mathbf{A}$ *is (RegEpi, Mono)-structured,*

(2) RegEpi($\mathbf{A}$*)=ExtrEpi(*$\mathbf{A}$*),*

(3) the class of regular epimorphisms in $\mathbf{A}$ *is closed under composition,*

(4) if $f \circ g$ *is a regular epimorphism, then so is* f.

4.14.2 Relationship to limits

Proposition. *(14.15) If* $\mathbf{A}$ *is (E,M)-structured, then* M *is closed under the formation of products and pullbacks, and* $M \cap Mono(\mathbf{A})$ *is closed under the formation of intersections.*[4]

Lemma (Factorization Lemma). *(14.16) Let* $\mathbf{A}$ *have intersections and equalizers, let* $C \xrightarrow{f} D$ *be an* $\mathbf{A}$*-morphism, and let* $M \subseteq Mono(\mathbf{A})$ *satisfy the following conditions:*

(a) intersection of families of M*-subobjects of* D *belong to* M*,*

(b) if $f = \hat{m} \circ g \circ h$ *with* $\hat{m} \in M$ *and* $g \in RegMono(\mathbf{A})$*, then* $\hat{m} \circ g \in M$*.*

Then there exists $m \in M$ *and* $e \in Epi(\mathbf{A})$*, such that*

(1) $f = m \circ e$*,*

(2) if $f = \overline{m} \circ g$ *with* $m \in M$*, then there exists a diagonal* d *that makes the diagram*

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "g"'] & \bullet \arrow[d, "m"] \arrow[dl, "d"'] \\ \bullet \arrow[r, "\overline{m}"'] & \bullet \end{tikzcd}$$

commute,

(3) if $e = \overline{m} \circ g$*, where* $m \circ \overline{m} \in M$*, then* $\overline{m} \in Iso(\mathbf{A})$*.*

Theorem. *(14.17) If* $\mathbf{A}$ *has finite limits and intersections, then* $\mathbf{A}$ *is (ExtrEpi, Mono)-structured.*

Proposition. *(14.18) If* $\mathbf{A}$ *has the (Epi, ExtrMono)-diagonalization property, then the class of extremal monomorphisms in* $\mathbf{A}$ *is closed under composition and intersections.*

Theorem. *(14.19) If* $\mathbf{A}$ *has equalizers and intersections, then* $\mathbf{A}$ *is (Epi, ExtrMono)-structured.*

Corollary. *(14.20) In a category with equalizers and intersections the class of extremal monomorphisms is the smallest class of morphisms that contains all regular monomorphisms and is closed under composition and intersections.*

Corollary. *(14.21) Every strongly complete category is (ExtrEpi, Mono)-structured and (Epi, ExtrMono)-structured.*

Proposition. *(14.22) A category with pullbacks and coequalizers is (RegEpi, Mono)-structured if and only if regular epimorphisms are closed under composition.*

4.15 Factorization structures for sources

Definition. (15.1) Let E be a class of morphisms and let $\mathbf{M}$ be a conglomerate of sources in a category $\mathbf{A}$. $(E, \mathbf{M})$ is called a *factorization structure* on $\mathbf{A}$, and $\mathbf{A}$ is called an $(E, \mathbf{M})$-*category* provided that

(1) each of E and $\mathbf{M}$ is closed under composition with isomorphisms,

(2) $\mathbf{A}$ has $(E, \mathbf{M})$-*factorizations (of sources)*; i.e. each source $\mathcal{S}$ in $\mathbf{A}$ has a factorization $\mathcal{S} = \mathcal{M} \circ e$ with $e \in E$ and $\mathcal{M} \in \mathbf{M}$, and

[4] In fact, M is closed under the formation of multiple pullbacks (cf. Exercise 11L).

(3) **A** has the *unique* $(E, \mathbf{M})$-*diagonalization property;* i.e. whenever $A \xrightarrow{e} B$ and $A \xrightarrow{f} C$ are **A**-morphism with $e \in E$ and $\mathcal{S} = (g_i)_I$ and $\mathcal{M} = (m_i)_I$ are sources in **A** with $\mathcal{M} \in \mathbf{M}$, such that $\mathcal{M} \circ f = \mathcal{S} \circ e$, then there exists a unique *diagonal*, i.e. a morphism $B \xrightarrow{d} C$ such that for each $i \in I$ the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g_i"] \arrow[dl, "d"'] \\ C \arrow[r, "m_i"'] & D_i \end{tikzcd}$$

commutes.

Top has a proper class (even an illegitimate conglomerate) of factorization structures. In particular, **Top** is an (Epi, ExtrMono-Source)-category, an (ExtrEpi, Mono-Source)-category, and a (Bimorphism, Initial Mono-Source)-category.

Theorem. *(15.4) If* **A** *is an* $(E, \mathbf{M})$*-category, then* $E \subseteq Epi(\mathbf{A})$*.*

Proposition. *(15.5) If* **A** *is an* $(E, \mathbf{M})$*-category, then the following hold:*

(1) $(E, \mathbf{M})$*-factorizations are essentially unique,*

(2) $E \subseteq Epi(\mathbf{A})$ *and* $ExtrMono - Source(\mathbf{A}) \subseteq \mathbf{M}$*,*

(3) $E \cap \mathbf{M} = Iso(\mathbf{A})$*,*

(4) each of E *and* **M** *is closed under composition,*

(5) if $f \circ g \in E$ *and* $g \in Epi(\mathbf{A})$*, then* $f \in E$*,*

(6) if $f \circ g \in E$ *and* $f \in Sect(\mathbf{A})$*, then* $g \in E$*,*

(7) if $(\mathcal{S}_i) \circ \mathcal{S} \in \mathbf{M}$*, then* $\mathcal{S} \in \mathbf{M}$*,*

(8) if a subsource of $\mathcal{S}$ *belongs to* **M***, then* $\mathcal{S}$ *belongs to* **M***,*

(9) E *and* **M** *determine each other via the diagonalization-property; moreover,*

(a) a source belongs to **M** *if and only if every* E*-morphism through which it factors is an isomorphism,*

(b) if **M** *consists of mono-sources only, then a morphism* f *belongs to* E *if and only if* $f = m \circ g$ *with* $m \in \mathbf{M}$ *implies than* $m \in Iso(\mathbf{A})$*.*

4.15.1 Relationship to special morphisms and special limits

Proposition. *(15.6) If* **A** *is a* $(RegEpi, \mathbf{M})$*-category, then* **M** *contains all mono-sources of* **A***.*

Proposition. *(15.7) For* $(E, \mathbf{M})$*-categories* **A***, the following are equivalent:*

(a) $\mathbf{M} \subseteq$ *Mono-Source*$(\mathbf{A})$*,*

(b) **A** *has coequalizers and* $RegEpi(\mathbf{A}) \subseteq E$*.*

Proposition. *(15.8) For* $(E, \mathbf{M})$*-categories* **A** *the following hold:*

(1) if $\mathbf{M} = \mathit{Mono\text{-}Source}(\mathbf{A})$*, then* $E = \mathit{ExtrEpi}(\mathbf{A})$*,*

(2) if $\mathbf{M} = \mathit{ExtrMono\text{-}Source}(\mathbf{A})$*, then* $E = \mathit{Epi}(\mathbf{A})$*,*

(3) if $E = \mathit{Epi}(\mathbf{A})$*, then the following conditions are equivalent:*

   (a) $\mathbf{M} = \mathit{ExtrMono\text{-}Source}(\mathbf{A})$*,*

   (b) $\mathbf{A}$ *has coequalizers,*

(4) if $E = \mathit{ExtrEpi}(\mathbf{A})$*, then the following conditions are equivalent:*

   (a) $\mathbf{M} = \mathit{Mono\text{-}Source}(\mathbf{A})$*,*

   (b) $\mathbf{A}$ *has coequalizers.*

4.15.2 Existence of factorization structures

Theorem. *(15.10) Every category that has (Epi, Mono-Source)-factorizations is an (ExtrEpi, Mono-Source)-category.*

Definition. (15.12) A category is said to *have regular factorizations* provided that it is (RegEpi, Mono-Source)-factorizable.

Proposition. *(15.13) If a category has regular factorizations, then it is a (RegEpi, Mono-Source)-category.*

Theorem. *(15.14) If* E *is a class of morphisms in* $\mathbf{A}$*, then* $\mathbf{A}$ *is an* $(E, \mathbf{M})$*-category of some* $\mathbf{M}$ *if and only if the following conditions are satisfied:*

(1) $\mathit{Iso}(\mathbf{A}) \subseteq E \subseteq \mathit{Epi}(\mathbf{A})$*,*

(2) E *is closed under composition,*

(3) for every $A \xrightarrow{e} B$ *in* E *and every morphism* $A \xrightarrow{f} C$ *there exists a pushout square*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "\overline{f}"] \\ C \arrow[r, "\overline{e}"'] & D \end{tikzcd}$$

for which $\overline{e} \in E$*,*

(4) for every source $(A \xrightarrow{e_i} A_i)_I$ *that consists of* E*-morphisms, there exists a cointersection*

$$A \xrightarrow{e} B = A \xrightarrow{e_i} A_i \xrightarrow{p_i} B$$

for which $e \in E$*.*

Corollary. *(15.15) Let* $\mathbf{A}$ *be a category with pushouts and cointersections. Then a class* E *of* $\mathbf{A}$*-morphisms is a part of a factorization structure on* $\mathbf{A}$ *if and only if* $\mathit{Iso}(\mathbf{A}) \subseteq E \subseteq \mathit{Epi}(\mathbf{A})$ *and* E *is closed under composition, pushouts, and cointersection.*

Corollary. *(15.16)*

(1) $\mathbf{A}$ *is an* $(\mathit{Epi}, \mathbf{M})$*-category for some* $\mathbf{M}$ *if and only if* $\mathbf{A}$ *has cointersections and has a pushout for every 2-source of the form* $\bullet \xleftarrow{f} \bullet \xrightarrow{e} \bullet$ *with epimorphic* e*.*

(2) $\mathbf{A}$ *is an (Epi, ExtrMono-Source)-category if and only if A has cointersections, pushouts for every 2-source of the form* $\bullet \xleftarrow{f} \bullet \xrightarrow{e} \bullet$ *with epimorphic e, and has coequalizers.*

Corollary. *(15.17) Every strongly cocomplete category is an (Epi, ExtrMono-Source)-category.* ∎

4.15.3 Extensions of factorization structures

Proposition. *(15.19)*

(1) *If $\mathbf{A}$ has products, then every factorization structure (E, M) for morphisms can be uniquely extended to a factorization structure $(E, \mathbf{M})$ for small sources.*

(2) *Conversely, if $\mathbf{A}$ has an initial object and each factorization structure (E, M) for morphisms on $\mathbf{A}$ can be extended to a factorization structure $(E, \mathbf{M})$ for small sources then $\mathbf{A}$ has products.*

Proposition. *(15.20) If $(E, \mathbf{M})$ is a factorization structure for small sources on $\mathbf{A}$ and $\mathbf{A}$ is E-co-wellpowered, then the following conditions are equivalent:*

(1) *$(E, \mathbf{M})$ can be uniquely extended to a factorization structure $(E, \mathbf{N})$ on $\mathbf{A}$,*

(2) *$E \subseteq Epi(\mathbf{A})$,*

(3) *$Sect(\mathbf{A}) \subseteq \mathbf{M}$,*

(4) *for each object A the 2-source (id_A, id_A) belongs to $\mathbf{M}$,*

(5) *whenever a subsource of a small source $\mathcal{S}$ belongs to $\mathbf{M}$, then so does $\mathcal{S}$.*

Corollary. *(15.21) In a co-wellpowered category $\mathbf{A}$ with products, every factorization structure (E, M) for morphisms with $E \subseteq Epi(\mathbf{A})$ can be uniquely extended to a factorization structure $(E, \mathbf{M})$ for sources.*

Proposition. *(15.24) In an $(E, \mathbf{M})$-category every factorization structure (C, N) for morphisms with $C \subseteq E$ can be uniquely extended to a factorization structure $(C, \mathbf{N})$ for sources.*

4.15.4 Factorization structures and limits

Theorem. *(15.25) Let $\mathbf{A}$ be a strongly complete, extremally co-wellpowered. Then the following hold:*

(1) $\mathbf{A}$ *is an (ExtrEpi, Mono-Source)-category.*

(2) *If $\mathbf{A}$ is co-wellpowered, then $\mathbf{A}$ is an (Epi, ExtrMono-Source)-category.*

(3) *If in $\mathbf{A}$ regular epimorphisms are closed under composition, then $\mathbf{A}$ is a (RegEpi, Mono-Source)-category.*

4.16 E-reflective subcategories

Definition. (16.1) Let $\mathbf{B}$ be a category and E be a class of $\mathbf{B}$-morphisms. An isomorphism-closed, full subcategory $\mathbf{A}$ of $\mathbf{B}$ is called *E-reflective* in $\mathbf{B}$ provided that each $\mathbf{B}$-object has an $\mathbf{A}$-reflection arrow in E. In particular, we use the terms *epireflective* (resp. *monoreflective*, *bireflective*) in case E is the class of all epimorphisms (resp. monomorphisms, bimorphisms) in $\mathbf{B}$. Likewise, regular epireflective, extremal epireflective.

