Prerequisites

Topological spaces. A set E in a space X is *σ-compact* if there exists a sequence of compact sets such that $E = \bigcup_{n=1}^{\infty} C_n$. A space X is *locally compact* if every point of X has a neighborhood whose closure is compact. A subset E of a locally compact space is *bounded* if there exists a compact set C such that $E \subset C$.

Topological groups. The set xE [or Ex] is called a *left translation* [or *right translation*.] If Y is a subgroup of X, the sets xY and Yx are called (left and right) *cosets* of Y.

A *topological group* is a group X which is a Hausdorff space such that the transformation (from $X \times X$ onto X) which sends (x, y) into $x^{-1}y$ is continuous. A class $\mathbf{N}$ of open sets containing e in a topological group is a *base at e* if

(a) for every x different form e there exists a set U in $\mathbf{N}$ such that $x \notin U$,

(b) for any two sets U and V in $\mathbf{N}$ there exists a set W in $\mathbf{N}$ such that $W \subset U \cap V$,

(c) for any set $U \in \mathbf{N}$ there exists a set $W \in \mathbf{N}$ such that $V^{-1}V \subset U$,

(d) for any set $U \in \mathbf{N}$ and any element $x \in X$, there exists a set $V \in \mathbf{N}$ such that $V \subset xUx^{-1}$, and

(e) for any set $U \in \mathbf{N}$ there exists a set $V \in \mathbf{N}$ such that $Vx \subset U$.

If $\mathbf{N}$ is a satisfies the conditions described above, and if the class of all translation of sets of $\mathbf{N}$ is taken for a base, then, with respect to the topology so defined, X becomes a topological group.

A subset E of a topological group X is *bounded* if, for every neighborhood U of e, there exists a finite set $\{x_1, \ldots, x_n\}$ such that $E \subset \bigcup_{i=1}^{n} x_i U$; if X is locally compact, then this definition of boundedness agrees with the one applicable in any locally compact space (i.e. the one that requires that the closure of E be compact). If a continuous, real valued function f on X is such that the set $N(f) = \{x : f(x) \neq 0\}$ is bounded then f is *uniformly continuous* in the sense that to every positive number ε there corresponds a neighborhood U of e such that $|f(x_1) - f(x_2)| < \varepsilon$ whenever $x_1 x_2^{-1} \in U$.

A topological group is *locally bounded* if there exists in it a bounded neighborhood of e. To every locally bounded topological group X, there corresponds a locally compact topological group X^*, called the *completion* of X (uniquely determined to within an isomorphism), such that X is a dense subgroup of X^*. Every closed subgroup and every quotient group of a locally compact group is a locally compact group.

1 Sets and classes

1.3 Limits, complements, and differences

If (E_n) is a sequence of subsets of X, the set E^* of all those point of X which belong to E_n for infinitely many values of n is called the *superior limit* of the sequence and is denoted by $E^* = \limsup_n E_n$.

The set E_* of all those points of X which belong to E_n for all but a finite number of values of n is called the *inferior limit* of the sequence and is denoted by $E_* = \liminf_n E_n$.

$\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$, $\liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$.

If it so happens that $E^* = E_*$, we shall use the notation $\lim_n E_n$. If (E_n) is a monotone sequence, then $\lim_n E_n$ exists and is equal to $\bigcup_n E_n$ or $\bigcap_n E_n$ according to as the sequence is increasing or decreasing.

The difference $E - F$ is called *proper* if $E \supset F$.

1.4 Ring and algebras

A *ring* (or *Boolean ring*) of sets is a non-empty class $\mathbf{R}$ of sets such that if $E, F \in \mathbf{R}$ then $E \cup F \in \mathbf{R}$ and $E - F \in \mathbf{R}$.

The empty set belongs to every ring $\mathbf{R}$. A non empty class of sets closed under the formation of unions and proper differences is a ring. If a non empty class of sets is closed under the formation of intersections, proper differences, and disjoint unions, then it is a ring.

An *algebra* (or *Boolean algebra*) of sets is a non empty class $\mathbf{R}$ of sets such that if $E, F \in \mathbf{R}$ then $E \cup F \in \mathbf{R}$, and if $E \in \mathbf{R}$, then $E' \in \mathbf{R}$.

Algebra = ring containing X.

A *semiring* is a non empty class $\mathbf{P}$ of sets such that:

a) if $E, F \in \mathbf{P}$ then $E \cap F \in \mathbf{P}$,

b) if $E, F \in \mathbf{P}$ and $E \subset F$ then there is a finite class $\{C_0, C_1, \dots, C_n\}$ of sets in $\mathbf{P}$ such that $E = C_0 \subset C_1 \subset \dots \subset C_n = F$ and $D_i = C_i - C_{i-1} \in \mathbf{P}$ for $i = 1, 2, \dots, n$.

1.5 Generated rings and σ-rings

Theorem A. *If $\mathbf{E}$ is any class of sets, then there exists a unique ring $\mathbf{R}_0$ such that $\mathbf{R}_0 \supset \mathbf{E}$ and such that if $\mathbf{R}$ is any other ring containing $\mathbf{E}$ then $\mathbf{R}_0 \subset \mathbf{R}$.*

The ring $\mathbf{R}_0$, the smallest ring containing $\mathbf{E}$, is called the ring *generated* by $\mathbf{E}$; it will be denoted by $\mathbf{R}(\mathbf{E})$.

Theorem B. *If $\mathbf{E}$ is any class of sets, then every set in $\mathbf{R}(\mathbf{E})$ may be covered by a finite union of sets in $\mathbf{E}$.*

Theorem C. *If $\mathbf{E}$ is a countable class of sets, then $\mathbf{R}(\mathbf{E})$ is countable.*

A *σ-ring* is a non empty class $\mathbf{S}$ of sets such that

(a) if $E \in \mathbf{S}$ and $F \in \mathbf{S}$, then $E - F \in \mathbf{S}$, and

(b) if $E_i \in \mathbf{S}$, $i = 1, 2, \dots$, then $\bigcup_{i=1}^{\infty} E_i \in \mathbf{S}$.

Equivalently a σ-ring is a ring closed under the formation of countable unions. A σ-ring is closed under the formation of countable intersections.

The σ-ring $\mathbf{S}(\mathbf{E})$ *generated* by a class $\mathbf{E}$ of sets is the smallest σ-ring containing $\mathbf{E}$.

Theorem D. *If $\mathbf{E}$ is any class of sets and E is any set in $\mathbf{S} = \mathbf{S}(\mathbf{E})$, then there exists a countable subclass $\mathbf{D}$ of $\mathbf{E}$ such that $E \in \mathbf{S}(\mathbf{D})$.*

Theorem E. *If $\mathbf{E}$ is any class of sets and if A is any subset of X, then*

$$\mathbf{S}(\mathbf{E}) \cap A = \mathbf{S}(\mathbf{E} \cap A).$$

1.6 Monotone classes

A non empty class $\mathbf{M}$ of sets is *monotone* if, for every monotone sequence (E_n) of sets in $\mathbf{M}$, we have $\lim_n E_n \in \mathbf{M}$.

The monotone class $\mathbf{M}(\mathbf{E})$ *generated* by a class $\mathbf{E}$ of sets is the smallest monotone class containing $\mathbf{E}$.

Theorem A. *A σ-ring is a monotone class; a monotone ring is a σ-ring.*

Theorem B. *If $\mathbf{R}$ is a ring, then $\mathbf{M}(\mathbf{R}) = \mathbf{S}(\mathbf{R})$. Hence if a monotone class contains a ring $\mathbf{R}$, then it contains $\mathbf{S}(\mathbf{R})$.*

2 Measures and outer measures

2.7 Measure on rings

A *set function* is a function whose domain of definition is a class of sets. An extended real valued set function μ defined on a class $\mathbf{E}$ is *additive* if, whenever

$$E \in \mathbf{E},\ F \in \mathbf{E},\ E \cup F \in \mathbf{E}, \quad \text{and} \quad E \cap F = 0,$$

then

$$\mu(E \cup F) = \mu(E) + \mu(F).$$

An extended real valued set function μ defined on a class $\mathbf{E}$ is *finitely additive* if, for every finite, disjoint class $\{E_1, \ldots, E_n\}$ of sets in $\mathbf{E}$ whose union is also in $\mathbf{E}$, we have

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i).$$

An extended real valued set function μ defined on a class $\mathbf{E}$ is *countably additive* if, for every disjoint sequence (E_n) of sets in $\mathbf{E}$ whose union is also in $\mathbf{E}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

A *measure* is an extended real valued, non negative, and countably additive set function μ, defined on a ring $\mathbf{R}$, and such that $\mu(0) = 0$.

If μ is a measure on a ring $\mathbf{R}$, a set E in $\mathbf{R}$ is said to have *finite measure* if $\mu(E) < +\infty$; the measure of $\mathbf{R}$ is *σ-finite* if there exists a sequence (E_n) of sets in $\mathbf{R}$ such that

$$E \subset \bigcup_{n=1}^{\infty} E_n \qquad \text{and} \qquad \mu(E_n) < +\infty, n = 1, 2, \ldots$$

If the measure of every set E in $\mathbf{R}$ is finite [or σ-finite], the measure μ is called *finite* [or *σ-finite*] on $\mathbf{R}$. If $X \in \mathbf{R}$ (i.e. if $\mathbf{R}$ is an algebra) and $\mu(X)$ is finite or σ-finite, then μ is called *totally finite* or *totally σ-finite* respectively. The measure μ is called *complete* if the conditions

$$E \in \mathbf{R},\ F \subset E, \qquad \text{and} \qquad \mu(E) = 0$$

imply that $F \in \mathbf{R}$.

