CHAPTER 5: TOPOLOGICAL FUNCTORS

1 Factorization structures

Definition (5.1.1). Let $\mathbf{C}$ be a category.

(i) If E is a class of $\mathbf{C}$-morphisms which is closed under composition with isomorphisms and M is a conglomerate of sources in $\mathbf{C}$ which is closed under composition with isomorphisms, then the pair (E, M) is called a *factorization structure* on $\mathbf{C}$ provided that the following are satisfied:

(a) For each source $(X, (f_i)_{i\in I})$ there exists $e\colon X \to Y$ in E and $(Y, (m_i)_{i\in I})$ in M such that $f_i = m_i \circ e$ for each $i \in I$; briefly each source has (E, M)*-factorization.*

(b) For any two $\mathbf{C}$-morphisms f and e and any two sources $(Y, (m_i)_{i\in I})$ and $(Z, (f_i)_{i\in I})$ in $\mathbf{C}$ such that $e \in E$ $(Y, (m_i)_{i\in I}) \in M$ and $f_i \circ e = m_i \circ f$ for each $i \in I$, there exists a unique $\mathbf{C}$-morphism $g\colon Z \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f"'] & Z \arrow[d, "f_i"] \arrow[dl, dashed, "g"'] \\ Y \arrow[r, "m_i"'] & X_i \end{tikzcd}$$

commutes for each $i \in I$; briefly: $\mathbf{C}$ satisfies the (E, M)*-diagonalization property.*

(ii) $\mathbf{C}$ is called (E, M)*-category* provided that (E, M) is a factorization structure on $\mathbf{C}$.

Let $\mathbf{C}$ be a topological category or $\mathbf{C} = \mathbf{Haus}$. E consists of all extremal epimorphisms and M consists of all mono-sources in $\mathbf{C}$. Then (E, M) is a factorization structure on $\mathbf{C}$.

Theorem (5.1.4). *Let* $\mathbf{C}$ *be a category and* (E, M) *a factorization structure on* $\mathbf{C}$*. Then the following are satisfied:*

(i) (E, M)*-factorizations are uniquely determined (up to isomorphisms).*

(ii) $E \cap M$ *is the class of all* $\mathbf{C}$*-isomorphisms.*

(iii) Every extremal source in $\mathbf{C}$ *belongs to* M*.*

(iv) If f, g *and* h *are* $\mathbf{C}$*-morphisms such that* $h = g \circ f$*, then the following are satisfied:*

(a) If $h \in E$ *and* f *is a* $\mathbf{C}$*-epimorphism, then* $g \in E$*.*

(b) $f \in E$ *and* $g \in E$ *imply* $h \in E$*, i.e.* E *is closed under composition.*

(v) If $(X, (f_i)_{i\in I})$ *is a source in* $\mathbf{C}$ *and* $(X, (g_j)_{j\in J})$, $(Z_j, (k_{j_i})_{i\in I_j})_{j\in J}$ *is a factorization of* $(X, (f_i)_{i\in I})$*, then the following hold:*

(a) $(X, (f_i)_{i\in I}) \in M$ *implies* $(X, (g_j)_{j\in J}) \in M$*.*

(b) $(X, (g_j)_{j\in J}) \in M$ *and* $(Z_j, (k_{j_i})_{i\in I_j}) \in M$ *for each* $j \in J$ *imply* $(X, (f_i)_{i\in I}) \in M$*.*

(vi) If $(X,(f_i)_{i\in I})$ is a source in $\mathbf{C}$ and there is some $J\subset I$ such that $(X,(f_j)_{j\in J})\in M$, then $(X,(f_i)_{i\in I})\in M$.

(vii) E and M determine each other by the diagonalization property.

2 Definition and properties of topological functors

Definition (5.2.1). Let $\mathbf{C}$ be a category supplied with a factorization structure (E,M), let $\mathbf{A}$ be any category and let $T\colon \mathbf{A}\to\mathbf{C}$ be a functor.

(i) A source $(A,(f_i : A\to A_i)_{i\in I})$ in $\mathbf{A}$ is called *T-initial* provided that for each source $(B,(g_i : B\to A_i)_{i\in I})$ in $\mathbf{A}$ and each $\mathbf{C}$-morphism $f\colon T(A)\to T(B)$ such that $T(f_i)\circ f = T(g_i)$ for each $i\in I$, there exists a unique $\mathbf{A}$-morphism $\overline{f}\colon B\to A$ with $T(\overline{f})=f$ and $f_i\circ\overline{f}=g_i$ for each $i\in I$.