HComp is reflective but not epireflective in **Top**, even though **HComp** is epireflective in **Haus** an **Haus** is epireflective in **Top**.

Proposition. *(16.3) Every monoreflective subcategory of* $\mathbf{B}$ *is bireflective in* $\mathbf{B}$*.*

Proposition. *(16.4) Every coreflective isomorphism-closed full subcategory of* $\mathbf{B}$ *that contains a* $\mathbf{B}$*-separator is bicoreflective in* $\mathbf{B}$*.*

Definition. (16.7) Let $\mathbf{M}$ be a conglomerate of sources in a category $\mathbf{B}$. A subcategory $\mathbf{A}$ of $\mathbf{B}$ is said to be *closed under the formation of* $\mathbf{M}$*-sources* provided that whenever $(B \xrightarrow{f_i} A_i)_I$ is a source in $\mathbf{M}$ such that all A_i belong to $\mathbf{A}$, then B belongs to $\mathbf{A}$.

Theorem. *(16.8) If* $\mathbf{A}$ *is a full subcategory of an* $(E, \mathbf{M})$*-category* $\mathbf{B}$*, then the following conditions are equivalent:*

(1) $\mathbf{A}$ *is E-reflective in* $\mathbf{B}$*.*

(2) $\mathbf{A}$ *is closed under the formation of* $\mathbf{M}$*-sources in* $\mathbf{B}$*.*

In the case that $\mathbf{B}$ *has products and is E-co-wellpowered, the above conditions are equivalent to:*

(3) $\mathbf{A}$ *is closed under the formation of products and* $\mathbf{M}$*-subobjects*[5] *in* $\mathbf{B}$*.*

Corollary. *(16.9) A full subcategory of a co-wellpowered, strongly complete category* $\mathbf{B}$ *is epireflective in* $\mathbf{B}$ *if and only if it is closed under the formation of products and extremal subobjects in* $\mathbf{B}$*.*

4.16.1 Subcategories defined by equations and implications

Definition. (16.12)

(1) (Regular) epimorphisms are called *(regular) implications.*

(2) An object Q *satisfies the implication* $A \xrightarrow{e} B$ provided that Q is $\{e\}$-injective (i.e., provided that for each morphism $A \xrightarrow{f} Q$ there exists a morphism $B \xrightarrow{\overline{f}} Q$ with $f = \overline{f} \circ e$).

(3) A full subcategory $\mathbf{A}$ of $\mathbf{B}$ is called *implicational* provided that there exists a class C of implications in $\mathbf{B}$ such that $\mathbf{A}$ consists precisely of those $\mathbf{B}$-objects that satisfy each implication in C. Construct that are concretely isomorphic to implicational subconstructs of $\mathbf{Alg}(\Omega)$ for some Ω are called *finitary quasivarieties.*

In case C can be chosen to be a subclass of some class E of implications in $\mathbf{B}$, $\mathbf{A}$ is called *E-implicational.*

[5] An $\mathbf{M}$-subobject is simply a singleton $\mathbf{M}$-source. It need not be a monomorphism.

16.13(2): T_1-spaces form an implicational subcategory of **Top**. If P is the Sierpinski space and P' is a singleton space, then a topological space is T_1 if and only if it satisfies the implication $P \to P'$.

Theorem. *(16.14) A full subcategory of an $(E, \mathbf{M})$-category $\mathbf{B}$ is E-implicational if and only if it is E-reflective in $\mathbf{B}$.*

Definition. (16.16)

(1) Let E be a class of epimorphism in a category $\mathbf{B}$. An implication in E with projective domain is called an *E-equation.* Regular epimorphic equations are called *regular equations.* A full subcategory $\mathbf{A}$ of $\mathbf{B}$ is called E-equational provided that there exists a class C of E-equations in $\mathbf{B}$ such that $\mathbf{A}$ consists precisely of those $\mathbf{B}$-objects that satisfy each E-equation in C.

(2) Let $\mathbf{B}$ be a construct. Regular implications with free domain are called *equations.* A full subcategory $\mathbf{A}$ of $\mathbf{B}$ is called *equational* provided that it can be defined as above by a class C of equations in $\mathbf{B}$.

(3) Construct that are concretely isomorphic to equational subconstructs of $\mathbf{Alg}(\Omega)$ for some Ω are called are called *finitary varieties.*

Theorem. *(16.17) Let $\mathbf{B}$ be an $(E, \mathbf{M})$-category with enough E-projectives (9.9 dual). Then the following condition are equivalent for any full subcategory $\mathbf{A}$ of $\mathbf{B}$:*

(1) $\mathbf{A}$ is E-equational in $\mathbf{B}$.

(2) $\mathbf{A}$ is closed under the formation of $\mathbf{M}$-sources and E-quotients in $\mathbf{B}$

In the case that $\mathbf{B}$ has products and is E-co-wellpowered, the above conditions are equivalent to:

(3) $\mathbf{A}$ is closed under the formation of products, $\mathbf{M}$-subobjects, and E-quotients in $\mathbf{B}$.

Theorem. *(16.18) Let $\mathbf{B}$ be a fibre-small, transportable, complete construct, that has free objects and for which the surjective morphisms are precisely the regular epimorphisms. Then for full subconstructs $\mathbf{A}$ of $\mathbf{B}$ the following conditions are equivalent:*

(1) $\mathbf{A}$ is equational in $\mathbf{B}$,

(2) $\mathbf{A}$ is regular-equational in $\mathbf{B}$,

(3) $\mathbf{A}$ is regular epireflective and closed under the formation of regular quotients (= homomorphic images) in $\mathbf{B}$,

(4) $\mathbf{A}$ is closed under the formation of products, subobjects, and homomorphic images in $\mathbf{B}$.

Corollary. *(16.19) For full subcategories $\mathbf{A}$ of $\mathbf{Alg}(\Omega)$, the following hold:*

(1) $\mathbf{A}$ is implicational in $\mathbf{Alg}(\Omega)$ if and only if $\mathbf{A}$ is closed under the formation of products and subalgebras.

(2) $\mathbf{A}$ is equational in $\mathbf{Alg}(\Omega)$ if and only if $\mathbf{A}$ is closed under the formation of products, subalgebras and homomorphic images.

4.16.2 E-reflective hulls

Proposition. *(16.20) For $(E,\mathbf{M})$-categories $\mathbf{B}$, the following hold:*

1. *The intersection of any conglomerate of E-reflective subcategories of $\mathbf{B}$ is E-reflective in $\mathbf{B}$.*
2. *For every full subcategory $\mathbf{A}$ of $\mathbf{B}$ there exists a smallest E-reflective subcategory of $\mathbf{B}$ that contains $\mathbf{A}$.*

Definition. (16.21) If $\mathbf{A}$ is a full subcategory of an $(E,\mathbf{M})$-subcategory $\mathbf{B}$, then the smallest E-reflective subcategory of $\mathbf{B}$ that contains $\mathbf{A}$ is called the *E-reflective hull* of $\mathbf{A}$ in $\mathbf{B}$.

Proposition. *(16.22) If $\mathbf{A}$ is a full subcategory of an $(E,\mathbf{M})$-category $\mathbf{B}$, then $\mathbf{B}$-objects B belongs to the E-reflective hull of $\mathbf{A}$ in $\mathbf{B}$ if and only if there exists a source $(B \xrightarrow{f_i} A_i)_I$ in $\mathbf{M}$ with all A_i in $\mathbf{A}$.*

Epireflective hulls in **Top**: $\mathbf{Top}_0$ of S, **Tych** of [0,1]; in **Haus**: **HComp** of [0,1].

Proposition (Reflectors as Composites of Epireflectors). *(16.24) If $\mathbf{A}$ is a full reflective subcategory of an (Epi, Mono-Source)-factorizable category $\mathbf{B}$, and if $\mathbf{C}$ is the extremally epireflective hull of $\mathbf{A}$ in $\mathbf{B}$, then $\mathbf{A}$ is epireflective in $\mathbf{C}$ and $\mathbf{C}$ is epireflective in $\mathbf{B}$.*

(16D) Full reflective subcategory of **Top** is co-wellpowered if and only if its epireflective hull in **Top** is co-wellpowered.

4.17 Factorization structures for functors

Definition. (17.1) Let $G\colon \mathbf{A} \to \mathbf{X}$ be a functor. A *G-structured source* $\mathcal{S}$ is a pair $(X,(f_i,A_i)_{i\in I})$ that consists of an $\mathbf{X}$-objects X and a family of G-structured arrows $X \xrightarrow{f_i} GA_i$ with domain X, indexed by some class I.

X is called the *domain* of $\mathcal{S}$ and the family $(A_i)_{i\in I}$ is called the *codomain* of $\mathcal{S}$.

Definition. (17.3) Let $G\colon \mathbf{A} \to \mathbf{X}$ be a functor, let E be a class of G-structured arrows, and let $\mathbf{M}$ be a conglomerate of $\mathbf{A}$-sources. $(E,\mathbf{M})$ is called a *factorization structure* for G, and G is called an *$(E,\mathbf{M})$-functor* provided that

(1) E and $\mathbf{M}$ are closed under composition with isomorphisms.

(2) G has *$(E,\mathbf{M})$-factorizations,* i.e., for each G-structured source $(X \xrightarrow{f_i} GA_i)_I$ there exists $X \xrightarrow{e} GA$ in E and $\mathcal{M} = (A \xrightarrow{m_i} A_i)_I \in \mathbf{M}$ such that

$$X \xrightarrow{f_i} GA_i = X \xrightarrow{e} GA \xrightarrow{Gm_i} GA_i \text{ for each } i \in I.$$

(3) G has the *unique $(E,\mathbf{M})$-diagonalization property,* i.e., whenever $X \xrightarrow{f} GA$ and $X \xrightarrow{e} GB$ are G-stuctured arrows with $(e,B) \in E$ and $\mathcal{M} = (A \xrightarrow{m_i} A_i)_I$ and $\mathcal{S} = (B \xrightarrow{f_i} A_i)_I$ are $\mathbf{A}$-sources with $\mathcal{M} \in \mathbf{M}$, such that $(Gm_i)\circ f = (Gf_i)\circ e$ for each $i \in I$, then there exists a unique *diagonal,* i.e. an $\mathbf{A}$-morphism $B \xrightarrow{d} A$ with $f = Gd \circ e$ and $\mathcal{S} = \mathcal{M}\circ d$, which will be expressed (imprecisely) by saying that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f"'] & GB \arrow[d, "Gf_i"] \arrow[dl, "Gd"'] \\ GA \arrow[r, "Gm_i"'] & GA_i \end{tikzcd}$$

Theorem. *(17.6) If G is an $(E, \mathbf{M})$-functor, then each member of E is generating.*

Theorem. *(17.7) If $G\colon \mathbf{A} \to \mathbf{B}$ is an $(E, \mathbf{M})$-functor, then the following hold:*

(1) $(E, \mathbf{M})$-factorizations are essentially unique,

(2) $\mathbf{M}$ determines E via the unique diagonalization property,

(3) if $\mathbf{A}$ is an $(\tilde{E}, \mathbf{M})$-category, $(e, A) \in E$ and $A \xrightarrow{\tilde{e}} B \in \tilde{E}$, then $((G\tilde{e}) \circ e, B) \in E$.