2.8 Measure on intervals

$\mathbf{P}$ = the class of all bounded, left closed and right opened intervals
$\mathbf{R}$ = all finite, dijoint unions of sets of $\mathbf{P}$

On the class $\mathbf{P}$ we define a set function μ by $\mu(\langle a, b)) = b - a$.

Theorem A. *If $\{E_1, \ldots, E_n\}$ is a finite, disjoint class of sets in $\mathbf{P}$, each contained in a given set E_0 in $\mathbf{P}$, then*

$$\sum_{i=1}^{n} \mu(E_i) \leq \mu(E_0).$$

Theorem B. *If a closed interval F_0, $F_0 = \langle a_0, b_0\rangle$, is contained in the union of a finite number of bounded, open intervals, $U_1, \ldots, U_n$, $U_i = (a_i, b_i)$, then $b_0 - a_0 < \sum_{i=1}^{n}(b_i - a_i)$.*

Theorem C. *If $(E_0, E_1, E_2, \ldots)$ is a sequence of sets in $\mathbf{P}$, such that*

$$E_0 \subset \bigcup_{i=1}^{\infty} E_i,$$

then

$$\mu(E_0) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Theorem D. *The set function μ is countably additive on $\mathbf{P}$.*

Theorem E. *There exists a unique, finite measure $\overline{\mu}$ on the ring $\mathbf{R}$ such that, whenever $E \in \mathbf{P}$, $\overline{\mu}(E) = \mu(E)$.*

2.9 Properties of measures

An extended real valued set function μ on a class $\mathbf{E}$ is *monotone* if, whenever $E, F \in \mathbf{E}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$. An extended real valued set function μ on a class $\mathbf{E}$ is *subtractive* if, whenever $E, F \in \mathbf{E}$, $E \subset F$, $F - E \in \mathbf{E}$ and $|\mu(E)| < +\infty$, then

$$\mu(F - E) = \mu(F) - \mu(E).$$

Theorem A. *If μ is a measure on a ring $\mathbf{R}$, then μ is monotone and subtractive.*

Theorem B. *If μ is a measure on a ring $\mathbf{R}$, if $E \in \mathbf{R}$, and if (E_i) is a finite or infinite sequence of sets in $\mathbf{R}$ such that $E \subset \bigcup_i E_i$, then $\mu(E) \leq \sum_i \mu(E_i)$.*

Theorem C. *If μ is a measure on a ring $\mathbf{R}$, if $E \in \mathbf{E}$, and if (E_i) is a finite or infinite disjoint sequence of sets in $\mathbf{R}$ such that $\bigcup_i E_i \subset E$, then*

$$\sum_i \mu(E_i) \leq \mu(E).$$

Theorem D. *If μ is a measure on a ring $\mathbf{R}$ and if (E_n) is an increasing sequence of sets in $\mathbf{R}$ for which $\lim_n E_n \in \mathbf{R}$, then $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.*

Theorem E. *If μ is a measure on a ring $\mathbf{R}$, and if (E_n) is a decreasing sequence of sets in $\mathbf{R}$ of which at least one has finite measure and for which $\lim_n E_n \in \mathbf{R}$, then $\mu(\lim_n E_n) = \lim_n \mu(E_n)$.*

We shall say that an extended real valued set function on a class $\mathbf{E}$ is *continuous from below* at a set E (in $\mathbf{E}$) if, for every increasing sequence (E_n) of sets in $\mathbf{E}$ for which $\lim_n E_n = E$, we have $\lim_n \mu(E_n) = \mu(E)$. Similarly μ is *continuous from above* at E if, for every decreasing sequence (E_n) of sets in $\mathbf{E}$ for which $|\mu(E_m)| < +\infty$ for at least one value of m and for which $\lim_n E_n = E$, we have $\lim_n \mu(E_n) = \mu(E)$.

Theorem F. *Let μ be a finite, non negative, and additive set function on a ring $\mathbf{R}$. If μ is either continuous from below at every E in $\mathbf{R}$, or continuous from above at 0, then μ is a measure on $\mathbf{R}$.*

2.10 Outer measures

A non empty class $\mathbf{E}$ of sets is *hereditary* if, whenever $E \in \mathbf{E}$ and $F \subset E$, then $F \in \mathbf{E}$.

If $\mathbf{E}$ is any class of sets, we shall denote the hereditary σ-ring *generated* by $\mathbf{E}$, i.e. the smallest hereditary σ-ring containing $\mathbf{E}$, by $\mathbf{H}(\mathbf{E})$.

An extended real valued set function μ^* defined on a class $\mathbf{E}$ of sets is *subadditive* if, whenever $E, F \in \mathbf{E}$, and $E \cup F \in \mathbf{E}$, then

$$\mu^*(E \cup F) \leq \mu^*(E) + \mu^*(F).$$

An extended real valued set function μ^* defined on a class $\mathbf{E}$ of sets is *finitely subadditive* if, for every finite class $\{E_1, \ldots, E_n\}$ of sets in $\mathbf{E}$ whose union is also in $\mathbf{E}$, we have

$$\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mu^*(E_i).$$

An extended real valued set function μ^* defined on a class $\mathbf{E}$ of sets is *countably subadditive* if, for sequence (E_i) of sets in $\mathbf{E}$ whose union is also in $\mathbf{E}$, we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

An *outer measure* is an extended real valued, non negative, monotone, and countably subadditive set function μ^*, defined on a hereditary σ-ring $\mathbf{H}$, and such that $\mu^*(0) = 0$.

Theorem A. *If μ is a measure on a ring $\mathbf{R}$ and if, for every E in $\mathbf{H}(\mathbf{R})$,*

$$\mu^*(E) = \inf\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in \mathbf{R}, n = 1, 2, \ldots, E \subset \bigcup_{n=1}^{\infty} E_n\},$$

then μ^* is an extension of μ to an outer measure on $\mathbf{H}(\mathbf{R})$; if μ is [totally] σ-finite, then so is μ^*.

2.11 Measurable sets

Let μ^* be an outer measure on a hereditary σ-ring $\mathbf{H}$. A set E in $\mathbf{H}$ is μ^*-*measurable* if, for every set A in $\mathbf{H}$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E').$$

Theorem A. *If μ^* is an outer measure on a hereditary σ-ring* $\mathbf{H}$ *and if* $\overline{\mathbf{S}}$ *is the class of all μ^*-measurable sets, then* $\overline{\mathbf{S}}$ *is a ring.*

Theorem B. *If μ^* is an outer measure on a hereditary σ-ring* $\mathbf{H}$ *and if* $\overline{\mathbf{S}}$ *is the class of all μ^*-measurable sets, then* $\overline{\mathbf{S}}$ *is a σ-ring. If $A \in \mathbf{H}$ and if (E_n) is a disjoint sequence of set in* $\overline{\mathbf{S}}$ *with $\bigcup_{n=1}^{\infty} = E$, then*

$$\mu^*(A \cap E) = \sum_{n=1}^{\infty} \mu^*(A \cap E_n).$$

Theorem C. *If μ^* is an outer measure on a hereditary σ-ring* $\mathbf{H}$ *and if* $\overline{\mathbf{S}}$ *is the class of all μ^*-measurable sets, then every set of outer measure zero belongs to* $\overline{\mathbf{S}}$ *and the set function $\overline{\mu}$, defined for E in* $\overline{\mathbf{S}}$ *by $\overline{\mu}(E) = \mu^*(E)$, is a complete measure on* $\overline{\mathbf{S}}$.

The measure $\overline{\mu}$ is called the measure *induced* by the outer measure μ^*.

An outer measure μ^* on the class $\mathbf{H}$ of all subsets of a metric space is called a *metric* outer measure if

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$$

whenever $\varrho(E, F) > 0$, where ϱ is the metric on X.

If μ^* is a metric outer measure, then every open set (and therefore every Borel set) is μ^*-measurable. If μ^* is an outer measure on the class of all subsets of a metric space X such that every open set is μ^*-measurable, then μ^* is a metric outer measure.

3 Extension of measures

3.12 Properties of induced measures

Throughout this section we shall assume that μ is a measure on a ring $\mathbf{R}$, μ^* is the induced outer measure on $\mathbf{H}(\mathbf{R})$, and $\overline{\mu}$ is the measure induced by μ^* on the σ-ring $\overline{\mathbf{S}}$ of all μ^*-measurable sets.

Theorem A. *Every set in $\mathbf{S}(\mathbf{R})$ is μ^*-measurable.*

Theorem B. *If $E \in \mathbf{H}(\mathbf{R})$, then*

$$\mu^*(E) = \inf\{\overline{\mu}(F) : E \subset F \in \overline{\mathbf{S}}\} = \inf\{\overline{\mu}(F) : E \subset F \in \mathbf{S}(\mathbf{R})\}.$$

If $E \in \mathbf{H}(\mathbf{R})$ and $F \in \mathbf{S}(\mathbf{R})$, we shall say that F is a *measurable cover* of E if $E \subset F$ and if, for every set $G \in \mathbf{S}(\mathbf{R})$ for which $G \subset F - E$, we have $\overline{\mu}(G) = 0$.