(ii) A source $(A,(f_i : A\to A_i)_{i\in I})$ in $\mathbf{A}$ *T-lifts* a source $(X,(g_i : X\to T(A_i))_{i\in I})$ in $\mathbf{C}$ provided that there exists an isomorphism $h\colon X\to T(A)$ in $\mathbf{C}$ with $T(f_i)\circ h=g_i$ for each $i\in I$.

(iii) T is called *(E,M)-topological* provided that for each family $(A_i)_{i\in I}$ of $\mathbf{A}$-objects and each source $(X,(m_i : X\to T(A_i))_{i\in I})$ in M, there exists a T-initial source $(A,(f_i : A\to A_i)_{i\in I})$ in $\mathbf{A}$ which T-lifts $(X,(m_i)_{i\in I})$.

(iv) T is called *absolutely topological* provided that T is topological for any factorization structure (E,M) on $\mathbf{C}$.

Theorem (5.2.4). *Let $\mathbf{C}$ be a category supplied with a factorization structure (E,M) and let $T\colon\mathbf{A}\to\mathbf{C}$ be an (E,M)-topological functor. Then T is faithful.*

Theorem (5.2.5). *Let $\mathbf{C}$ be an (E,M)-category and $T\colon\mathbf{A}\to\mathbf{C}$ an (E,M)-topological functor. If E_T denotes the class of all morphisms f in $\mathbf{A}$ with $T(f)\in E$ and M_T denotes the conglomerate of all T-initial sources $(A,(f_i)_{i\in I})$ in $\mathbf{A}$ with $(T(A),(T(f_i))_{i\in I})\in M$, then $\mathbf{A}$ is an (E_T,M_T)-category.*

Theorem (5.2.8). *Let $T\colon\mathbf{A}\to\mathbf{C}$ be an (E,M)-topological functor, let $D\colon\mathbf{I}\to\mathbf{A}$ be a diagram and let $(L,(l_i : L\to D(i))_{i\in|\mathbf{I}|})$ be a source in $\mathbf{A}$. Then the following are equivalent:*

(i) $(L,(l_i)_{i\in|\mathbf{I}|})$ is a limit of D.

(ii) $(L,(l_i)_{i\in|\mathbf{I}|})$ is T-initial and $(T(L),(T(l_i))_{i\in|\mathbf{I}|})$ is a limit of $T\circ D$.

Corollary (5.2.9). *If $T\colon\mathbf{A}\to\mathbf{C}$ is an (E,M)-topological functor and $\mathbf{C}$ is complete, then $\mathbf{A}$ is complete.*

Proposition (5.2.10). *Let $T\colon\mathbf{A}\to\mathbf{C}$ be an (E,M)-topological functor. Then for each family $(A_i)_{i\in I}$ of $\mathbf{A}$-objects and each sink $((f_i\colon T(A_i)\to X)_{i\in I},X)$ in $\mathbf{C}$, there exists a sink $((t_i\colon A_i\to A)_{i\in I},A)$ in $\mathbf{A}$ and a morphism $e\colon X\to T(A)$ in E such that $T(t_i)=e\circ f_i$ for each $i\in I$ and such that the following condition is satisfied:*
(F) For each sink $((g_i\colon A_i\to B),B)$ in $\mathbf{A}$ and each morphism $g\colon X\to T(B)$ with $T(g_i)=g\circ f_i$ for each $i\in I$, there exists a morphism $k\colon A\to B$ with $T(k)\circ e=g$.

Theorem (5.2.11). *Let $T\colon \mathbf{A} \to \mathbf{C}$ be an (E, M)-topological functor and let $D\colon \mathbf{I} \to \mathbf{A}$ be a diagram such that $T \circ D$ has a colimit. Then D has a colimit.*

Corollary (5.2.12). *Let $T\colon \mathbf{A} \to \mathbf{C}$ be an (E, M)-topological functor. If $\mathbf{C}$ is cocomplete, then $\mathbf{A}$ is cocomplete.*

3 Initially structured categories

A functor $F\colon \mathbf{A} \to \mathbf{B}$ is called *amnestic* provided that any $\mathbf{A}$-isomorphism f is an $\mathbf{A}$-identity iff $F(f)$ is a $\mathbf{B}$-identity.