Proposition. *(17.9) If G is an (E, Mono-Source)-functor, then E consists precisely of those structured arrows that are extremally generating.*

Proposition. *(17.10) If a functor G has (Generating, Mono-Source)-factorizations, then G is an (Extremally Generating, Mono-Source)-functor.*

4.17.1 Factorization structures and limits

Theorem. *(17.11) Let $\mathbf{A}$ be a strongly complete category and let $\mathbf{A} \xrightarrow{G} \mathbf{X}$ be a functor that preserves strong limits.*

(1) If G is extremally co-wellpowered or if $\mathbf{A}$ has a coseparator, then G is an (ExtrGen, Mono-Source)-functor.

(2) If G is faithful and concretely co-wellpowered, then G is a (ConGen, Initial Mono-Source)-functor

Proposition. *(17.12) If G-structure 2-sources have (Generating, $-$)-factorizations, then G preserves mono-sources.*

Proposition. *(17.13) If $G\colon \mathbf{A} \to \mathbf{B}$ is a functor such that G-structured 2-sources have (Generating, Mono-Source)-factorizations, then the following conditions are equivalent:*

(1) G reflects isomorphism,

(2) each mono-source is G-initial,

(3) G is faithful and reflects extremal epimorphisms,

(4) G reflects limits,

(5) G reflects equalizers.

Observe that in the diagram below, all arrows indicate implications that hold without any assumptions, whereas those labeled 1, 2, 3 and 4 are equivalences under the hypothesis of the preceding Proposition.

$$
\begin{tikzcd}
\boxed{\text{monomorphisms are initial}} \arrow[d, "1"] & \\
\boxed{\text{reflect limits}} \arrow[d, "2"] & \\
\boxed{\text{reflects equalizers}} \arrow[r] \arrow[d, "3"] & \boxed{\text{faithful}} \\
\boxed{\text{reflects isomorphisms}} & \boxed{\text{faithful and reflects extremal epimorphisms}} \arrow[l, "4"'] \arrow[u]
\end{tikzcd}
$$

Proposition. *(17.15) A functor $G\colon \mathbf{A} \to \mathbf{X}$ is faithful if and only if for each $\mathbf{A}$-object A the G-structured source $(GA \xleftarrow{id} GA \xrightarrow{id} GA)$ is (Generating, Initial Source)-factorizable.*

Corollary. *(17.16) If G-structured 2-sources are (Generating, Initial Source)-factorizable, then G is faithful.*

5 Adjoints and monads

5.18 Adjoint functors

Perhaps the most successful concept of category theory is that of *adjoint functor*. Adjoint functors occur frequently in many branches of mathematics and the "adjoint functor theorems" have a surprising range of applications

Definition. (18.1) A functor $G\colon \mathbf{A} \to \mathbf{B}$ is said to be *adjoint* for every $\mathbf{B}$-object B there exists a G-universal arrow with domain B.

Dual notion: *co-adjoint*

Examples: reflective, coreflective; The forgetful functor U of a concrete category $(\mathbf{A}, U)$ over $\mathbf{X}$ is adjoint if and only if for each $\mathbf{X}$-object X there exists a free object over X. The forgetful functors of the constructs **Rel**, **Top** and $\mathbf{Alg}(\Sigma)$ are both adjoint and co-adjoint (cf. 8.23 and 8.41). The forgetful functors of the constructs **CLat** and **CBoo** are neither adjoint nor co-adjoint.

Proposition. *(18.3) For a functor $G\colon \mathbf{A} \to \mathbf{B}$ the following conditions are equivalent:*

(1) G is adjoint,

(2) G has (Generating[6],$-$)-factorizations,

(3) G is an $(E, \mathbf{M})$-functor for some E and $\mathbf{M}$,

(4) G is a (Universal[7], Source)-functor.

[6] "Generating" in this context denotes the class of all generating G-structured arrows

[7] "Universal" in this context denotes the class of all universal G-structured arrows

Proposition. *(18.4) If* $\mathbf{A}$ *is* $(Epi, \mathbf{M})$*-category, then for any functor* $G\colon \mathbf{A} \to \mathbf{B}$ *the following are equivalent:*

(1) G *is adjoint,*

(2) G *is a (Generating,*$\mathbf{M}$*)-functor.*

5.18.1 Properties of adjoint functors

Proposition. *(18.5) Composition of adjoint functors is adjoint.*

Proposition. *(18.6) Adjoint functors preserve mono-sources.*

Corollary. *(18.7) Embeddings of reflective subcategories preserve and reflect mono-sources.*

Proposition. *(18.9) Adjoint functors preserve limits.*

Corollary. *(18.10) If* $(\mathbf{A}, U)$ *is a concrete category over* $\mathbf{X}$ *that has free objects, then the following hold:*

(1) all limits in $(\mathbf{A}, U)$ *are concrete,*

(2) U *preserves and reflects mono-sources,*

(3) if $(\mathbf{A}, U)$ *is fibre-small and transportable, then wellpoweredness of* $\mathbf{X}$ *implies wellpoweredness of* $\mathbf{A}$*.*

Proposition. *(18.11) If* $G\colon \mathbf{A} \to \mathbf{B}$ *is an adjoint functor and* $\mathbf{A}$ *is co-wellpowered, or extremally co-wellpowered, then so is* G*.*

5.18.2 Adjoint functor theorems

Theorem (Adjoint functor theorem). *(18.12) A functor* $G\colon \mathbf{A} \to \mathbf{B}$*, whose domain* $\mathbf{A}$ *is complete, is adjoint if and only if* G *satisfies the following conditions:*

(1) G *preserves small limits,*

(2) for each $\mathbf{B}$*-object* B *there exists a* G-solution set, *i.e., a set-indexed* G*-structured source* $(B \xrightarrow{f_i} GA_i)_I$ *through which each* G*-structured arrow factors (in the sense that given any* $B \xrightarrow{f} GA$*, there exists a* $j \in I$ *and* $A_j \xrightarrow{g} A$ *such that*

$$\begin{tikzcd} B \arrow[r, "f_j"] \arrow[dr, "f"'] & GA_j \arrow[d, "Gg"] \\ & GA \end{tikzcd}$$

commutes.)

Theorem. *(18.14) If* $\mathbf{A}$ *is strongly complete and (extremally) co-wellpowered, then the following conditions are equivalent for any functor* $G\colon \mathbf{A} \to \mathbf{B}$*:*

(1) G *is adjoint,*

(2) G *preserves small limits and is (extremally) co-wellpowered.*

Corollary. *(18.15) Fibre-small, concretely co-wellpowered constructs that are concretely complete have free objects.*

Theorem (Special adjoint functor theorem). *(18.17) If $\mathbf{A}$ is a strongly complete category with a coseparator, then for any functor $G\colon \mathbf{A} \to \mathbf{B}$, the following conditions are equivalent:*

(1) G is adjoint,

(2) G preserves strong limits.

Set, **Vec**, **Pos**, **Top**, and **HComp**, are complete, wellpowered, and have coseparators, so that the above theorem applies to them. Since many familiar categories have separators but fail to have coseparators, the dual of the Special Adjoint Functor Theorem is applicable even more often than the theorem itself.

Theorem (Concrete adjoint functor theorem). *(18.19) Let $G\colon (\mathbf{A}, U) \to (\mathbf{B}, V)$ be a concrete functor. If $(\mathbf{A}, U)$ is complete, wellpowered, co-wellpowered, and has free objects, then G is adjoint if and only if G preserves small limits.*

18A: Functors that are simultaneously adjoint and co-adjoint
Let $\mathbf{A} \xrightarrow{G} \mathbf{B}$. Show that
If G is an equivalence then G is adjoint and co-adjoint.
If G is adjoint and co-adjoint and $\mathbf{A} = \mathbf{B} = \mathbf{Set}$, then G is an equivalence.
If G is adjoint and co-adjoint and $\mathbf{A}$ and $\mathbf{B}$ are monoids, considered as categories, then G is an equivalence.
If $\mathbf{A} = 1$ and G maps the single object of $\mathbf{A}$ to a zero object in $\mathbf{B}$, then G is adjoint and co-adjoint.
The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is adjoint and co-adjoint.
If $\mathbf{A}$ is small and $\mathbf{C}$ is the category that is complete and cocomplete, then the functor $[\mathbf{B}, \mathbf{C}] \xrightarrow{[G,id]} [\mathbf{A}, \mathbf{C}]$ is adjoint and co-adjoint.

18E: Show that the covariant power-set functor $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ is neither adjoint nor co-adjoint, but that the contravariant power-set functor $\mathcal{Q}\colon \mathbf{Set}^{\mathbf{op}} \to \mathbf{Set}$

5.19 Adjoint situations

Theorem. *(19.1) Let $G\colon \mathbf{A} \to \mathbf{B}$ be an adjoint functor, and for each $\mathbf{B}$-object B let $\eta_B\colon B \to G(A_B)$ be a G-universal arrow. Then there exists a unique functor $F\colon \mathbf{B} \to \mathbf{A}$ such that $F(B) = A_B$ for each $\mathbf{B}$-object B, and $id_{\mathbf{B}} \xrightarrow{\eta=(\eta_B)} G \circ F$ is a natural transformation.*

Moreover there exists a unique natural transformation $F \circ G \xrightarrow{\varepsilon} id_{\mathbf{A}}$ that satisfies the following conditions:

(1) $G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = G \xrightarrow{id_G} G$,

(2) $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = F \xrightarrow{id_F} F$.

Definition. (19.3) An *adjoint situation* $(\eta, \varepsilon)\colon F \dashv G\colon \mathbf{A} \to \mathbf{B}$ consists of functors $G\colon \mathbf{A} \to \mathbf{B}$ and $F\colon \mathbf{B} \to \mathbf{A}$ and natural transformations $id_{\mathbf{B}} \xrightarrow{\eta} GF$ (called the *unit*) and $FG \xrightarrow{\varepsilon} id_{\mathbf{A}}$ (called the *co-unit*) that satisfy the following conditions:

(1) $G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = G \xrightarrow{id_G} G$,

(2) $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = F \xrightarrow{id_F} F$.

Theorem (Duality theorem for adjoint situations). *(19.6) If $(\eta, \varepsilon)\colon F \dashv G\colon \mathbf{A} \to \mathbf{B}$ is an adjoint situation, then $(\varepsilon^{op}, \eta^{op}\colon G^{op} \dashv F^{op}\colon \mathbf{B}^{op} \to \mathbf{A}^{op})$ is an adjoint situation.*

Proposition. *(19.7) If $(\eta, \varepsilon)\colon F \dashv G\colon \mathbf{A} \to \mathbf{B}$ is an adjoint situation, then the following hold:*

(1) G is an adjoint functor,

(2) for each $\mathbf{B}$-object B, $B \xrightarrow{\eta_B} GFB$ is a G-universal arrow,

(3) F is a co-adjoint functor,

(4) for each $\mathbf{A}$-object A, $FGA \xrightarrow{\varepsilon_A} A$ is a F-co-universal arrow.