Theorem C. *If a set E in $\mathbf{H}(\mathbf{R})$ is of σ-finite outer measure, then there exists a set F in $\mathbf{S}(\mathbf{R})$ such that $\mu^*(E) = \overline{\mu}(F)$ and such that F is a measurable cover of E.*

Theorem D. *If $E \in \mathbf{H}(\mathbf{R})$ and F is a measurable cover of E, then $\mu^*(E) = \overline{\mu}(F)$; if both F_1 and F_2 are measurable covers of E, then $\overline{\mu}(F_1 \triangle F_2) = 0$.*

Theorem E. *If μ on $\mathbf{R}$ is σ-finite, then so are the measures $\overline{\mu}$ on $\mathbf{S}(\mathbf{R})$ and $\overline{\mu}$ on $\overline{\mathbf{S}}$.*

Suppose that we start with an outer measure μ^*, form the induced measure $\overline{\mu}$, and then form the outer measure $\overline{\mu}^*$ induced by $\overline{\mu}$. In general these two set functions are not the same; if, however, the induced outer measure $\{\overline{\mu}\}^*$ does coincide with the original outer measure μ^*, then μ^* is called *regular*.

If X is a metric space, p is a positive real number, and E is a subset of X, then the p-dimensional *Hausdorff* (outer) *measure* of E is defined to be the number

$$\mu_p^*(E) = \sup_{\varepsilon>0} \inf\{\sum_{i=1}^{\infty}(\delta(E_i))^p : E = \bigcup_{i=1}^{\infty} E_i, \delta(E_i) < \varepsilon, i = 1, 2, \ldots\},$$

where $\delta(E)$ denotes the diameter of E.

The set function μ_p^* is a metric outer measure. The outer measure μ_p^* is regular.

3.13 Extension, completion and approximation

Theorem A. *If μ is a σ-finite measure on a ring $\mathbf{R}$, then there is a unique measure $\overline{\mu}$ on the σ-ring $\mathbf{S}(\mathbf{R})$ such that, for E in $\mathbf{R}$, $\overline{\mu}(E) = \mu(E)$; the measure $\overline{\mu}$ is σ-finite.*

The measure $\overline{\mu}$ is called the *extension* of μ; except when it is likely to lead to confusion, we shall write $\mu(E)$ instead of $\overline{\mu}(E)$ even for sets E in $\mathbf{S}(\mathbf{R})$.

Theorem B. *If μ is a measure on a σ-ring $\mathbf{S}$, then the class $\overline{\mathbf{S}}$ of all sets of the form $E \triangle N$, where $E \in \mathbf{S}$ and N is a subset of a set of measure zero in $\mathbf{S}$, is a σ-ring, and the set function $\overline{\mu}$ defined by $\overline{\mu}(E \triangle N) = \mu(E)$ is a complete measure on $\overline{\mathbf{S}}$.*

The measure $\overline{\mu}$ is called the *completion* of μ.

Theorem C. *If μ is a σ-finite measure on a ring $\mathbf{R}$, and if μ^* is the outer measure induced by μ, then the completion of the extension of μ to $\mathbf{S}(\mathbf{R})$ is identical with μ^* on the class of all μ^*-measurable sets.*

Theorem D. *If μ is a σ-finite measure on a ring $\mathbf{R}$, then, for every set E of finite measure in $\mathbf{S}(\mathbf{R})$ and for every positive number ε, there exists a set E_0 in $\mathbf{R}$ such that $\mu(E \triangle E_0) \leq \varepsilon$.*

3.14 Inner measures

For every E in $\mathbf{H}(\mathbf{S})$ we write

$$\mu_*(E) = \sup\{\mu(F) : E \supset F \in \mathbf{S}\}.$$

μ_* = *inner measure* μ_* *induced by* μ

Throughout this section we shall assume that μ is a σ-finite measure on a σ-ring $\mathbf{S}$, μ^* and μ_* are the outer and the inner measure induced by μ, respectively, and $\overline{\mu}$ on $\overline{\mathbf{S}}$ is the completion of μ.

Theorem A. *If $E \in \mathbf{H}(\mathbf{S})$, then*

$$\mu_*(E) = \sup\{\overline{\mu}(F) : E \supset F \in \overline{\mathbf{S}}\}.$$

If $E \in \mathbf{H}(\mathbf{S})$ and $F \in \mathbf{S}$, we shall say that F is a *measurable kernel* of E, if $F \subset E$ and if, for every set G in $\mathbf{S}$ for which $G \subset E - F$, we have $\mu(G) = 0$. Loosely speaking a measurable kernel of a set E in $\mathbf{H}(\mathbf{S})$ is a maximal set in $\mathbf{S}$ which is contained in E.

Theorem B. *Every set E in $\mathbf{H}(\mathbf{S})$ has a measurable kernel.*

Theorem C. *If $E \in \mathbf{H}(\mathbf{S})$ and F is a measurable kernel of E, then $\mu(F) = \mu_*(E)$; if both F_1 and F_2 are measurable kernels of E, then $\mu(F_1 \triangle F_2) = 0$.*

Theorem D. *If (E_n) is a disjoint sequence of sets in $\mathbf{H}(\mathbf{S})$, then*

$$\mu_*\left(\bigcup_{n=1}^{\infty} E_n\right) \geq \sum_{n=1}^{\infty} \mu_*(E_n).$$

Theorem E. *If $A \in \mathbf{H}(\mathbf{S})$ and if (E_n) is a disjoint sequence of sets in $\overline{\mathbf{S}}$ with $\bigcup_{n=1}^{\infty} E_n = E$, then*

$$\mu_*(A \cap E) = \sum_{n=1}^{\infty} \mu_*(A \cap E_n).$$

Theorem F. *If $E \in \overline{\mathbf{S}}$, then*

$$\mu^*(E) = \mu_*(E) = \overline{\mu}(E),$$

and conversely, if $E \in \mathbf{H}(\mathbf{S})$ and

$$\mu^*(E) = \mu_*(E) < \infty,$$

then $E \in \overline{\mathbf{S}}$.

Theorem G. *If E and F are disjoint sets in $\mathbf{H}(\mathbf{S})$, then*

$$\mu_*(E \cup F) \leq \mu_*(E) + \mu^*(F) \leq \mu^*(E \cup F).$$

Theorem H. *If $E \in \overline{\mathbf{S}}$, then, for every subset A of X,*

$$\mu_*(A \cap E) + \mu^*(A' \cap E) = \overline{\mu}(E).$$

3.15 Lebesgue measure

Throughout this section we will assume that X is the real line, $\mathbf{P}$ is the class of all bounded, semiclosed intervals of the form $\langle a, b)$, $\mathbf{S}$ is the σ-ring generated by $\mathbf{P}$, and μ is the set function on $\mathbf{P}$ defined by $\mu(\langle a, b)) = b - a$.

The sets of the σ-ring $\mathbf{S}$ are called the *Borel sets* of the line; according to the extension theorems 8.E and 13.A we may assume that μ is defined for all Borel sets. If $\overline{\mu}$ on $\overline{\mathbf{S}}$ is the completion of μ on $\mathbf{S}$, the sets of $\overline{\mathbf{S}}$ are the *Lebesgue measurable sets* of the line; the measure $\overline{\mu}$ is *Lebesgue measure.*

Measures μ on $\mathbf{S}$ and $\overline{\mu}$ on $\overline{\mathbf{S}}$ are totally σ-finite.

Theorem A. *Every countable set is a Borel set of measure zero.*

Theorem B. *The class* $\mathbf{S}$ *of all Borel sets coincides with the* σ*-ring generated by the class* $\mathbf{U}$ *of all open sets.*

Theorem C. *If* $\mathbf{U}$ *is the class of all open sets, then, for every* E *in* X*,*

$$\mu^*(E) = \inf\{\mu(U) : E \subset U \in \mathbf{U}\}.$$

Theorem D. *Let* T *be the one to one transformation of the entire real line onto itself, defined by* $T(x) = \alpha x + \beta$*, where* α *and* β *are real numbers and* $\alpha \neq 0$*. If, for every subset* E *of* X*,* $T(E)$ *denotes the set of all points of the form* $T(x)$ *with* x *in* E*,* $T(E)$ *denotes the set of all points of the form* $T(x)$ *with* x *in* E*, i.e.* $T(E) = \{\alpha x + \beta; x \in E\}$*, then*

$$\mu^*(T(E)) = |\alpha|\mu^*(E) \qquad \text{and} \qquad \mu_*(T(E)) = |\alpha|\mu_*(E).$$

The set $T(E)$ *is a Borel set or a Lebesgue measurable set if and only if* E *is a Borel set or a Lebesgue measurable set, respectively.*

$C=$ the set of all those numbers x in whose ternary expansion the digit 1 is not needed. The set C is called the *Cantor set.* $\mu(C) = 0$. C is nowhere dense, perfect. C has the cardinal number of continuum.

3.16 Non measurable sets

The symbol $D(E)$ will be used to denote the *difference set* of E, i.e. the set of all numbers of the form $x - y$ with $x, y \in E$.