A functor $F\colon \mathbf{A} \to \mathbf{B}$ is called *transportable* provided that for each $\mathbf{A}$-object A, each $\mathbf{B}$-object B and each isomorphism $q\colon B \to F(A)$, there exists a unique $\mathbf{A}$-object C and isomorphism $\overline{q}\colon C \to A$ with $F(\overline{q}) = q$.

An object X in a category $\mathbf{C}$ is called *terminal* provided thet for every object Y in $\mathbf{C}$, the set of morphisms $[Y, X]_{\mathbf{C}}$ is a singleton.

Definition (5.3.1). A pair $(\mathbf{A}, T)$ is called an *initially structured* category provided that $\mathbf{A}$ is a category and $T\colon \mathbf{A} \to \mathbf{Set}$ is a functor which is amnestic and transportable such that the following hold:

(i) T is (epi, mono-source)-topological.

(ii) T has small fibres, i.e. for each $X \in \mathbf{Set}$ $\{A \in |\mathbf{A}| : T(A) = X\}$ is a set.

(iii) There is precisely one object P in $\mathbf{A}$ (up to isomorphism) such that $T(P)$ is a terminal separator in $\mathbf{Set}$, i.e. $T(P)$ is a singleton.

Remark (5.3.3). Obviously the condition (i) can be replaced by:
Every source $(X, (f_i\colon X \to T(A_i))_{i\in I})$ is $\mathbf{Set}$ has an (epi, mono-source) factorization

$$\begin{tikzcd} X \arrow[rr, "f_i"] \arrow[dr, "e"'] & & T(A_i) \\ & T(B) \arrow[ur, "T(g_i)"'] & \end{tikzcd}$$

Theorem (5.3.4). *Let $(\mathbf{A}, T)$ be an initially structured category. Then the following are satisfied:*

(i) T is faithful.

(ii) A source $(A, (f_i\colon A \to D_i)_{i\in I})$ in $\mathbf{A}$ is a limit of a diagram $D\colon \mathbf{I} \to \mathbf{A}$ with $|\mathbf{I}| = I$ if and only if if this source is T-initial and $(T(A), (T(f_i)\colon T(A) \to T(D_i))_{i\in I})$ is limit of $T \circ D$.

(iii) For any sink $((f_i\colon D_i \to A)_{i\in I, A})$ in $\mathbf{A}$ and an epimorphism $e\colon X \to T(A)$ with $e \circ f_i = T(a_i)$ for each $i \in I$ such that the following condition (F) is satisfied: For each sink $((b_i : D_i \to B)_{i\in I}, B)$ in $\mathbf{A}$ and each morphism $d\colon X \to T(B)$ there exists a (unique) morphism $c\colon A \to B$ such that $T(c) \circ e = d$.

(iv) **A** *is complete and cocomplete.*

Proposition (5.3.5). *Let* $(\mathbf{A}, T)$ *be an initially structured category. Any sink* $((f_i\colon A_i \to C)_{i\in I}, C)$ *in* **A** *has*

(i) a factorization

$$\begin{tikzcd} A_i \arrow[rr, "f_i"] \arrow[dr, "a_i"'] & & C \\ & A \arrow[ur, "c"'] & \end{tikzcd}$$

such that $T(c)$ *is an isomorphism and* $((a_i\colon A_i \to A)_{i\in I}, A)$ *is* T-final *(i.e. for any sink* $((b_i\colon A_i \to B)_{i\in I}, B)$ *and any morphism* $f\colon T(A) \to T(B)$ *with* $f \circ T(a_i) = T(b_i)$ *for each* $i \in I$*, there exists a unique morphism* $\overline{f}\colon A \to B$ *with* $T(\overline{f}) = f$ *and* $\overline{f} \circ a_i = b_i$ *for each* $i \in I$*), and*

(ii) a factorization $f_i = c \circ a_i$ *where* c *is a monomorphism and* $((a_i\colon A_i \to A)_{i\in I}, A)$ *is a* T*-final epi-sink.*

Proposition (5.3.6). *Let* $(\mathbf{A}, T)$ *be an initially structured category. Then the following are satisfied:*

(i) The object P *in* **A** *given by* IS_3 *(definition of initially structured category, (iii)) is terminal and a separator in* **A**.