Proposition. *(19.9) Adjoint situation associated with a given adjoint functor $G\colon \mathbf{A} \to \mathbf{B}$ are essentially unique, i.e., for each adjoint situation $(\eta, \varepsilon)\colon F \dashv G\colon \mathbf{A} \to \mathbf{B}$, the following hold:*

(1) if $(\overline{\eta}, \overline{\varepsilon})\colon \overline{F} \dashv G\colon \mathbf{A} \to \mathbf{B}$ is an adjoint situation, then there exists a natural isomorphism with $F \xrightarrow{\tau} \overline{F}$ for which $\overline{\eta} = G\tau \circ \eta$ and $\overline{\varepsilon} = \varepsilon \circ \tau^{-1}G$,

(2) if $\overline{F}\colon \mathbf{B} \to \mathbf{A}$ is a functor and $F \xrightarrow{\tau} \overline{F}$ is a natural isomorphism, then $(G\tau \circ \eta, \varepsilon \circ \tau^{-1}G)\colon \overline{F} \dashv G\colon \mathbf{A} \to \mathbf{B}$ is an adjoint situation.

Definition. (19.10) Let $G\colon \mathbf{A} \to \mathbf{B}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be functors. Then F is called a *co-adjoint for G* and G is called an *adjoint for F* (in symbols $F \dashv G$) provided there exist natural transformations η and ε such that $(\eta, \varepsilon)\colon F \dashv G\colon \mathbf{A} \to \mathbf{B}$ is an adjoint situation.

The reader should be aware that the following alternative terminology is also used:
G is right adjoint = G has a left adjoint = G is adjoint
F is left adjoint = F has a right adjoint = F is co-adjoint

Proposition. *(19.13) Adjoint situations can be composed, specifically, if $(\eta, \varepsilon)\colon F \dashv G\colon \mathbf{A} \to \mathbf{B}$ and $(\overline{\eta}, \overline{\varepsilon})\colon \overline{F} \dashv \overline{G}\colon \mathbf{B} \to \mathbf{C}$ are adjoint situations then so is*

$$(\overline{G}\eta\overline{F} \circ \overline{\eta}, \varepsilon \circ F\overline{\varepsilon}G)\colon F \circ \overline{F} \dashv \overline{G} \circ G\colon \mathbf{A} \to \mathbf{C}.$$

Theorem. *(19.14) If $(\eta, \varepsilon)\colon F \dashv G\colon \mathbf{A} \to \mathbf{B}$ is an adjoint situation, then the following hold:*

(1) The following are equivalent:

- *(a) G is faithful,*
- *(b) G reflects epimorphisms,*
- *(c) ε is an Epi-transformation.*

(2) The following are equivalent:

- *(a) G is faithful and reflect isomorphism,*
- *(b) G reflects extremal epimorphisms,*
- *(c) ε is an (Extremal Epi)-transformation.*

(3) G is full if and only if ε is a Section-transformation.

(4) G is full and faithful if and only if ε is a natural isomorphism.

(5) If G reflects regular epimorphism, then each mono-source is G-initial.

5.20 Monads

5.20.1 Monads and algebras

Definition. (20.1) A *monad* is a triple $\mathbf{T} = (T, \eta, \mu)$ consisting of a functor $T\colon \mathbf{X} \to \mathbf{X}$ and natural transformations

$$\eta\colon id_{\mathbf{X}} \to T \quad \text{and} \quad \mu\colon T \circ T \to T$$

such that the diagrams

$$\begin{tikzcd} T \circ T \circ T \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T \circ T \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "id"'] & T \circ T \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, "id"] \\ & T & \end{tikzcd}$$

Example. (20.2) In **Set** the *word-monad* $\mathbf{T} = (T, \eta, \mu)$ is defined as follows: $TX = \bigcup\limits_{n \in \mathbb{N}} X^n$ ("word" over X), $\eta_X(x) = (x)$ (one-letter word), $\mu_X : T(TX) \to TX$ is given by concatenation. $(\mathbf{X}^{\mathbf{T}}, U^{\mathbf{T}})$ is concretely isomorphic to the construct **Mon** of all monoids.

In **Set** the *power-set monad* $\mathbf{T} = (\mathcal{P}, \eta, \mu)$ is defined as follows: $\mathcal{P}$ is the power-set functor, $\eta_x(x) = \{x\}$, $\mu_X(Z) = \bigcup(Z)$. $(\mathbf{X}^{\mathbf{T}}, U^{\mathbf{T}})$ is concretely isomorphic to the construct **JCPos**.

Proposition. *(20.3) Each adjoint situation $(\eta, \varepsilon)\colon F \dashv G\colon \mathbf{A} \to \mathbf{X}$ gives rise to the* associated monad (T, η, μ) *on* $\mathbf{X}$*, defined by*

$$T = G \circ F\colon \mathbf{X} \to \mathbf{X} \quad \textit{and} \quad \mu = G\varepsilon F\colon T \circ T \to T.$$

Definition. (20.4) Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on $\mathbf{X}$. The full concrete subcategory of $\mathbf{Alg}(T)$ consisting of all algebras $TX \xrightarrow{x} X$ that satisfy

(1) $x \circ \eta_X = id_X$, and

(2) $x \circ Tx = x \circ \mu_X : T(TX) \to X$

is denoted by $(\mathbf{X}^{\mathbf{T}}, U^{\mathbf{T}})$ an is called the *Eilenberg-Moore category* of the monad $\mathbf{T}$, or the *category of* $\mathbf{T}$*-algebras.*

Proposition. *(20.7) Every monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathbf{X}$ gives rise to an* associated adjoint situation *$(\eta, \varepsilon)\colon F^{\mathbf{T}} \dashv U^{\mathbf{T}}\colon \mathbf{X}^{\mathbf{T}} \to X$, where*

(1) $\mathbf{X}^{\mathbf{T}}$ and $U^{\mathbf{T}}$ are defined as in Definition 20.4,

(2) $F^{\mathbf{T}}\colon \mathbf{X} \to \mathbf{X}^{\mathbf{T}}$ is defined by $F^{\mathbf{T}}(X \xrightarrow{f} Y) = (TX, \mu_X) \xrightarrow{Tf} (TY, \mu_Y)$, in particular, (TX, μ_X) is a free object over X in $(\mathbf{X}^{\mathbf{T}}, U^{\mathbf{T}})$.

(3) $F^{\mathbf{T}} U^{\mathbf{T}} \xrightarrow{\varepsilon} id_{\mathbf{X}^{\mathbf{T}}}$ is defined by $\varepsilon_{(X,x)} = x$.

Moreover, the monad associated with the above adjoint situation (20.3) is $\mathbf{T}$ itself.

5.20.2 Monadic categories and functors

Definition. (20.8)

(1) A concrete category over $\mathbf{X}$ is called *monadic* provided that it is concretely isomorphic to $(\mathbf{X}^{\mathbf{T}}, U^{\mathbf{T}})$ for some monad $\mathbf{T}$ on $\mathbf{X}$.

(2) A functor $U\colon \mathbf{A} \to \mathbf{X}$ is called *monadic* provided that U is faithful and $(\mathbf{A}, U)$ is monadic.

Lemma. *(20.11) The Eilenberg-Moore category of a monad $\mathbf{T} = (T, \eta, \mu)$ is closed under the formation of mono-sources in $\mathbf{Alg}(T)$.*

Proposition. *(20.12) For monadic functor $U\colon \mathbf{A} \to \mathbf{X}$ the following hold:*

(1) U is faithful,

(2) $(\mathbf{A}, U)$ is fibre-small,

(3) U is adjoint, i.e., $(\mathbf{A}, U)$ has free objects,

(4) $(\mathbf{A}, U)$ is uniquely transportable, hence amnestic,

(5) U creates isomorphisms, hence reflects them,

(6) U reflects epimorphisms and extremal epimorphisms,

(7) U preserves and reflects mono-sources,

(8) in $(\mathbf{A}, U)$ mono-sources are initial

(9) U detect wellpoweredness*, i.e., if $\mathbf{X}$ is wellpowered, then so is $\mathbf{A}$,*

(10) U creates limits.

Definition. (20.14) A fork $A \overset{p}{\underset{q}{\rightrightarrows}} B \xrightarrow{c} C$ is called a *congruence fork* provided that (p, q) is a congruence relation of c and c is a coequalizer of p and q.

A fork $A \overset{p}{\underset{q}{\rightrightarrows}} B \xrightarrow{c} C$ is called a *split fork* and c is called a *split coequalizer* of (p, q) provided that there exist morphisms s and t such that the diagram

$$\begin{tikzcd}
B \arrow[rrr, "c"] \arrow[dr, "t"] \arrow[dd, "id"'] & & & C \arrow[dl, "s"'] \arrow[dd, "id"] \\
& A \arrow[r, "p"] \arrow[dl, "q"] & B \arrow[dr, "c"'] & \\
B \arrow[rrr, "c"] & & & C
\end{tikzcd}$$

commutes.

A colimit $\mathcal{K}$ of a diagram $D\colon \mathbf{I} \to \mathbf{A}$ is called an *absolute colimit* provided that for each functor $G\colon \mathbf{A} \to \mathbf{B}$ the sink $G\mathcal{K}$ is a colimit of $G \circ D$.

In particular, c is called an *absolute coequalizer* of p and q in $\mathbf{A}$ provided that for each functor $G\colon \mathbf{A} \to \mathbf{B}$, Gc is a coequalizer of Gp and Gq in $\mathbf{B}$.

A functor $G\colon \mathbf{A} \to \mathbf{B}$ is said to *create absolute colimits* provided that for each diagram $D\colon \mathbf{I} \to \mathbf{A}$ and each absolute colimit $\mathcal{K}$ of $G \circ D$ there exists a unique sink $\mathcal{C} = (D_i \xrightarrow{c_i} C)_{\mathrm{Ob}(\mathbf{I})}$ such that $G\mathcal{C} = \mathcal{K}$ and, moreover, $\mathcal{C}$ is (not necessarily absolute) colimit of D.

Proposition. *(20.16) Each monadic functor U creates absolute colimits.*

Theorem (Characterization theorem for monadic functors). *(20.17) For any functor $U\colon \mathbf{A} \to \mathbf{X}$ the following conditions are equivalent:*

(1) U is monadic,

(2) U is adjoint and creates absolute coequalizers,

(3) U is adjoint and creates split coequalizers.

Proposition. *(20.18) Each construct of the form $\mathbf{Alg}(\Omega)$ is monadic.*

Proposition. *(20.19) Let $(\mathbf{A}, U)$ be a monadic category over $\mathbf{X}$. Then each concrete full reflective subcategory of $\mathbf{A}$ that is closed under the formation of regular quotients is also monadic over $\mathbf{X}$.*

Proposition. *(20.20) Each finitary variety is a monadic construct.*

5.20.3 E-monads and E-monadic categories and functors

Definition. (20.21) A monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathbf{X}$ is called an *E-monad* provided that $\mathbf{X}$ is an $(E, \mathbf{M})$ category for some $\mathbf{M}$ and $T[E] \subseteq E$.

RegEpi-monads in categories with regular factorizations are called *regular monads*.

A concrete category $(\mathbf{A}, U)$ over $\mathbf{X}$ (or a faithful functor $U : \mathbf{A} \to \mathbf{X}$) is called *$E$-monadic* provided that $(\mathbf{A}, U)$ is concretely isomorphic to $(\mathbf{X}^{\mathbf{T}}, u^{\mathbf{T}})$ for some E monad $\mathbf{T}$ on $\mathbf{X}$. If $E =$RegEpi, then E-monadic is called *regularly monadic.*

Proposition. *(20.22) Every monad on* **Set** *is regular.*

Definition. (20.23) A functor $U : \mathbf{A} \to \mathbf{X}$ *lifts $(E, \mathbf{M})$-factorizations uniquely* provided that for any source $\mathcal{S}$ in $\mathbf{A}$ and any $(E, \mathbf{M})$-factorization $U\mathcal{S} = \mathcal{M} \circ e$ in $\mathbf{X}$ there exists a unique factorization $\mathcal{S} = \hat{\mathcal{M}} \circ \hat{e}$ with $U\hat{\mathcal{M}} = \mathcal{M}$ and $U\hat{e} = e$.