Theorem A. *If* E *is a Lebesgue measurable set of positive finite measure, and if* $0 \leq \alpha < 1$*, then there exists an open interval* U *such that* $\overline{\mu}(E \cap U) \geq \alpha\mu(U)$*.*

Theorem B. *If* E *is a Lebesgue measurable set of positive measure, then there exists an open interval containing the origin and entirely contained in the difference set* $D(E)$*.*

Theorem C. *If* ξ *is an irrational number, then the set* A *of all numbers of the form* $n + m\xi$*, where* n *and* m *are arbitrary integers, is everywhere dense on the line; the same is true of the subset* B *of all numbers of the form* $n + m\xi$ *with* n *even, and the subset* C *of numbers of the form* $n + m\xi$ *with* n *odd.*

Theorem D. *There exists at least one set* E_0 *which is not Lebesgue measurable.*

Theorem E. *There exists a subset* M *of the real line such that, for every Lebesgue measurable set* E*,*

$$\mu_*(M \cap E) = 0 \qquad \text{and} \qquad \mu^*(M \cap E) = \overline{\mu}(E).$$

4 Measurable functions

4.17 Measure spaces

A *measurable space* is a set X and a σ-ring $\mathbf{S}$ of subsets of X with the property that $\bigcup \mathbf{S} = X$. A subset E of X is called *measurable* if and only if it belongs to the σ-ring $\mathbf{S}$.

A *measure space* is a measurable space $(X, \mathbf{S})$ and a measure μ on $\mathbf{S}$. The measure space X is called [totally] finite, σ-finite, or complete, according as the measure μ is [totally] finite, σ-finite or complete.

A subset X_0 of a measure space $(X, \mathbf{S}, \mu)$ is *thick* if $\mu_*(E - X_0) = 0$ for every measurable set E.

Theorem A. *If X_0 is a thick subset of a measure space $(X, \mathbf{S}, \mu)$, if $\mathbf{S}_0 = \mathbf{S} \cap X_0$, and if, for $E \in \mathbf{S}$, $\mu_0(E \cap X_0) = \mu(E)$, then $(X_0, \mathbf{S}_0, \mu_0)$ is a measure space.*

4.18 Measurable functions

$(X, \mathbf{S})$ is a measurable space. We put $N(f) = \{x : f(x) \neq 0\}$. If a real valued function f is such that, for every Borel subset M of the real line the set $N(f) \cap f^{-1}(M)$ is measurable, then f is called a *measurable function.*

Borel measurable functions - $\mathbf{S}$ =Borel sets

Lebesgue measurable functions - $\mathbf{S}$ =Lebesgue measurable sets

Theorem A. *A real function f on a measurable space $(X, \mathbf{S})$ is measurable if and only if, for every real number c, the set $N(f) \cap \{x : f(x) < c\}$ is measurable.*

4.19 Combination of measurable functions

Theorem A. *If f and g are extended real valued measurable functions on a measurable space $(X, \mathbf{S})$, and if c is any real number, then each of the three sets*

$$\begin{aligned} A &= \{x : f(x) < g(x) + c\}, \\ B &= \{x : f(x) \leq g(x) + c\}, \\ C &= \{x : f(x) = g(x) + c\}, \end{aligned}$$

has a measurable intersection with every measurable set.

Theorem B. *If ϕ is an extended real valued Borel measurable function on the extended real line such that $\phi(0) = 0$, and if f is an extended real valued measurable function on a measurable space X, then the function $\overline{f}$, defined by $\overline{f} = \phi(f(x))$, is a measurable function on X.*

Theorem C. *If f and g are extended real valued measurable functions on a measurable space X, then so also are $f + g$ and fg.*

$$f^+ = \max(f, 0) \text{ and } f^- = -\min(f, 0) = \max(-f, 0)$$

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^-$$

$$f^+ = \textit{positive part}, \quad f^- = \textit{negative part}$$

Cantor function ψ:
$x \notin C \Rightarrow \psi(x) = (0.\frac{\alpha_1}{2}\frac{\alpha_2}{2}\frac{\alpha_3}{2}\ldots 1)_2$,
$x \in C \Rightarrow \psi(x) = (0.\frac{\alpha_1}{2}\frac{\alpha_2}{2}\frac{\alpha_3}{2}\ldots)_2$.
ψ is nondecreasing, continuous.

4.20 Sequences of measurable functions

Theorem A. *If* (f_n) *is a sequence of extended real valued, measurable functions on a measurable space* X*, then each of the four functions* h*,* g*,* f^* *and* f_**, defined by*

$$\begin{aligned} h(x) &= \sup\{f_n(x) : n = 1, 2, \ldots\}, \\ g(x) &= \inf\{f_n(x) : n = 1, 2, \ldots\}, \\ f^*(x) &= \limsup f_n(x), \\ f_*(x) &= \liminf f_n(x), \end{aligned}$$

is measurable.

A function f, defined on a measurable space X, is called *simple* if there is a finite, disjoint class $\{E_1, \ldots, E_n\}$ of measurable sets and a finite set $\{\alpha_1, \ldots, \alpha_n\}$ of real numbers such that

$$f(x) = \begin{cases} \alpha_i & \text{if } x \in E_i, i = 1, \ldots, n, \\ 0 & \text{if } x \notin E_1 \cup \ldots \cup E_n. \end{cases}$$

Theorem B. *Every extended real valued measurable function* f *is the limit of a sequence* (f_n) *of simple functions; if* f *is non negative, then each* f_n *may be taken non negative and the sequence* (f_n) *may be assumed increasing.*

4.21 Pointwise convergence

If a certain proposition is true for *almost every* point (i.e. with the exception at most of a measurable set of measure zero) we say that it is true *almost everywhere.*

A function f is called *essentially bounded* if it is bounded a.e., i.e. if there exists a positive, finite constant c such that $\{x : |f(x)| > c\}$ is a set of measure zero. The infimum of the values of c for which this statement is true is called the *essential supremum* of $|f|$, abbreviated to ess. sup. $|f|$.

The sequence (f_n) converges to f *uniformly a.e.* if there exists a set E_0 of measure zero, such that, for every $\varepsilon > 0$, an integer $n_0 = n_0(\varepsilon)$ can be found with the property that

$$|f_n(x) - f(x)| < \varepsilon, \text{ if } n \geq n_0 \text{ and } x \notin E_0,$$

in other words if the sequence of functions converges uniformly to f (in the ordinary sense of the phrase) on the set $X - E_0$.

Theorem A (Egoroff's theorem). *If* E *is a measurable set of finite measure, and if* (f_n) *is a sequence of a.e. finite valued measurable functions which converges a.e. on* E *to a finite valued measurable function* f*, then, for every* $\varepsilon > 0$ *there exists a measurable subset* F *of* E *such that* $\mu(F) < \varepsilon$ *and such that the sequence* (f_n) *converges to* f *uniformly on* $E - F$*.*

A sequence (f_n) of a.e. finite valued measurable functions will be said to converge to the measurable function f almost uniformly if, for every $\varepsilon > 0$,there exists a measurable set F such that $\mu(F) < \varepsilon$ and such that the sequence (f_n) converges to f uniformly on F'. In this language Egoroff's theorem asserts that on a set of finite measure convergence a.e. implies almost uniform convergence.

Theorem B. *If (f_n) is a sequence of measurable functions which converges to f almost uniformly, then (f_n) converges to f a.e.*

4.22 Convergence in measure

Theorem A. *Suppose that f and f_n, $n = 1, 2, \ldots$, are real valued functions on a set E of finite measure, and write, for every $\varepsilon > 0$,*

$$E_n(\varepsilon) = \{x : |f_n(x) - f(x)| \geq \varepsilon\}, \; n = 1, 2, \ldots$$

The sequence (f_n) converges to f a.e. if and only if

$$\lim_{n \to \infty} \mu(E \cap \bigcup_{m=n}^{\infty} E_m(\varepsilon)) = 0$$

for every $\varepsilon > 0$.

A sequence (f_n) of a.e. finite valued, measurable function *converges in measure* to the measurable function f if, for every $\varepsilon > 0$, $\lim\limits_{n \to \infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$. A sequence (f_n) of a.e. finite valued, measurable function is *fundamental in measure* if, for every $\varepsilon > 0$,

$$\mu(\{x : |f_n(x) - f_m(x)|\}) \to 0 \qquad \text{as} \qquad n \text{ and } m \to \infty.$$

If a sequence of finite valued measurable functions converges a.e. to a finite limit [or is fundamental a.e.] on a set E of finite measure, then it converges in measure [or is fundamental in measure] on E.

Theorem B. *Almost uniform convergence implies convergence in measure.*

Theorem C. *If (f_n) converges in measure to f, then (f_n) is fundamental in measure. If also (f_n) converges in measure to g, then $f = g$ a.e.*

Theorem D. *If (f_n) is a sequence of measurable functions which is fundamental in measure, then some subsequence (f_{n_k}) is almost uniformly fundamental.*

Theorem E. *If (f_n) is a sequence of measurable functions which is fundamental in measure, then there exists a measurable function f such that (f_n) converges in measure to f.*

5 Integration

5.23 Integrable simple functions

A simple function $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ on a measure space $(X, \mathbf{S}, \mu)$ is *integrable* if $\mu(E_i) < \infty$ for every index i for which $\alpha_i \neq 0$. The *integral* of f, in symbols

$$\int f(x) d\mu(x) \text{ or } \int f d\mu$$

is defined by $\int f d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i)$.