(ii) If X *and* Y *are* **A***-objects and* $g\colon T(X) \to T(Y)$ *is a constant morphism, then there exists a unique* **A***-m0orphism* $f\colon X \to Y$ *with* $T(f) = g$.

Definition (5.3.7). Let $(\mathbf{A}, T)$ be an initially structured category. An **A**-morphism $f\colon A \to B$ is called a

(i) T*-embedding* provided that $(A, f\colon A \to B)$ is a T-initial mono-source.

(ii) T*-quotient map* provided that $(A, f\colon A \to B)$ is a T-final epi-sink.

Proposition (5.3.8). *Let* $(\mathbf{A}, T)$ *be an initially structured category. Then the following are satisfied:*

(i) f *is an* **A***-monomorphism if and only if* $T(f)$ *is a monomorphism in* **Set**.

(ii) f *is an* **A***-epimorphism whenever* $T(f)$ *is an epimorphism in* **Set**.

(iii) Every extremal monomorphism in **A** *is a* T*-embedding.*

Proposition (5.3.10). *Let* $(\mathbf{A}, T)$ *be an initially structured category. Then the following are satisfied:*

(i) If $((f_i\colon A_i \to A)_{i\in I}, A)$ *is a* T*-final epi-sink in* **A***, then* $((T(f_i)\colon T(A_i) \to T(A))_{i\in I}, T(A))$ *is an epi-sink in* **Set**.

(ii) A sink in **A** *is an extremal epi-sink if and only if it is a* T*-final epi-sink.*

Proposition (5.3.11). *For every $\mathbf{A}$-morphism $f\colon A \to B$ in an initially structured category $(\mathbf{A}, T)$ the following are equivalent:*

(i) f is a T-quotient map.

(ii) f is a regular epimorphism.

(iii) f is an extremal epimorphism.

Proposition (5.3.13). *Every initially structured category $(\mathbf{A}, T)$ is well-powered.*

Proposition (5.3.15). *Let $(\mathbf{A}, T)$ be an initially structured category. Then $\mathbf{A}$ is a category with a factorization structure (E, M) where E consists of all extremal epimorphisms and M of all mono-sources.*

Theorem (5.3.16). *Every E-reflective (full and isomorphism-closed) subcategory of an initially structured category $(\mathbf{A}, T)$ is initially structured provided that E consists of all $\mathbf{A}$-morphisms f for which $T(f)$ is an epimorphism in $\mathbf{Set}$.*

Remark (5.3.17). Every extremal epireflective (resp. epireflective) full and isomorphism-closed subcategory of an initially structure (resp. topological) category is initially structured.

Proposition (5.3.18). *For each T-final epi-sink $(f_i\colon A_i \to A)_{i\in I}$ in an initially structured category, there is a set $J \subset I$ such that $(f_j\colon A_j \to A)_{j\in J}$ is likewise a T-final epi-sink.*

Proposition (5.3.19). *Let $(\mathbf{A}, T)$ and $(\mathbf{A}', T')$ be initially structured categories and let $F\colon \mathbf{A} \to \mathbf{A}'$ be a functor preserving colimits. If $(f_i\colon A_i \to A)$ is a T-final epi-sink in $\mathbf{A}$, then $(T(f_i)\colon T(A_i) \to T(A))$ is a T'-final epi-sink in $\mathbf{A}'$.*

Theorem (5.3.20). *Let $(\mathbf{A}, T)$ be an initially structured category. $\mathbf{A}$ is cartesian closed if and only if $A \times -$ preserves T-final epi-sinks for each $A \in |\mathbf{A}|$.*

Theorem (5.3.22). *Every extremal epireflective (full and isomorphism-closed) subcategory $\mathbf{B}$ of an initially structured cartesian closed category $(\mathbf{A}, T)$ is cartesian closed.*

Theorem (5.3.24). *For a concrete category $\mathbf{C}$ (in the sense of 1.1.1) the following are equivalent:*

(i) $\mathbf{C}$ is initially structured.

(ii) $\mathbf{C}$ is an epireflective (full and isomorphism-closed) subcategory of a topological category.

(iii) $\mathbf{C}$ is an extremal epireflective (full and isomorphism-closed) subcategory of a topological category.