Proposition. *(20.24) If $\mathbf{X}$ is an $(E, \mathbf{M})$-category and $U : \mathbf{A} \to \mathbf{X}$ is E-monadic, then U lifts $(E, \mathbf{M})$-factorizations uniquely.*

Proposition. *(20.25) Let $\mathbf{A}$ be a full subcategory of an $(E, \mathbf{M})$-category $\mathbf{B}$. Then the inclusion functor $U : \mathbf{A} \to \mathbf{B}$ is E-monadic if and only if the following conditions are satisfied:*

(1) $\mathbf{A}$ is reflective in $\mathbf{B}$,

(2) if $A \xrightarrow{e} B \xrightarrow{m} A'$ is an $(E, \mathbf{M})$-factorization of an $\mathbf{A}$-morphism $A \xrightarrow{m \circ e} A'$, then B belongs to $\mathbf{A}$.

Corollary. *(20.26) If $\mathbf{A}$ is an E-reflective subcategory of an $(E, \mathbf{M})$-category $\mathbf{B}$, then the inclusion functor $\mathbf{A} \to \mathbf{B}$ is E-monadic.*

Proposition. *(20.28) If $(\mathbf{A}, U)$ is an E-monadic category over an $(E, \mathbf{M})$-category $\mathbf{X}$, then the following hold:*

(1) Every $\mathbf{A}$-morphism F with $Uf \in E$ is final in $(\mathbf{A}, U)$.

(2) $\mathbf{A}$ is an $(U^{-1}[E], U^{-1}[\mathbf{M}])$-category.

Corollary. *(20.29) If $(\mathbf{A}, U)$ is E-monadic over an E-co-wellpowered category, then $\mathbf{A}$ is $U^{-1}[E]$-co-wellpowered.*

Proposition. *(20.30) If $(\mathbf{A}, U)$ is regularly monadic, the $\mathbf{A}$ has regular factorizations and U preserves and reflects regular and extremal epimorphisms.*

Corollary. *(20.31) Regularly monadic functor detect extremal co-wellpoweredness.*

Theorem (Characterization theorem for regularly monadic functors). *(20.32) A functor $U\colon \mathbf{A} \to \mathbf{X}$ is regularly monadic if and only if the following conditions hold:*

(1) U is monadic,

(2) $\mathbf{X}$ has regular factorizations,

(3) U preserves regular epimorphisms.

Proposition. *(20.33) Regularly monadic functors detect colimits.*

5.20.4 Monadic constructs

Proposition. *(20.34) Monadic constructs are complete, cocomplete, wellpowered, extremally co-wellpowered, and have regular factorizations.*

Theorem (Characterization theorem for monadic constructs). *(20.35) For constructs $(\mathbf{A}, U)$ the following conditions are equivalent:*

(1) U is monadic,

(2) U is regularly monadic,

(3) U is adjoint and creates finite limits and coequalizers of congruence relations,

(4) U is extremally co-wellpowered and creates limits and coequalizers of congruence relations.

5.20.5 The comparison functor

Proposition. *(20.37) If $(\eta, \varepsilon)\colon F \dashv U\colon \mathbf{A} \to \mathbf{X}$ is an adjoint situation and $(\mathbf{X}^{\mathbf{T}}, U^{\mathbf{T}})$ is the associated category of algebras, the there exists a unique functor $K\colon \mathbf{A} \to \mathbf{X}^{\mathbf{T}}$ such that the diagram*

$$\begin{tikzcd} \mathbf{X} \arrow[r, "F"] \arrow[d, "F^{\mathbf{T}}"'] & \mathbf{A} \arrow[d, "U"] \arrow[dl, "K"'] \\ \mathbf{X}^{\mathbf{T}} \arrow[r, "U^{\mathbf{T}}"'] & \mathbf{X} \end{tikzcd}$$

commutes.

Definition. (20.38)

(1) For each adjoint situation $(\eta, \varepsilon)\colon F \dashv U\colon \mathbf{A} \to \mathbf{X}$, the unique functor $K\colon \mathbf{A} \to \mathbf{X}^{\mathbf{T}}$ of the above proposition is called its *comparison functor.*

(2) For each adjoint functor $U\colon \mathbf{A} \to \mathbf{X}$ (resp. each concrete category $(\mathbf{A}, U)$ that has free objects) the comparison functor of an associated adjoint situation is called *comparison functor for U* (resp. for $(\mathbf{A}, U)$.)

Proposition. *(20.40) An adjoint functor is monadic if and only if the associated comparison functor is a concrete isomorphism.*

Theorem. *(20.42) Let $K\colon (\mathbf{A}, \mathbf{U}) \to (\mathbf{X}^{\mathbf{T}}, \mathbf{U}^{\mathbf{T}})$ be a comparison functor. If $\mathbf{A}$ has coequalizers, then K is adjoint.*

Theorem. *(20.43) Let* $(\eta, \varepsilon)\colon F \dashv U\colon \mathbf{A} \to \mathbf{X}$ *be an adjoint situation with associated comparison functor* $K\colon \mathbf{A} \to \mathbf{X}^{\mathbf{T}}$*. Then:*

(1) K *is faithful if and only if* U *is faithful,*

(2) K *is full and faithful if and only if* ε *is a RegEpi-transformation*

Corollary. *(20.44) The comparison functor* $K\colon (\mathbf{A}, U) \to (\mathbf{X}^{\mathbf{T}}, U^{\mathbf{T}})$ *of a uniquely transportable concrete category, for which* U *reflects regular epimorphisms, is an isomorphism-closed full embedding.*

5.20.6 Deficiencies of monadic functors

5.20.7 Varietors and free monads

Definition. (20.53) A functor $T\colon \mathbf{X} \to \mathbf{X}$ is called a *varietor* provided that the concrete category $\mathbf{Alg}(T)$ has free objects.

Definition. (20.55)

(1) Given monads $\mathbf{T} = (T, \eta, \mu)$ and $\mathbf{T}' = (T', \eta', \mu')$ over $\mathbf{X}$, a natural transformation $\tau\colon T \to T'$ is called a *monad morphism* (denoted by $\tau\colon \mathbf{T} \to \mathbf{T}'$) provided that $\eta' = \tau \circ \eta$ and $\tau \circ \mu = \mu' \circ \tau T' \circ T\tau$.

(2) A *free monad* generated by a functor $T\colon \mathbf{X} \to \mathbf{X}$ is a monad $\mathbf{T}^{\#} = (T^{\#}, \eta^{\#}, \mu^{\#})$ together with a natural transformation $\lambda\colon T \to T^{\#}$ that has the following universal property: for every monad $\mathbf{T}' = (T', \eta', \mu')$ and every natural transformation $\tau\colon T \to T'$ there exists a unique monad morphism $\tau^{\#}\colon T^{\#} \to T'$ with $\tau = \tau^{\#} \circ \lambda$.

Theorem. *(20.56) If* $T\colon \mathbf{X} \to \mathbf{X}$ *is a varietor, then* $\mathbf{Alg}(T)$ *is monadic over* $\mathbf{X}$ *and the associated monad is a free monad generated by* T*.*

Corollary. *(20.57) If* $T\colon \mathbf{X} \to \mathbf{X}$ *is a varietor, then the category* $\mathbf{Alg}(T)$ *is concretely isomorphic to* $\mathbf{X}^{\mathbf{T}^{\#}}$ *for a free monad* $\mathbf{T}^{\#}$*.*

Theorem. *(20.59) If* $\mathbf{X}$ *is a strongly complete category, then every functor* $T\colon \mathbf{X} \to \mathbf{X}$ *that generates a free monad is a varietor.*

6 Topological and algebraic categories

6.21 Topological categories

6.21.1 Topological functors

Definition. (21.1) A functor $G\colon \mathbf{A} \to \mathbf{B}$ is called *topological* provided that every G-structured source $(B \xrightarrow{f_i} GA_i)_I$ has unique G-initial lift $(A \xrightarrow{\overline{f}_i} A_i)_I$.

Examples: **Top**, **Unif**,

Theorem. *(21.3) Topological functors are faithful.*

Proposition. *(21.5) If* $G\colon \mathbf{A} \to \mathbf{B}$ *is a functor such that every* G*-structured source has* G*-initial lift, then the following conditions are equivalent:*

(1) G *is topological,*

(2) $(\mathbf{A}, G)$ *is uniquely transportable,*

(3) $(\mathbf{A}, G)$ *is amnestic.*

Proposition. *(21.6) If* $G\colon \mathbf{A} \to \mathbf{B}$ *and* $F\colon \mathbf{B} \to \mathbf{C}$ *are topological, then so is* $F \circ G\colon \mathbf{A} \to \mathbf{C}$.

6.21.2 Topological categories

Definition. (21.7) A concrete category $(\mathbf{A}, U)$ is called *topological*, provided that U is topological.

Examples: **Top**, **Unif**

All functor-structured categories $\mathbf{Spa}(T)$ and all functor-costructured categories $(\mathbf{Spa}(T))^{op}$ are topological.

TopGrp is topological if it is considered as a concrete category over **Grp**, but not over **Set** or **Top**.

$\mathbf{Top}_1$ is not topological.

Theorem (Topological duality theorem). *(21.9) If* $(\mathbf{A}, U)$ *is topological over* $\mathbf{X}$*, then* $(\mathbf{A}^{op}, U^{op})$ *is topological over* $\mathbf{X}^{op}$ *(i.e., the existence of unique* U*-initial lifts of* U*-structured sources implies the existence of unique* U*-final lifts of* U*-structured sinks).*

Proposition. *(21.11) Topological categories are fibre-complete. The smallest (resp. largest) member of each fibre is discrete (resp. indiscrete).*

Proposition. *(21.12) If* $(\mathbf{A}, U)$ *is topological over* $\mathbf{X}$*, then*

(1) U *is an adjoint functor; its co-adjoint* $F\colon \mathbf{X} \to \mathbf{A}$ *(the discrete functor) is a full embedding satisfying* $U \circ F = id_{\mathbf{X}}$.

(2) U *is a co-adjoint functor; its adjoint* $G\colon \mathbf{X} \to \mathbf{A}$ *(the indiscrete functor) is a full embedding satisfying* $U \circ G = id_{\mathbf{X}}$.

Proposition. *(21.13) If* $(\mathbf{A}, U)$ *is topological over* $\mathbf{X}$*, then the following hold:*

(1) U *preserves and reflects mono-sources and epi-sinks.*

(2) An $\mathbf{A}$*-morphism is an extremal (resp. regular) monomorphism if and only if it is initial and extremal (resp. regular)* $\mathbf{X}$*-monomorphism.*

(3) An $\mathbf{A}$*-morphism is an extremal (resp. regular) epimorphism if and only if it is initial and extremal (resp. regular)* $\mathbf{X}$*-epimorphism.*

In particular, in topological constructs, the following hold:

(4) embedding = extremal monomorphisms = regular monomorphisms.

(5) quotient morphisms = extremal epimorphisms = regular epimorphisms.

Proposition. *(21.14) If* $(\mathbf{A}, U)$ *is topological over an* $(E, \mathbf{M})$*-category* $\mathbf{X}$*, then the following holds:*

(1) $\mathbf{A}$ *is* $(E, \mathbf{M}_{init})$*-category, where* $\mathbf{M}_{init}$ *consists of all initial sources in* $\mathbf{M}$.