If E is a measurable set and f is an integrable simple function we define the *integral* of f *over* E by

$$\int_E f d\mu = \int \chi_E f d\mu.$$

Theorem A. *If f and g are integrable functions and α and β are real numbers, then*

$$\int(\alpha f+\beta g)d\mu=\alpha\int fd\mu+\beta\int gd\mu.$$

Theorem B. *If an integrable function f is non negative a.e., then $\int fd\mu\geq 0$.*

Theorem C. *If f and g are integrable functions such that $f\geq g$ a.e., then*

$$\int fd\mu\geq\int gd\mu.$$

Theorem D. *If f and g are integrable functions, then*

$$\int|f+g|d\mu\leq\int|f|d\mu+\int|g|d\mu.$$

Theorem E. *If f is an integrable function, then*

$$\left|\int fd\mu\right|\leq\int|f|d\mu.$$

Theorem F. *If f is an integrable function, α and β are real numbers, and E is a measurable set such that, for x in E, $\alpha\leq f(x)\leq\beta$, then*

$$\alpha\mu(E)\leq\int_E fd\mu\leq\beta\mu(E).$$

The *indefinite integral* of an integrable function f is the set function ν, defined for every measurable set E by $\nu(E)=\int_E fd\mu$.

Theorem G. *If an integrable function f is non negative a.e., then its indefinite integral is monotone.*

A finite valued set function of the class of all measurable sets of a measure space $(X,\mathbf{S},\mu)$ is *absolutely continuous* if for every positive number ε there exists a positive number δ such that $|\nu(E)|<\varepsilon$ for every measurable set E for which $\mu(E)<\delta$.

Theorem H. *The indefinite integral of an integrable function is absolutely continuous.*

Theorem I. *The indefinite integral of an integrable function is countably additive.*

If f and g are integrable functions, we define the *distance*, $\varrho(f,g)$, between them by the equation

$$\varrho(f,g)=\int|f-g|d\mu.$$

5.24 Sequences of integrable simple functions

A sequence (f_n) of integrable functions is *fundamental in the mean,* or *mean fundamental,* if

$$\varrho(f_n,f_m)\to 0\qquad\text{as}\qquad n\text{ and }m\to\infty.$$

Theorem A. *A mean fundamental sequence (f_n) of integrable functions is fundamental in measure.*

Theorem B. *If (f_n) is a mean fundamental sequence of integrable functions, and if the indefinite integral of f_n is ν_n, $n = 1, 2, \ldots$, then*

$$\nu(E) = \lim_{n\to\infty} \nu_n(E)$$

exists for every measurable set E, and the set function ν is finite valued and countably additive.

If (ν_n) is a sequence of finite valued set functions defined for all measurable sets, we say that the terms of the sequence are *uniformly absolutely continuous* whenever for every positive number ε there exists a positive number δ such that $|\nu_n(E)| < \varepsilon$ for every measurable set E for which $\mu(E) < \delta$, and for every positive integer n.

Theorem C. *If (f_n) is a mean fundamental sequence of integrable functions, and if the indefinite integral of f_n is ν_n, $n = 1, 2, \ldots$, then the set functions ν_n are uniformly absolutely continuous.*

Theorem D. *If (f_n) and (g_n) are mean fundamental sequences of integrable simple functions which converge in measure to the same measurable function f, if the indefinite integrals of f_n and g_n are ν_n and λ_n respectively, and if, for every measurable set E,*

$$\nu(E) = \lim_{n\to\infty} \nu_n(E) \qquad \text{and} \qquad \lambda(E) = \lim_{n\to\infty} \lambda_n(E),$$

then the set function ν and λ are identical.

5.25 Integrable functions

An a.e. finite valued, measurable function f on a measure space $(X, \mathbf{S}, \mu)$ is *integrable* if there exists a mean fundamental sequence (f_n) of integrable simple functions which converges in measure to f. The *integral* of f, in symbols

$$\int f(x) d\mu(x) \quad \text{or} \int f d\mu$$

is defined by $\int f d\mu = \lim\limits_{n\to\infty} f_n d\mu$.

We define the *integral of f over E* by

$$\int_E f d\mu = \int \chi_E f d\mu.$$

We shall say that a sequence (f_n) of integrable functions *converges in the mean,* or *mean converges,* to an integrable function f if

$$\varrho(f_n, f) = \int |f_n - f| d\mu \to 0 \qquad \text{as} \qquad n \to \infty.$$

Theorem A. *If (f_n) is a sequence of integrable functions which converges in the mean to f, then (f_n) converges to f in measure.*

Theorem B. *If f is an a.e. non negative integrable function, then a necessary and sufficient condition that $\int f d\mu = 0$ is that $f = 0$ a.e.*

Theorem C. *If f is an integrable function and E is a set of measure zero, then*

$$\int_E f d\mu = 0.$$

Theorem D. *If f is an integrable function which is positive a.e. on a measurable set E, and if $\int_E f d\mu = 0$, then $\mu(E) = 0$.*

Theorem E. *If f is an integrable function such that $\int_F f d\mu = 0$ for every measurable set F, then $f = 0$ a.e.*

Theorem F. *If f is an integrable function, then the set $N(f) = \{x : f(x) \neq 0\}$ has σ-finite measure.*

5.26 Sequences of integrable functions

Theorem A. *If (f_n) is a mean fundamental sequence of integrable simple functions which converges in measure to the integrable function f, then*

$$\varrho(f, f_n) = \int |f - f_n| d\mu \to 0 \qquad as \qquad n \to \infty;$$

hence, to every integrable function f and to every positive number ε, there corresponds an integrable simple function g such that $\varrho(f, g) < \varepsilon$.

Theorem B. *If (f_n) is a mean fundamental sequence of integrable functions, then there exists an integrable function f such that $\varrho(f_n, f) \to 0$ (and consequently $\int f_n d\mu \to \int f d\mu$) as $n \to \infty$.*

If (ν_n) is a sequence of finite valued set functions on a class $\mathbf{E}$, then the terms of the sequence are *equicontinuous* from above at 0 if, for every decreasing sequence (E_n) of sets in $\mathbf{E}$ for which $\lim_n E_n = 0$, and for every positive number ε, there exists a positive integer m_0 such that if $m \geq m_0$, then $|\nu_n(E_m)| < \varepsilon$, $n = 1, 2, \ldots$

Theorem C. *A sequence (f_n) of integrable functions converges in the mean to the integrable function f if and only if (f_n) converges in measure to f and the indefinite integrals of $|f_n|$, $n = 1, 2, \ldots$, are uniformly absolutely continuous and equicontinuous from above at 0.*

Theorem D (Lebesgue's bounded convergence theorem). *If (f_n) is a sequence of integrable functions which converges in measure to f [or else converges to f a.e.], and if g is an integrable function such that $|f_n(x)| \leq |g(x)|$ a.e., $n = 1, 2, \ldots$, then f is integrable and the sequence (f_n) converges to f in the mean.*

5.27 Properties of integrals

Theorem A. *If f is measurable, g is integrable, and $|f| \leq |g|$ a.e., then f is integrable.*

Theorem B. *If* (f_n) *is an increasing sequence of extended real valued non negative measurable functions and if* $\lim_{n\to\infty} f_n(x) = f(x)$ *a.e., then* $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$.

Theorem C. *A measurable function is integrable if and only if its absolute value is integrable.*

Theorem D. *If* f *is integrable and* g *is an essentially bounded measurable function, then* fg *is integrable.*

Theorem E. *If* f *is an essentially bounded measurable function and* E *is a measurable set of finite measure, then* f *is integrable over* E.

Theorem F (Fatou's lemma). *If* (f_n) *is a sequence of non negative integrable functions for which*

$$\liminf \int f_n d\mu < \infty,$$

then the function f*, defined by*

$$f(x) = \liminf f_n(x),$$

is integrable and

$$\int f d\mu \leq \liminf \inf f_n d\mu.$$

6 General set functions

6.28 Signed measures

We define a *signed measure* as an extended real valued, countably additive set function μ on the class of all measurable sets of a measurable space $(X, \mathbf{S})$, such that $\mu(0) = 0$, and such that μ assumes at most one of the values $+\infty$ and $-\infty$.

The words "[totally] finite"" and "[totally] σ-finite"" will be used for signed measures just as for measures, except the $\mu(E)$ has to be replaced by $|\mu(E)|$, or, equivalently, $\mu(E) < \infty$ has to be replaced by $-\infty < \mu(E) < \infty$.

Theorem A. *If* E *and* F *are measurable sets and* μ *is a signed measure such that*

$$E \subset F \qquad \textit{and} \qquad |\mu(F)| < \infty,$$

then $|\mu(E)| < \infty$.

Theorem B. *If* μ *is a signed measure and* (E_n) *is a disjoint sequence of measurable sets such that* $|\mu\left(\bigcup_{n=1}^{\infty} E_n\right)| < \infty$*, then the series* $\sum_{n=1}^{\infty} \mu(E_n)$ *is absolutely convergent.*

Theorem C. *If* μ *is a signed measure, if* (E_n) *is a monotone sequence of measurable sets, and if, in case* (E_n) *is a decreasing sequence,* $|\mu(E_n)| < \infty$ *for at least one value of* n*, then*

$$\mu(\lim_n E_n) = \lim_n \mu(E_n).$$

6.29 Hahn and Jordan decomposition

If μ is a signed measure on the class of all measurable sets of a measurable space $(X, \mathbf{S})$, we shall call a set E *positive* if, for every measurable set F, $E \cap F$ is measurable and $\mu(E \cap F) \geq 0$; similarly we call E *negative* if, for every measurable set F, $E \cap F$ is measurable and $\mu(E \cap F) \leq 0$.