(2) $\mathbf{A}$ *is an* $(E_{fin}, \mathbf{M})$*-category, where* E_{fin} *consists of all final* E*-morphisms.*

Proposition. *(21.15) If* $(\mathbf{A}, U)$ *is topological over* $\mathbf{X}$*, then* U *uniquely lifts both limits (via initiality) and colimits (via finality), and it preserves both limits and colimits.*

Theorem. *(21.16) If* $(\mathbf{A}, U)$ *is topological over* $\mathbf{X}$*, then the following hold:*

(1) $\mathbf{A}$ *is (co)complete if and only if* $\mathbf{X}$ *is (co)complete.*

(2) $\mathbf{A}$ *is (co-)wellpowered if and only if* $(\mathbf{A}, U)$ *is fibre-small and* $\mathbf{X}$ *is (co-)wellpowered.*

(3) $\mathbf{A}$ *is extremally (co-)wellpowered if and only if* $\mathbf{X}$ *is extremally (co-)wellpowered.*

(4) $\mathbf{A}$ *is (Epi,Mono-Source)-factorizable if and only if* $\mathbf{X}$ *is (Epi, Mono-Source)-factorizable.* $\blacksquare$

(5) $\mathbf{A}$ *has regular factorizations if and only if* $\mathbf{X}$ *has regular factorizations.*

(6) $\mathbf{A}$ *has a (co)separator if and only if* $\mathbf{X}$ *has a (co)separator.*

Corollary. *(21.17) Each topological construct*

(1) is complete and cocomplete,

(2) is wellpowered (resp. co-wellpowered) if and only if it is fibre-small,

(3) is an (Epi, Extremal Mono-Source)-category,

(4) has regular factorizations,

(5) has separators and coseparators.

Theorem (Internal topological characterization theorem). *(21.18) A concrete category* $(\mathbf{A}, U)$ *over* $\mathbf{X}$ *is topological if and only if it satisfies the following conditions:*

(1) U *lifts limits uniquely,*

(2) $(\mathbf{A}, U)$ has indiscrete structures, *i.e., every* $\mathbf{X}$*-object has an indiscrete lift.*

Theorem (External topological characterization theorem). *(21.21) Let* $\mathbf{CAT}(\mathbf{X})$ *be the quasicategory of all concrete categories and concrete functors over a fixed category* $\mathbf{X}$*. If* $\mathcal{M}$ *is the conglomerate of all full functors in* $\mathbf{CAT}(\mathbf{X})$*, then for each concrete category* $(\mathbf{A}, U)$ *over* $\mathbf{X}$ *the following are equivalent:*

(1) $(\mathbf{A}, U)$ *is topological over* $\mathbf{X}$*.*

(2) $(\mathbf{A}, U)$ *is an* $\mathcal{M}$*-injective object in* $\mathbf{CAT}(\mathbf{X})$*.*

6.21.3 Initiality-preserving concrete functors

preservation of initial sources - see Definition 10.47

Proposition. *(21.23) Initiality-preserving concrete functors preserve indiscrete objects.*

Theorem (Galois correspondence theorem). *(21.24) For concrete functors* $(\mathbf{A}, U) \xrightarrow{G} (\mathbf{B}, V)$ *with topological domain* $(\mathbf{A}, U)$ *the following conditions are equivalent:*

(1) G *preserves initial sources,*

(2) G *is adjoint and has a concrete co-adjoint* $(\mathbf{B}, V) \xrightarrow{F} (\mathbf{A}, U)$*,*

(3) there exists a (unique) $(\mathbf{B},V) \xrightarrow{F} (\mathbf{A},U)$ *such that* (F,G) *is a Galois correspondence.*

Definition. (21.26) Let $U\colon \mathbf{A} \to \mathbf{X}$ and $V\colon \mathbf{B} \to \mathbf{Y}$ be functors. An adjoint situation $(\hat{\eta},\hat{\varepsilon})\colon \hat{F} \dashv \hat{G}\colon \hat{\mathbf{A}} \to \hat{\mathbf{B}}$ is said to *lift an adjoint situation* $(\eta,\varepsilon)\colon F \dashv G\colon \mathbf{X} \to \mathbf{U}$ *along* U *and* V provided that the following conditions are satisfied:

(1) the diagrams

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\hat{G}"] \arrow[d, "U"'] & \mathbf{B} \arrow[d, "V"] \\ \mathbf{X} \arrow[r, "G"'] & \mathbf{Y} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{B} \arrow[r, "\hat{F}"] \arrow[d, "V"'] & \mathbf{A} \arrow[d, "U"] \\ \mathbf{Y} \arrow[r, "F"'] & \mathbf{X} \end{tikzcd}$$

commute,

(2) $V\hat{\eta} = \eta V$,

(3) $U\hat{\varepsilon} = \varepsilon U$.

Theorem (Taut lift theorem). *(21.28) Let* $(\mathbf{A},U)$ *be a topological category over the base category* $\mathbf{X}$ *and* $(\mathbf{B},V)$ *be a concrete category over the base category* $\mathbf{Y}$*. If* $\hat{G}\colon \mathbf{A} \to \mathbf{B}$ *is a functor and* $G\colon \mathbf{X} \to \mathbf{Y}$ *is an adjoint functor with* $V \circ \hat{G} = G \circ U$*, then the following conditions are equivalent:*

(1) $\hat{G}$ *sends* U*-initial sources into* V*-initial sources,*

(2) every adjoint situation $(\eta,\varepsilon)\colon F \dashv G\colon \mathbf{X} \to \mathbf{Y}$ *can be lifted along* U *and* V *to an adjoint situation* $(\hat{\eta},\hat{\varepsilon})\colon \hat{F} \dashv \hat{G}\colon \mathbf{A} \to \mathbf{B}$.

6.21.4 Topological subcategories

Definition. (21.29) A full concrete subcategory $(\mathbf{A},U)$ of a concrete category $(\mathbf{B},V)$ is called *initially closed* in $(\mathbf{B},V)$ provided that every V-initial source whose codomain is a family of $\mathbf{A}$-objects has its domain in $\mathbf{A}$.

Dual notion: *finally closed subcategory.*

Proposition. *(21.30) An initially closed subcategory of a topological category is topological.*

Proposition. *(21.31) For any full concrete subcategory* $(\mathbf{A},U)$ *of a concrete category* $(\mathbf{B},V)$ *the following conditions are equivalent:*

(1) $(\mathbf{A},U)$ *is initially closed in* $(\mathbf{B},V)$*,*

(2) $(\mathbf{A},U)$ *is concretely reflective in* $(\mathbf{B},V)$*.*

Definition. (21.32) A full concrete subcategory $(\mathbf{A},U)$ of a concrete category $(\mathbf{B},V)$ is called *finally dense* in $(\mathbf{B},V)$ provided that for every $\mathbf{B}$-object B there exists a V-final sink $(A_i \xrightarrow{f_i} B)_I$ with each A_i in $\mathbf{A}$.

Dual notion: *initially dense subcategory*

Proposition. *(21.33) If* $(\mathbf{A},U)$ *is a finally dense full concrete subcategory of* $(\mathbf{B},V)$*, then the associated inclusion functor* $(\mathbf{A},U) \xrightarrow{E} (\mathbf{B},V)$ *preserves initial sources.*

Proposition. *(21.34) If a topological category* $(\mathbf{A},U)$ *is a finally dense full concrete subcategory of* $(\mathbf{B},V)$*, then* $(\mathbf{A},U)$ *is concretely reflective in* $(\mathbf{B},V)$*.*

Theorem. *(21.35) For a full concrete subcategory* $(\mathbf{A}, U)$ *of a topological category* $(\mathbf{B}, V)$ *the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ *is topological,*

(2) there exists a concretely reflective subcategory $(\mathbf{C}, W)$ *of* $(\mathbf{B}, V)$ *such that* $(\mathbf{A}, U)$ *is concretely coreflective in* $(\mathbf{C}, W)$*,*

(3) there exists a concretely coreflective subcategory $(\mathbf{C}, W)$ *of* $(\mathbf{B}, V)$ *such that* $(\mathbf{A}, U)$ *is concretely reflective in* $(\mathbf{C}, W)$*,*

(4) there exists a concrete functor $(\mathbf{B}, V) \xrightarrow{R} (\mathbf{A}, U)$ *that leaves each* $\mathbf{A}$*-object fixed.*

6.21.5 Fibre-small topological categories

Proposition. *(21.36) For fibre-small concrete categories* $(\mathbf{A}, U)$*, the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ *is topological,*

(2) every small structured source $(X \xrightarrow{f_i} UA_i)_I$ *has a unique initial lift,*

(3) every small structures sink $(UA_i \xrightarrow{f_i} X)_I$ *has a unique final lift.*

Proposition. *(21.37) A fibre-small concrete category* $(\mathbf{A}, U)$ *over a category* $\mathbf{X}$ *with products is topological if and only if it satisfies the following conditions:*

(1) $(\mathbf{A}, U)$ *has concrete products,*

(2) $(\mathbf{A}, U)$ has initial subobjects, *i.e., every structured* $\mathbf{X}$*-monomorphisms* $X \xrightarrow{m} UA$ *has a unique initial lift,*

(3) $(\mathbf{A}, U)$ *has indiscrete objects.*

Proposition. *(21.38) In a fibre-small topological category, a source* $(A \xrightarrow{f_i} A_i)_I$ *is initial if and only if there exists a subset* J *of* I *such that* $(A \xrightarrow{f_j} A_j)_J$ *is initial.*

Proposition. *(21.39) Let* $(\mathbf{A}, U)$ *be a full concrete subcategory of a fibre-small topological category* $(\mathbf{B}, V)$ *over a category* $\mathbf{X}$ *with products. Then* $(\mathbf{A}, U)$ *is concretely reflective in* $(\mathbf{B}, V)$ *if and only if it is closed under the formation of*

(a) products,

(b) initial subobjects, and

(c) indiscrete objects.

6.21.6 M-topological and monotopological categories

Definition. (21.40) A concrete category $(\mathbf{A}, U)$ over an $(E, \mathbf{M})$-category $\mathbf{X}$ is said to be *M-topological* provided that every structured source in $\mathbf{M}$ has a unique initial lift. If $\mathbf{M} = Mono - Sources$, the term *monotopological* is used.

Theorem (M-topological characterization theorem). *(21.42) A concrete category* $(\mathbf{A}, U)$ *over an* $(E, \mathbf{M})$*-category* $\mathbf{X}$ *is* $\mathbf{M}$*-topological if and only if* $(\mathbf{A}, U)$ *is an* E*-reflective concrete subcategory of some topological category over* $\mathbf{X}$*.*

Theorem. *(21.44) Let* $(\mathbf{A}, U)$ *be a fibre-small concrete category over an* E*-co-wellpowered* $(E, \mathbf{M})$*-category with products. Then* $(\mathbf{A}, U)$ *is* $\mathbf{M}$*-topological if and only if it satisfies the following conditions:*

(1) $(\mathbf{A}, U)$ *has concrete products,*

(2) $(\mathbf{A}, U)$ *has* M-initial subobjects, *i.e. every structured* M*-morphism* $X \xrightarrow{m} UA$ *has a unique initial lift.*

6.22 Topological structure theorems

6.22.1 Topological axioms

Definition. (21.1) Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$.

(1) Each identity-carried morphism $P \xrightarrow{p} P'$ is called a *topological axiom* in $(\mathbf{A}, U)$. An $\mathbf{A}$-object A us said to *satisfy* the axiom p provided that A is $\{p\}$-injective; that is, each $\mathbf{A}$-morphism $f\colon P \to A$ is also an $\mathbf{A}$-morphism $P' \xrightarrow{f} A$.

(2) A full subcategory $\mathbf{B}$ of $\mathbf{A}$ is said to be *definable by topological axioms* in $(\mathbf{A}, U)$ provided that it is E-implicational in $\mathbf{A}$, where E is a class of topological axioms in $(\mathbf{A}, U)$; i.e., the objects in $\mathbf{B}$ are precisely those $\mathbf{A}$-objects that satisfy each of the axioms in E.

Theorem (Topological structure theorem). *(22.3) For concrete categories* $(\mathbf{A}, U)$*, the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ *is fibre-small and topological,*

(2) $(\mathbf{A}, U)$ *is concretely isomorphic to an initially closed full subcategory of a functor-structured category,*

(3) $(\mathbf{A}, U)$ *is concretely isomorphic to an isomorphism-closed concretely reflective subcategory of some functor-structured category,*

(4) $(\mathbf{A}, U)$ *is concretely isomorphic to a subcategory of a functor-structured category of* $\mathbf{Spa}(T)$ *that is definable by topological axioms in* $\mathbf{Spa}(T)$*.*

Corollary. *(22.4) For a construct* $(\mathbf{A}, U)$ *the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ *is fibre-small and topological,*

(2) $(\mathbf{A}, U)$ *can be concretely embedded in a functor-structured construct as a full subconstruct that is closed under the formation of:*

(a) products,

(b) initial subobjects, and

(c) indiscrete objects.

Definition. (22.6) Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$.

(1) Each identity-carried morphism $P' \xrightarrow{p} P$ is called a *topological co-axiom* in $(\mathbf{A}, U)$.[8] An $\mathbf{A}$-object A is said to *satisfy* the co-axiom p provided that A is $\{p\}$-projective; that is, each $\mathbf{A}$-morphism $f\colon A \to P$ is also an $\mathbf{A}$-morphism $f\colon A \to P'$.

[8] Observe that f is a topological co-axiom if and only if it is a topological axiom. However, the concept "A satisfies the topological co-axiom f" is dual to the concept "A satisfies the topological axiom f."

(2) A full subcategory $\mathbf{B}$ of $\mathbf{A}$ is said to be *definable by topological co-axioms* in $\mathbf{A}$ if there exists a class of topological co-axioms in $\mathbf{A}$ such that an $\mathbf{A}$-object A satisfies each of these co-axioms if and only if $A \in Ob(\mathbf{B})$.

Theorem. *(22.8) For a concrete category $(\mathbf{A}, U)$ the following are equivalent:*

(1) $(\mathbf{A}, U)$ is fibre-small and topological,

(2) $(\mathbf{A}, U)$ is concretely isomorphic to a finally closed full subcategory of some functor-costructured category,

(3) $(\mathbf{A}, U)$ is concretely isomorphic to a full concretely coreflective subcategory of some functor-costructured category.

(4) $(\mathbf{A}, U)$ is concretely isomorphic to a subcategory of some fuctor-costructured category $\mathbf{Spa}(T)^{op}$ that is definable by topological axioms in $\mathbf{Spa}(T)^{op}$.

6.22.2 M-topological structure theorems

Theorem (M-topological structure theorem). *(22.9) For concrete categories $(\mathbf{A}, U)$ over an E-co-wellpowered $(E, \mathbf{M})$-category the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ is fibre-small and $\mathbf{M}$ -topological,

(2) $(\mathbf{A}, U)$ is concretely isomorphic to an $\mathbf{M}$ -initially closed full subcategory of a functor-structured category,

(3) $(\mathbf{A}, U)$ is concretely isomorphic to an isomorphism-closed E-reflective subcategory of a functor-structured category.

(4) $(\mathbf{A}, U)$ is concretely isomorphic to an E-implicational subcategory of a functor-structured category.

Corollary. *(22.10) For constructs $(\mathbf{A}, U)$ the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ is fibre-small and monotopological,

(2) $(\mathbf{A}, U)$ is concretely isomorphic to a full subcategory of a functor-structured conctruct that is closed under the formation of products and initial subobjects,

(3) $(\mathbf{A}, U)$ is concretely isomorphic to an implicational subconstruct of a functor-structured constructs.

7 Cartesian closedness and partial morphisms

7.27 Cartesian closed categories

Definition. (27.1) A category $\mathbf{A}$ is called *cartesian closed* provided that it has finite products and for each $\mathbf{A}$-object A the functor $(A \times -)\colon \mathbf{A} \to \mathbf{A}$ is co-adjoint.

The essential uniqueness of products and of co-universal arrows allows us to introduce the following standard notation for cartesian closed categories: "The" adjoint functor for $(A \times -)$ is denoted on objects by $B \mapsto B^A$, and "the" associated co-universal arrows are denoted by

$$ev\colon A \times B^A \to B.$$

Thus, a category with finite products is cartesian closed if and only if for each pair (A, B) of objects there exists an object B^A and a morphism $ev\colon A \times B^A \to B$ with the following universal property: for each morphism $f\colon A \times C \to B$ there exists a unique morphism $\hat{f}\colon C \to B^A$ such that

$$\begin{tikzcd} A \times C \arrow[d, "id_A \times \hat{f}"'] \arrow[dr, "f"] & \\ A \times B^A \arrow[r, "ev"'] & B \end{tikzcd}$$

commutes. We shall call the objects B^A *power objects,*, the morphism $ev\colon A \times B^A \to B$ *evaluation morphism* and the morphism $\hat{f}$, associated with f, the *exponential morphism* for f.

Set, **Rel**, **Pos**, **Alg**(1), **Cat** are cartesian closed.

A poset A considered as a category is cartesian closed if and only if A has finite meets and for each pair (a, b) of elements the set $\{x \in A : a \wedge x \leq b\}$ has a largest member. In particular, a complete lattice is cartesian closed if and only if it satisfies the distributive law: $a \wedge \bigvee b_i = \bigvee(a \wedge b_i)$; i.e. if and only if it is a frame.

Top is not cartesian closed (since $\mathbb{Q} \times -\colon$ **Top** $\to$ **Top** does not preserve quotients, and hence does not preserve coequalizers). However, **Top** has cartesian closed supercategories (**Conv**, **PsTop**) as well as cartesian closed subcategories (**kTop** of all (compact Hausdorff)-generated topological spaces.).

Theorem (Characterization theorem for cartesian closed categories). *(27.4) Let* **A** *be a cocomplete and co-wellpowered category that has a separator. Then* **A** *is cartesian closed if and only if it has finite products, and for each* **A** *object A the functor $(A \times -)$ preserves colimits.*

Definition. (27.5) Let **A** be a cartesian closed category. For each object C

(1) "the" *covariant exponential functor for C*, denoted by $(-)^C\colon \mathbf{A} \to \mathbf{A}$, is "the" adjoint functor for $(C \times -)$ and is defined (for an **A**-morphism $A \xrightarrow{f} B$) by:

$$(-)^C(A \xrightarrow{f} B) = A^C \xrightarrow{f^C} B^C,$$

where f^C is the unique **A**-morphism that makes the diagram

$$\begin{tikzcd} C \times A^C \arrow[r, "ev"] \arrow[d, "id \times f^C"'] & A \arrow[d, "f"] \\ C \times B^C \arrow[r, "ev"'] & B \end{tikzcd}$$

commute;

(2) "the" *contravariant exponential functor for C,* denoted by $C^{(-)}\colon \mathbf{A^{op}} \to \mathbf{A}$, is defined by

$$C^{(-)}(A \xrightarrow{f} B) = C^B \xrightarrow{C^f} C^A,$$

where C^f is the unique **A**-morphism that makes the diagram

$$\begin{tikzcd} A \times C^B \arrow[r, "f \times id"] \arrow[d, "id \times C^f"'] & B \times C^B \arrow[d, "ev"] \\ A \times C^A \arrow[r, "ev"'] & C \end{tikzcd}$$

commute.

Observe that the exponential functors are not determined uniquely, but that any two covariant (resp. contravariant) functors are naturally isomorphic.

Proposition. *(27.7) In a cartesian closed category* $\mathbf{A}$*, every contravariant exponential functor* $C^{(-)}$ *is an adjoint functor and has its own dual* $(C^{(-)})^{op}$ *as a co-adjoint.*

Proposition. *(27.8) In a cartesian closed category the following hold:*

(1) ***First Exponential Law:*** $A^{B\times C} \cong (A^B)^C$,

(2) ***Second Exponential Law:*** $(\prod A_i)^B \cong \prod A_i^B$,

(3) ***Third Exponential Law:*** $A^{\coprod B_i} \cong \prod A^{B_i}$,

(4) ***Distributive Law:*** $A \times \coprod B_i \cong \coprod (A \times B_i)$,

(5) Finite products of epimorphisms are epimorphisms.

7.27.1 Cartesian closed subcategories

Proposition. *(27.9) Let* $\mathbf{A}$ *be an isomorphism-closed full subcategory of a cartesian closed category* $\mathbf{B}$*.*

(1) If $\mathbf{A}$ *is reflective in* $\mathbf{B}$ *and the* $\mathbf{A}$*-reflector preserves finite products, then* $\mathbf{A}$ *is closed under the formation of finite products and powers in* $\mathbf{B}$*, and hence is cartesian closed.*

(2) If $\mathbf{A}$ *is coreflective in* $\mathbf{B}$ *and is closed under the formation of finite products in* $\mathbf{B}$*, then* $\mathbf{A}$ *is cartesian closed.*

7.27.2 Cartesian closed concrete categories

Definition. (27.11) A concrete category $(\mathbf{A}, U)$ over $\mathbf{X}$ is called *concretely cartesian closed* provided that the following hold:

(1) $\mathbf{A}$ and $\mathbf{X}$ are cartesian closed,

(2) U preserves finite products, power objects, and evaluation; in particular, whenever $A \times B^A \xrightarrow{ev} B$ is an evaluation in $\mathbf{A}$, then

$$U(A \times B^A \xrightarrow{ev} B) = UA \times UB \xrightarrow{ev} UB$$

is an evaluation in $\mathbf{X}$.

Proposition. *(27.14) If* $(\mathbf{A}, U)$ *is topological category over* $\mathbf{X}$ *and if* $\mathbf{A}$ *is cartesian closed, then so is* $\mathbf{X}$*.*

Theorem (Characterization theorem for concretely cartesian closed topological categories). *(27.15) For a topological category* $(\mathbf{A}, U)$ *over a cartesian closed category* $\mathbf{X}$ *the following are equivalent:*

(1) $(\mathbf{A}, U)$ *is concretely cartesian closed,*

(2) $\mathbf{A}$ *is cartesian closed and every* $\mathbf{A}$*-morphism with a discrete codomain has a discrete domain.*

(3) for each $\mathbf{A}$*-object* A *the functor* $(A \times -)$ *preserves final sinks.*

7.27.3 Cartesian closed constructs

Proposition. *(27.16) Every concretely cartesian closed amnestic construct with discrete terminal object is, up to concrete isomorphism, a full subconstruct of* **Set***.*

Definition. (27.17) A construct $(\mathbf{A}, U)$ is said to *have function spaces* provided that the following holds:

(1) $(\mathbf{A}, U)$ has finite concrete products,

(2) $\mathbf{A}$ is cartesian closed and the evaluation morphisms $A \times B^A \xrightarrow{ev} B$ can be chosen in such a way that $U(B^A) = \hom_{\mathbf{A}}(A, B)$ and ev is the restriction of the canonical evaluation map in **Set**

Proposition. *(27.18) Let $(\mathbf{A}, U)$ be a construct with finite concrete products. If $\mathbf{A}$ is cartesian closed, then the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ has functions spaces,

(2) terminal $\mathbf{A}$-objects are discrete

(3) each constant function[9] between $\mathbf{A}$-objects is an $\mathbf{A}$-morphism.

Definition. (27.20) A construct is called *well-fibred* provided that it is fibre-small and for each set with at most one element, the corresponding fibre has exactly one element.