Theorem A. *If μ is a signed measure, then there exist two disjoint sets A and B whose union is X, such that A is positive and B is negative with respect to μ.*

The sets A and B are said to form a *Hahn decomposition* of X with respect to μ. The equations

$$\mu^+(E) = \mu(E \cap A) \qquad \text{and} \qquad \mu^-(E) = -\mu(E \cap B)$$

unambiguously define two set functions μ^+ and μ^- on the class of all measurable sets, called, respectively, the *upper variation* and the *lower variation* of μ. The set $|\mu|$, defined for every measurable set E by $|\mu|(E) = \mu^+(E) + \mu^-(E)$, is the *total variation* of μ.

Theorem B. *The upper, lower, and total variations of a signed measure μ are measures and $\mu(E) = \mu^+(E) - \mu^-(E)$ for every measurable set E. If μ is [totally] finite of σ-finite, then so also are μ^+ and μ^-; at least one of the measures μ^+ and μ^- is always finite.*

The representation of μ as the difference of its upper and lower variations is called the *Jordan decomposition* of μ.

If $(X, \mathbf{S}, \mu)$ is a measure space and f is is an integrable function on X, then the set function ν, defined by $\nu(E) = \int_E f(x) d\mu(x)$, is a finite signed measure, and $\nu^+(E) = \int_E f^+ d\mu$, $\nu^-(E) = \int_E f^- d\mu$.

6.30 Absolute continuity

If $(X, \mathbf{S})$ is a measurable space and μ and ν are signed measures on $\mathbf{S}$, we say that ν is *absolutely continuous* with respect to μ, in symbols $\nu \ll \mu$, if $\nu(E) = 0$ for every measurable set E for which $|\mu|(E) = 0$.

Theorem A. *If μ and ν are signed measures, then the conditions*

(a) $\nu \ll \mu$,

(b) $\nu^+ \ll \mu$ *and* $\nu^- \ll \mu$,

(c) $|\nu| \ll |\mu|$,

are mutually equivalent.

Theorem B. *If ν is a finite signed measure and if μ is a signed measure such that $\nu \ll \mu$, then, corresponding to every positive number ε, there is a positive number δ such that $|\nu|(E) < \varepsilon$ for every measurable set E for which $|\mu|(E) < \delta$.*

Two signed measures μ and ν for which both $\nu \ll \mu$ and $\mu \ll \nu$ are called *equivalent,* in symbols $\mu \equiv \nu$.

If $(X, \mathbf{S})$ is a measurable space and μ and ν are signed measures on $\mathbf{S}$, we say that μ and ν are *mutually singular,* or more simply that μ and ν are *singular,* in symbols $\mu \perp \nu$, if there exist two disjoint sets A and B whose union is X such that, for every measurable set E, $A \cap E$ and $B \cap E$ are measurable and $|\mu|(A \cap E) = |\nu|(B \cap E) = 0$.

If, for each point x of a measurable space $(X, \mathbf{S})$, $\pi(x)$ is a proposition concerning x, and μ is a signed measure on $\mathbf{S}$, then the symbol

$$\pi(x)\,[\mu] \qquad \text{or} \qquad \pi\,[\mu]$$

shall mean that $\pi(x)$ is true for almost every x with respect to the measure $|\mu|$. The symbol $[\mu]$ may be read as "modulo" μ.

6.31 The Radon-Nikodym theorem

Theorem A. *If μ and ν are totally finite measures such that $\nu \ll \mu$ and ν is not identically zero, then there exists a positive number ε and a measurable set A such that $\mu(A) > 0$ and such that A is a positive set for the signed measure $\nu - \varepsilon\mu$.*

Theorem B (Radon-Nikodym theorem). *If $(X, \mathbf{S}, \mu)$ is a totally σ-finite measure space and if a σ-finite measure ν on $\mathbf{S}$ is absolutely continuous with respect to μ, then there exists a finite valued measurable function f on X such that*

$$\nu(E) = \int_E f d\mu$$

for every measurable set E. The function f is unique in the sense that if also $\nu(E) = \int_E g d\mu$, $E \in \mathbf{S}$, then $f = g\,[\mu]$.

6.32 Derivatives of signed measures

If μ is a totally σ-finite measure and if $\nu(E) = \int_E f d\mu$ for every measurable set E, we shall write

$$f = \frac{d\nu}{d\mu} \qquad \text{or} \qquad d\nu = f d\mu.$$

All the properties of Radon-Nikodym integrands (which we may also call *Radon-Nikodym derivatives*), which are suggested by the well known differential formalism, correspond to true theorems.

Theorem A. *If λ and μ are totally σ-finite measures such that $\mu \ll \lambda$ and if ν is a totally σ-finite signed measure such that $\nu \ll \mu$, then*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\,[\lambda].$$

Theorem B. *If λ and μ are totally σ-finite measures such that $\mu \ll \lambda$, and if f is a finite valued measurable function for which $\int f d\mu$ is defined, then $\int f d\mu = \int f \frac{d\mu}{d\lambda} d\lambda$.*

Theorem C. *If $(X, \mathbf{S})$ is a measurable space and μ and ν are totally σ-finite signed measures on $\mathbf{S}$, then there exist two uniquely determined totally σ-finite signed measures ν_0 and ν_1 whose sum is ν, such that $\nu_0 \perp \mu$ and $\nu_1 \ll \mu$.*

*(*Lebesgue decomposition *of a totally σ-finite signed measure into an absolutely continuous part and a singular part with respect to another totally σ-finite measure)*

7 Product spaces

7.33 Cartesian products

Cartesian product, rectangle=set $E = A \times B$, sides A and B

Theorem A. *A rectangle is empty if and only if one of its sides is empty.*

Theorem B. *If $E_1 = A_1 \times B_1$ and $E_2 = A_2 \times B_2$ are non empty rectangles, then $E_1 \subset E_2$ if and only if*

$$A_1 \subset A_2 \qquad \text{and} \qquad B_1 \subset B_2.$$

Theorem C. *If $A_1 \times B_1 = A_2 \times B_2$ is a non empty rectangle, then $A_1 = A_2$ and $B_1 = B_2$.*

Theorem D. *If $E = A \times B$, $E_1 = A_1 \times B_1$, and $E_2 = A_2 \times B_2$ are non empty rectangles, then a necessary and sufficient condition that E be the disjoint union of E_1 and E_2, is that either A is the disjoint union of A_1 and A_2 and $B = B_1 = B_2$, or else B is the disjoint union of B_1 and B_2, and $A = A_1 = A_2$.*

Theorem E. *If $\mathbf{S}$ and $\mathbf{T}$ are rings of subsets of X and Y respectively, then the class $\mathbf{R}$ of all finite, disjoint unions of rectangles of the form $A \times B$, where $A \in \mathbf{S}$ and $B \in \mathbf{T}$, is a ring.*

Suppose now that in addition to the two sets X and Y we are also given two σ-rings $\mathbf{S}$ and $\mathbf{T}$ of subsets of X and Y respectively. We shall denote by $\mathbf{S} \times \mathbf{T}$ the σ-ring of subsets of $X \times Y$ generated by the class of all sets of the form $A \times B$, where $A \in \mathbf{S}$ and $B \in \mathbf{T}$.

Theorem F. *If $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ are measurable spaces, then $(X \times Y, \mathbf{S} \times \mathbf{T})$ is a measurable space.*

The measurable space $(X \times Y, \mathbf{S} \times \mathbf{T})$ is the *Cartesian product* of the two given measurable spaces.

measurable rectangle: A rectangle in the Cartesian product of two measurable spaces $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ is measurable if it belongs to $\mathbf{S} \times \mathbf{T}$. Equivalently: $A \in \mathbf{S}$ and $B \in \mathbf{T}$.

7.34 Sections

$E_x = \{y : (x, y) \in E\}$= a *section* of E determined by x

X-section, Y-section

If f is any function defined on a subset E of the product space $X \times Y$ and x is any point of X, we shall call the function f_x, defined on the section E_x by

$$f_x(y) = f(x, y),$$

a *section* of f, ore more precisely an *X-section* of f determined by x.

Theorem A. *Every section of a measurable set is a measurable set.*

Theorem B. *Every section of a measurable function is a measurable function.*

7.35 Product measures

Theorem A. *If $(X, \mathbf{S}, \mu)$ and $(Y, \mathbf{T}, \nu)$ are σ-finite measure spaces, and if E is any measurable subset of $X \times Y$, then the functions f and g defined on X and Y respectively by $f(X) = \nu(E_x)$ and $g(y) = \mu(E^y)$, are non negative measurable functions such that $\int f d\mu = \int g d\nu$.*

Theorem B. *If $(X, \mathbf{S}, \mu)$ and $(Y, \mathbf{T}, \nu)$ are σ-finite measure spaces, then the set function λ, defined for every set E in $\mathbf{S} \times \mathbf{T}$ by*

$$\lambda(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y),$$

is a σ-finite measure with the property that, for every measurable rectangle $A \times B$,

$$\lambda(A \times B) = \mu(A).\nu(B).$$

The latter condition determines λ uniquely.

The measure λ is called the *product* of the given measures μ and ν, in symbols $\lambda = \mu \times \nu$; the measure space is the *Cartesian product* of the given measure spaces.

7.36 Fubini's theorem

Throughout this section we shall assume that $(X, \mathbf{S}, \mu)$ and $(Y, \mathbf{T}, \nu)$ are σ-finite measure spaces and λ is the product measure $\mu \times \nu$ on $\mathbf{S} \times \mathbf{T}$.

If a function h on $X \times Y$ is such that its integral is defined, then the integral is denoted by

$$\int h(x, y) d\lambda(x, y) \qquad \text{or} \int h(x, y) d(\mu \times \nu)(x, y)$$

and is called the *double integral* of h. If h_x is such that

$$\int h_x(y) d\nu(y) = f(x)$$

is defined, and if it happens that $\int f d\mu$ is also defined, it is customary to write

$$\int f d\mu = \iint h(x, y) d\nu(y) d\mu(x) = \int d\mu(x) \int h(x, y) d\nu(y).$$

The symbols $\iint h(f, y) d\mu(x) d\nu(y)$ and $\int d\nu(y) \int h(x, y) d\mu(x)$ are defined similarly, as the integral (if it exists) of the function g on Y, defined by $g(y) = \int h^y(x) d\mu(x)$. The integrals $\iint h d\mu d\nu$ and $\iint h d\nu d\mu$ are called the *iterated integrals* of h.

Theorem A. *A necessary and sufficient condition that a measurable subset of $X \times Y$ have measure zero is that almost every X-section [or almost every Y-section] have measure zero.*

Theorem B. *If h is a non negative, measurable function on $X \times Y$, then*

$$\int h d(\mu \times \nu) = \iint h d\mu d\nu = \iint h d\nu d\mu.$$

Theorem C (Fubini's theorem). *If h is an integrable function on $X \times Y$, then almost every section of h is integrable. If the functions f and g are defined by $f(x) = \int h(x,y)d\nu(y)$ and $g(y) = \int h(x,y)d\mu(x)$, then f and g are integrable and*

$$\int hd(\mu \times \nu) = \int fd\mu = \int gd\nu.$$

7.37 Finite dimensional product spaces

We may prove the analogs of the theorems of §33 by mathematical induction on n.

If $(X_i, \mathbf{S}_i)$, $i = 1, 2, \ldots, n$, are measurable spaces, we shall denote by

$$\mathbf{S}_1 \times \ldots \times \mathbf{S}_n \qquad \text{or} \qquad \bigtimes_{i=1}^{n} \mathbf{S}_i \qquad \text{or} \qquad \bigtimes \{\mathbf{S}_i : i = 1, 2, \ldots, n\}$$

the σ-ring generated by the class of all those rectangles $\bigtimes_{i=1}^{n} A_i$ for which $A_i \in \mathbf{S}_i$, $i = 1, \cdots, n$, and we define the *Cartesian product* of the given measurable spaces as $(X_1 \times \cdots \times X_n, \mathbf{S}_1 \times \cdots \times \mathbf{S}_n)$. Proceeding by mathematical induction, it is now trivial to define the Cartesian product of σ-finite measure spaces $(X_i, \mathbf{S}_i, \mu_i)$, $i = 1, 2, \ldots, n$; there is one and only one measure μ (denoted $\mu_1 \times \ldots \times \mu_n$) on $\mathbf{S}_1 \times \ldots \times \mathbf{S}_n$ such that

$$\mu(A_1 \times \ldots \times A_n) = \prod_{i=1}^{n} \mu_i(A_i)$$

for every measurable rectangle $A_1 \times \ldots \times A_n$.

It is customary to refer to a product space $X = \bigtimes_{i=1}^{n} X_i$ as *n-dimensional.*

7.38 Infinite dimensional product spaces

Suppose that, for each $i = 1, 2, \ldots$, X_i is a set, $\mathbf{S}_i$ is a σ-algebra of subsets of X_i, and μ_i is a measure on $\mathbf{S}_i$ such that $\mu_i(X_i) = 1$. In this case we define a *rectangle* as a set of the form $\bigtimes_{i=1}^{\infty} A_i$, where $A_i \subset X_i$ for all i and $A_i = X_i$ for all but finite number of values of i. We define a *measurable rectangle* as a rectangle $\bigtimes_{i=1}^{\infty} A_i$ for which each A_i is a measurable subset of X_i. A subset of $\bigtimes_{i=1}^{\infty} X_i$ will be called *measurable* if it belongs to the σ-ring $\mathbf{S}$ (which is in fact a σ-algebra generated by the class of all measurable rectangles; we shall write $\mathbf{S} = \bigtimes_{i=1}^{\infty} \mathbf{S}_i$.

Suppose that J is any subset of the set I of all positive integers; we shall say that two points

$$x = (x_1, x_2, \ldots) \qquad \text{and} \qquad y = (y_1, y_2, \ldots)$$

agree on J, in symbols $x \equiv y\,(J)$, if $x_j = y_j$ for every $j \in J$. A set E in X is called a *J-cylinder* if $x \equiv y\,(J)$ implies that x and y belong or do not belong to E simultaneously.

$$X^{(n)} = \bigtimes_{i=n+1}^{\infty} X_i, \qquad n = 0, 1, 2, \ldots$$

Theorem A. *If $J = \{1, \ldots, n\}$ and if a subset E of X is a [measurable] J-cylinder, then $E = A \times X^{(n)}$, where A is a [measurable] subset of $X_1 \times \ldots \times X_n$.*

Theorem B. *If $\{(X_i, \mathbf{S}_i, \mu_i)\}$ is a sequence of totally finite measure spaces with $\mu_i(X_i) = 1$, then there exists a unique measure μ on the σ-algebra $\mathbf{S} = \times_{i=1}^{\infty} \mathbf{S}_i$ with the property that, for every measurable set E of the form $A \times X^{(n)}$,*

$$\mu(E) = (\mu_1 \times \ldots \mu_n)(A).$$

The measure μ is called the *product* of the given measures μ_i, $\mu = \times_{i=1}^{\infty} \mu_i$; the measure space

$$\left(\mathop{\times}_{i=1}^{\infty} X_i, \mathop{\times}_{i=1}^{\infty} \mathbf{S}_i, \mathop{\times}_{i=1}^{\infty} \mu_i \right)$$

is the *Cartesian product* of the given measure spaces.

8 Transformations and functions

8.39 Measurable transformations

A *transformation* is a function T defined for every point of a set X and taking values in a set Y.

domain, range, into, onto, image, inverse image, one to one

Theorem A. *If T is a transformation from X into Y, if g is a function on Y, and if M is any subset of the space in which the values of g lie, then*

$$\{x : (gT)(x) \in M\} = T^{-1}(\{y : g(y) \in M\}).$$

If $(X, \mathbf{S})$ and $(Y, \mathbf{T})$ are measurable space, we shall say that T is a measurable transformation *if the inverse image of every measurable set is measurable.*

Theorem B. *If T is a measurable transformation from $(X, \mathbf{S})$ into $(Y, \mathbf{T})$, and if g is an extended real valued measurable function on Y, then gT is measurable with respect to the σ-ring $T^{-1}(\mathbf{T})$.*

Theorem C. *If T is a measurable transformation from a measure space $(X, \mathbf{S}, \mu)$ into a measurable space $(Y, \mathbf{T})$, and if g is an extended real valued measurable function on Y, then*

$$\int g d(\mu T^{-1}) = \int (gT) d\mu,$$

in the sense that if either integral exists, then so does the other and the two are equal.

Theorem D. *If T is a measurable transformation from a measure space $(X, \mathbf{S}, \mu)$ into a totally σ-finite measure space $(Y, \mathbf{T}, \nu)$, such that μT^{-1} is absolutely continuous with respect to ν, then there exists a non negative measurable function ϕ on Y such that*

$$\int g(T(x)) d\mu(x) = \int g(y) \phi(y) d\nu(y)$$

for every measurable function g, in the sense that if either integral exists, then so does the other and the two are equal.

The function ϕ plays the similar role as the Jacobian (or the absolute value of the Jacobian) in the theory of transformations of multiple integrals.

If T is a one to one transformation from a measurable space $(X, \mathbf{S})$ *onto* a measurable space $(Y, \mathbf{T})$, and if both T and T^{-1} are measurable, we shall say that T is *measurability preserving.* A measurability preserving transformation T from a measure space $(X, \mathbf{S}, \mu)$ onto a measure space $(Y, \mathbf{T}, \nu)$ is *measure preserving* if $\mu T^{-1} = \nu$.

8.40 Measure rings

A *Boolean ring* is a ring in the usual algebraic sense, with the property that every element is idempotent. Equivalently, a Boolean ring is a set $\mathbf{R}$ and two algebraic operations (called addition and multiplication) defined for pairs of elements of $\mathbf{R}$, subject to the following restrictions. (a) Both addition and multiplication are commutative and associative, and multiplication is distributive with respect to addition. (b) There exists in $\mathbf{R}$ a unique element (denoted by 0) such the result of adding 0 to any element E is E. (c) The result of adding any element to itself is 0. (d) The result of multiplying any element E by itself is E.

A typical example of a Boolean ring is a ring of subsets of a set X with $E \triangle F$ and $E \cap F$ playing the roles of the sum and the product of E and f, respectively. We shall always denote addition and multiplication in Boolean rings by $\triangle$ and $\cap$.