Theorem. *(27.22) For well-fibred topological constructs the following conditions are equivalent:*

(1) $\mathbf{A}$ is cartesian closed,

(2) $(\mathbf{A}, U)$ has function spaces,

(3) for each $\mathbf{A}$-object A the functor $(A \times -)$ preserves final epi-sinks,

(4) for each $\mathbf{A}$-object A the functor $(A \times -)$ preserves colimits,

(5) for each $\mathbf{A}$-object the functor $(A \times -)$ preserves (a) coproducts and (b) quotients.

Proposition. *(27.24) For cartesian closed, well-fibred topological constructs, the following hold:*

(1) products with discrete factors A are coproducts:

$$A \times B \cong {}^{|A|}B = \coprod_{x \in |A|} B,$$

(2) power objects with discrete exponents A are powers:

$$B^A \cong B^{|A|} = \prod_{x \in |A|} B.$$

[9] A function is called *constant* provided that it factors through a one-element set.

27C: **Coreflective Hulls and Cartesian Closedness.** Let $(\mathbf{A}, U)$ be a cartesian closed topological construct and let $\mathbf{B}$ be a full subcategory of $\mathbf{A}$ that is closed under the formation of finite products. The bicoreflective hull of $\mathbf{B}$ in $\mathbf{A}$ is cartesian closed.
27E: **Composition as a Morphism.** In cartesian closed constructs describe explicitly the unique morphism *comp*: $B^A \times C^B \to C^A$ that makes the diagram

$$\begin{tikzcd} A \times (B^A \times C^B) \arrow[r, "ev \times id_{C^B}"] \arrow[d, "id_A \times comp"'] & B \times C^B \arrow[d, "ev"] \\ A \times C^A \arrow[r, "ev"'] & C \end{tikzcd}$$

commute.
27F: **(Concretely) Cartesian Closed Topological Categories as Injective Objects** Let $\mathbf{X}$ be a cartesian closed category and let $\mathbf{CAT_p}(\mathbf{X})$ be the quasicategory whose objects are the amnestic concrete categories over $\mathbf{X}$ with finite concrete products, and whose morphisms are the concrete functors over $\mathbf{X}$ that preserve finite products.

(a) Show that the injective objects in $\mathbf{CAT_p}(\mathbf{X})$ (with respect to full embeddings) are precisely the concretely cartesian closed topological categories over $\mathbf{X}$.

Let $\mathbf{CONST_p}$ be the quasicategory whose objects are the amnestic well-fibred constructs with finite products, and whose morphisms are the concrete functors that preserve finite products.

(b) Show that the injective objects in $\mathbf{CONST_p}$ (with respect to full embeddings) are precisely the well-fibred topological constructs that have function spaces.
27G: **(Concretely) Cartesian Closed Topological Hulls.**

Let $\mathbf{X}$ be a cartesian closed category. A morphism $(\mathbf{A}, U) \xrightarrow{E} (\mathbf{B}, V)$ in $\mathbf{CAT_p}(\mathbf{X})$ is called a *concretely cartesian closed topological hull* (shortly: a CCCT hull) of $(\mathbf{A}, U)$ provided that the following are satisfied:

(1) E is a full embedding,

(2) $E[\mathbf{A}]$ is finally dense in $(\mathbf{B}, V)$,

(3) $\{EA^{E\overline{A}} | A, \overline{A} \in \mathrm{Ob}\mathbf{A}\}$ is initially dense in $(\mathbf{B}, V)$,

(4) $(\mathbf{B}, V)$ is a concretely cartesian closed topological category.

(a) Show that the injective hulls in $\mathbf{Cat_p}(\mathbf{X})$ are precisely the CCCT hulls.

A morphism $(\mathbf{A}, U) \xrightarrow{E} (\mathbf{B}, V)$ in $\mathbf{CONST_p}$ is called a *cartesian closed topological hull* (shortly: a CCT hull) of $(\mathbf{A}, U)$ provided that the above conditions (1), (2), (3) and the following condition (4*) holds:

4* $(\mathbf{B}, V)$ is a cartesian closed topological category.

(b) The injective hulls in $\mathbf{CONTS_p}$ are precisely the CCT hulls.
(c) The concrete embedding $\mathbf{PrTop} \hookrightarrow \mathbf{PsTop}$ is a CCT hull of $\mathbf{PrTop}$.
27H: **Well-Fibred Functor-Structured Constructs.** Show that a functor-structured construct $\mathbf{Spa}(T)$ is well-fibred only if T is the constant functor, defined by $T(X \xrightarrow{f} Y) = \emptyset \xrightarrow{id_\emptyset} \emptyset$, i.e., only if $\mathbf{Spa}(T)$ is concretely isomorphic to the construct $\mathbf{Set}$.
27I: **Cartesian Closed Functor-Structured Categories.**

(a) Prove that if $\mathbf{Spa}(T)$ is cartesian closed, then it is concretely cartesian closed.

(b) Prove that $\mathbf{Spa}(T)$ is (concretely) cartesian closed whenever $\mathbf{X}$ is cartesian closed and T *weakly preserves pullbacks,* i.e., for each 2-sink $\bullet \xrightarrow{f} \bullet \xleftarrow{g} \bullet$ the factorizing morphism of the T-image of the pullback of (f,g) through the pullback of (Tf,Tg) is a retraction.

(c) Verify that the **Set**-functor $\mathcal{S}^n$ and $\mathcal{P}$ weakly preserve pullbacks.

7.28 Partial morphisms, quasitopoi, and topological universes

7.28.1 Representations of partial morphisms

Definition. (28.1) Let M be a class of morphisms in $\mathbf{A}$.

(1) A 2-source ($A \xleftarrow{m} \bullet \xrightarrow{f} B$) with $m \in M$ is called an *M-partial morphism* from A into B. (Extremal) Mono-partial morphisms are called *(extremal) partial morphism.*

(2) An M-morphism $B \xrightarrow{m_B} B^*$ is said to *represent* M-partial morphisms into B provided that the following two conditions are satisfied:

(a) for every morphism $\bullet \xrightarrow{f} B^*$ there exists a pullback

$$\begin{tikzcd} \circ \arrow[r, "\overline{m}"] \arrow[d, "\overline{f}"'] & \bullet \arrow[d, "f"] \\ B \arrow[r, "m_B"'] & B^* \end{tikzcd}$$

and every such $\overline{m}$ belongs to M,

(b) for every M-partial morphism ($\bullet \xleftarrow{m} \circ \xrightarrow{f} \bullet$) there exists a unique morphism $\bullet \xrightarrow{f^*} B^*$ such that

$$\begin{tikzcd} \circ \arrow[r, "m"] \arrow[d, "f"'] & \bullet \arrow[d, "f^*"] \\ B \arrow[r, "m_B"'] & B^* \end{tikzcd}$$

is a pullback.

(c) $\mathbf{A}$ is said to have *representable M-partial morphisms* provided that for each $\mathbf{A}$-object B there exists some $B \xrightarrow{m_B} B^*$ that represents M-partial morphisms into B.

Proposition. *(28.3) If $\mathbf{A}$ has representable M-partial morphisms, then the following hold:*

(1) $Iso(\mathbf{A}) \subseteq M \subseteq RegMono(\mathbf{A})$.

(2) Pullbacks along M-morphisms exist and belong to M.

Proposition. *(28.5) If $\mathbf{A}$ has finite products and representable M-partial morphisms, where M is a family that contains all sections, then the following hold:*

(1) $\mathbf{A}$ is finitely complete.

(2) $M = RegMono(\mathbf{A})$

Corollary. *(28.6) If* $\mathbf{A}$ *has finite products and representable (extremal) partial morphisms, then the following hold:*

(1) $\mathbf{A}$ *is finitely complete.*

(2) $(Extr)Mono(\mathbf{A}) = RegMono(\mathbf{A})$.

(3) In $\mathbf{A}$ *regular monomorphisms are closed under composition.*

7.28.2 Quasitopoi

Definition. (28.7) Let M be a class of morphisms in a category $\mathbf{A}$. Then $\mathbf{A}$ is called *M-topos* provided that it:

(1) has representable M-partial morphisms,

(2) is cartesian closed, and

(3) is finitely complete.

Mono-topoi are called *topoi*, and ExtrMono-topoi are called *quasitopoi.*

Proposition. *(28.8) Topoi are precisely the balanced quasitopoi.*

Proposition. *(28.10) Each quasitopos is (Epi, RegMono)-structured.*

7.28.3 Concrete quasitopoi

Proposition. *(28.12) Let* $(\mathbf{A}, U)$ *be a topological category over* $\mathbf{X}$*, let* M *be a class of morphisms in* $\mathbf{X}$*, and let* M_{init} *be the class of all initial* $\mathbf{A}$*-morphisms* M *with* $Um \in M$*. If* $\mathbf{A}$ *has representable* M_{init}*-partial morphisms, then* $\mathbf{X}$ *has representable* M*-partial morphisms and* U *preserves these representations.*

Definition. (28.13)

(1) Let $B \xrightarrow{M} A$ be a morphisms and let $\mathcal{S} = (A_i \xrightarrow{f_i} A)$ be a sink. If for each $i \in I$ the diagram

$$\begin{tikzcd} B_i \arrow[r, "m_i"] \arrow[d, "\hat{f}_i"'] & A_i \arrow[d, "f_i"] \\ B \arrow[r, "m"'] & A \end{tikzcd}$$

is a pullback, then the sink $(B_i \xrightarrow{\hat{f}_i} B)_I$ is called a *pullback of $\mathcal{S}$ along m.*

(2) Let M be a class of morphisms and let $\mathcal{C}$ be a conglomerate of sinks in a category $\mathbf{A}$. $\mathcal{C}$ is called *stable along pullback along M* provided that every pullback of a sink in $\mathcal{C}$ along a M-morphism is a member of $\mathcal{C}$. In particular, $\mathcal{C}$ is called *pullback-stable* provided that $\mathcal{C}$ is stable under pullbacks along Mor($\mathbf{A}$). In the case that M is a class of monomorphisms, $\mathcal{C}$ is called *reducible* provided that it is stable under pullbacks along M. When M is the class of all extremal monomorphisms, we say that $\mathcal{C}$ is *extremally reducible.*

Theorem. *(28.15) If* $\mathbf{X}$ *has representable* M*-partial morphisms, then for each topological category* $(\mathbf{A}, U)$ *over* $\mathbf{X}$ *the following conditions are equivalent:*

(1) $\mathbf{A}$ *has representable* M_{init}*-partial morphisms,*

(2) final sinks in $(\mathbf{A}, U)$ *are* M_{init}*-reducible.*

Definition. (28.16) A concrete category is called *universally topological* provided that it is topological and final sinks are pullback-stable.

Theorem. *(28.18) For topological categories* $(\mathbf{A}, U)$ *over a quasitopos* $\mathbf{X}$ *the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ *is universally topological,*

(2) $(\mathbf{A}, U)$ *is a concrete quasitopos, i.e.,* $\mathbf{A}$ *is a quasitopos and* U *preserves power objects and representations of extremal partial morphisms.*

7.28.4 Topological universes

Theorem. *(28.19) For well-fibred topological constructs, the following conditions are equivalent:*

(1) extremal partial morphisms are representable,

(2) final sinks are extremally reducible,

(3) final epi-sinks are extremally reducible,

(4) coproducts and quotients are extremally reducible.

Definition. (28.21) A well-fibred topological construct $(\mathbf{A}, U)$ for which $\mathbf{A}$ is a quasitopos is called a *topological universe.*

Theorem. *(28.22) For a well-fibred topological construct* $(\mathbf{A}, U)$ *the following conditions are equivalent:*

(1) $(\mathbf{A}, U)$ *is a topological universe,*

(2) $(\mathbf{A}, U)$ *has function-spaces and representable extremal partial morphisms,*

(3) in $(\mathbf{A}, U)$ *final epi-sinks are pullback-stable.*

TODO conretely complete = ?

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