A *Boolean σ-ring* is a Boolean ring $\mathbf{S}$ with the property that every countable set of elements in $\mathbf{S}$ has a union.

A *Boolean algebra* is a Boolean ring $\mathbf{R}$ in which there exists an element different from 0 (which, for obvious reasons, we shall denote by X), with the property that $E \subset X$ for every E in $\mathbf{R}$. A *Boolean σ-algebra* is a Boolean σ-ring which is a Boolean algebra.

The definitions of the concepts of additivity, measure, σ-finiteness, etc. for functions defined on a Boolean ring are the same as the corresponding definitions for set functions on a ring of sets. A measure μ on a Boolean ring is *positive* if it vanishes for the zero element only.

If $(X, \mathbf{S}, \mu)$ is a measure space, we shall use the symbol $\mathbf{S}(\mu)$ to denote the σ-ring $\mathbf{S}$ with equality interpreted modulo μ.

A *measure ring* $(\mathbf{S}, \mu)$ is a Boolean σ-ring $\mathbf{S}$ and a positive measure μ on $\mathbf{S}$. If $(X, \mathbf{S}, \mu)$ is a measure space, then $(\mathbf{S}(\mu), \mu)$ is a measure ring; we shall call it the measure ring *associated* with X or simply the measure ring of X. A *measure algebra* is a Boolean algebra which is at the same time a measure ring. The phrases [totally] finite and σ-finite are used for measure rings and measure algebras in the same way as for measure spaces.

An *isomorphism* between two measure rings $(\mathbf{S}, \mu)$ and $(\mathbf{T}, \nu)$ is a one to one transformation T from $\mathbf{S}$ onto $\mathbf{T}$ such that

$$T(E - F) = T(E) - T(F),\ T\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} T(E_n),$$

and

$$\mu(E) = \nu(T(E)),$$

whenever E, F, and E_n are elements of $\mathbf{S}$, $n = 1, 2, \ldots$ Two measure rings are *isomorphic* if there exists an isomorphism between them. Two measure spaces are isomorphic if their associated measure rings are isomorphic.

An *atom* of a measure ring $(\mathbf{S}, \mu)$ [or of a measure μ] is an element E different from 0 such that if $F \subset E$, then either $F = 0$ or $F = E$; a measure ring with no atoms is *non atomic.*

If $(\mathbf{S}, \mu)$ is a measure ring, we shall denote by $\mathcal{S}$ [or $\mathcal{S}(\mu)$] the set of elements of finite measure in $\mathbf{S}$ and, for any two elements E and F in $\mathcal{S}$, we shall write

$$\varrho(E, F) = \mu(E \triangle F).$$

The function ϱ is a metric for $\mathcal{S}$; we shall call $\mathcal{S}$ the metric space *associated* with $(\mathbf{S}, \mu)$, or, simply, the metric space of $(\mathbf{S}, \mu)$. We shall also use the symbol $\mathcal{S}(\mu)$ for the metric space space associated with the measure ring $(\mathbf{S}(\mu), \mu)$ of a measure space $(X, \mathbf{S}, \mu)$. A measure ring or a measure space is called *separable* if the associated metric space is separable.

Theorem A. *If $\mathcal{S}$ is the metric space of a measure ring $(\mathbf{S}, \mu)$, and if*

$$f(E, F) = E \cup F \qquad \text{and} \qquad g(E, F) = E \cap F,$$

then f, g, and also μ, are all uniformly continuous functions of their arguments.

Theorem B. *If $(X, \mathbf{S}, \mu)$ is a σ-finite measure space such that the σ-ring $\mathbf{S}$ has a countable set of generators, then the metric space $\mathcal{S}(\mu)$ of measurable sets of finite measure is separable.*

8.41 The isomorphism theorem

In what follows we restrict our attention to totally finite measure algebras. If $(\mathbf{S}, \mu)$ is a totally finite measure algebra, then, if not specified, the symbol X will denote the maximal element of $\mathbf{S}$. The algebra $\mathbf{S}$ and the measure μ are called *normalized* if $\mu(X) = 1$. A *partition* of an element E of $\mathbf{S}$ is a finite set $\mathbf{P}$ of disjoint elements of $\mathbf{S}$ whose union is E. The *norm* of a partition $\mathbf{P} = \{E_1, \ldots, E_k\}$, denoted by $|\mathbf{P}|$, is the maximum of the numbers $\mu(E_1), \ldots, \mu(E_k)$.

If $\mathbf{P}_1$ and $\mathbf{P}_2$ are partitions, we shall write $\mathbf{P}_1 \leq \mathbf{P}_2$ if each element of $\mathbf{P}_1$ is contained in some element of $\mathbf{P}_2$; a sequence $\{P_n\}$ is *decreasing* if $\mathbf{P}_{n+1} \leq \mathbf{P}_n$ for $n = 1, 2, \ldots$ A sequence $\{\mathbf{P}_n\}$ of partitions is *dense* if to every element E of $\mathbf{S}$ and to every positive number ε there corresponds a positive integer n and an element E_0 of $\mathbf{S}$ which is equal to a union of elements of $\mathbf{P}_n$ and is such that $\varrho(E, E_0) = \mu(E \triangle E_0) < \varepsilon$.

Theorem A. *If $(\mathbf{S}, \mu)$ is a totally finite, non atomic measure algebra, and if $\{\mathbf{P}_n\}$ is a dense, decreasing sequence of partitions of X, then* $\lim_n |\mathbf{P}_n| = 0$.

Theorem B. *If Y is the unit interval, $\mathbf{T}$ is the class of all Borel subsets of Y, and ν is the Lebesgue measure on $\mathbf{T}$, and if $\{Q_n\}$ is a sequence of partitions into intervals of the maximal element Y of the measure algebra $(\mathbf{T}, \nu)$, such that* $\lim_n |\mathbf{Q}_n|$ *is a sequence of partitions into intervals of the maximal element Y of the measure algebra $(\mathbf{T}, \nu)$, such that* $\lim_n |\mathbf{Q}_n| = 0$, *then $\{\mathbf{Q}_n\}$ is dense.*

Theorem C. *Every separable, non atomic, normalized measure algebra $(\mathbf{S}, \mu)$ is isomorphic to the measure algebra $(\mathbf{T}, \nu)$ of the unit interval.*

8.42 Function spaces

L_1, L_p for real number $p > 1$.

$$\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$$

Theorem A (Hölder's inequality). *If p and q are real numbers greater than 1 such that $\frac{1}{p} + \frac{1}{q} = 1$, and if $f \in L_p$ and $g \in L_q$, then $fg \in L_1$ and $\|fg\| \leq \|f\|_p . \|g\|_q$.*

Theorem B (Minkowski's inequality). *If p is a real number greater than 1, and if f and g are in L_p, then $f + g \in L_p$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Another useful space is the set $\mathcal{M}$ of all essentially bounded measurable function. If we write, for any $f \in \mathcal{M}$

$$\|f\|_\infty = \text{esssup}\{|f(x)| : x \in X\},$$

then $\mathcal{M}$ becomes a Banach space.

8.43 Set functions and point functions

Throughout this section we shall assume that X is the real line, $\mathbf{S}$ is the class of all Borel sets, and μ is Lebesgue measure on $\mathbf{S}$. Monotone, non-decreasing functions will be referred for brevity just as *monotone* functions.

If f is a bounded monotone function, then the following limits exists

$$\lim_{x \to -\infty} f(x), \quad \lim_{x \to +\infty} f(x) \quad \text{and}$$

will be denoted as $f(-\infty)$ and $f(+\infty)$ respectively.

Theorem A. *If μ is a finite measure on $\mathbf{S}$ and if, for every real number x,*

$$f_\nu(x) = \nu(\{t : -\infty < t < x\}),$$

then f_ν is a bounded monotone function, continuous on the left and such that $f_\nu(-\infty) = 0$.

Theorem B. *If f is a bounded monotone function, continuous on the left and such that $f(-\infty) = 0$, then there exists a unique finite measure ν on $\mathbf{S}$ such that $f = f_\nu$.*

Theorem C. *If ν is a finite measure on $\mathbf{S}$, then a necessary and sufficient condition that f_ν be continuous is that $\nu(\{x\}) = 0$ for every point x.*

A real valued function f of a real variable is called *absolutely continuous* if for every positive number ε there corresponds a positive number δ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for every finite disjoint class $\{(a_i, b_i) : i = 1, \ldots, n\}$ of bounded open intervals for which $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

Theorem D. *If ν is a finite measure on* $\mathbf{S}$, *then a necessary and sufficient condition that f_ν be absolutely continuous is that ν be absolutely continuous with respect to μ.*

We shall say that a finite measure ν on $\mathbf{S}$ is *purely atomic* if there exists a countable set C such that $\nu(X - C) = 0$.

Theorem E. *If ν is a finite measure measure on* $\mathbf{S}$, *then there exist three uniquely determined measures ν_1, ν_2 and ν_3 on* $\mathbf{S}$ *whose sum is ν and which are such that ν_1 is absolutely continuous to μ, ν_2 is purely atomic, and ν_3 is singular with respect to μ but $\nu_3(\{x\}) = 0$ for every point x.*

9 Probability

9.44 Heuristic introduction

9.45 Independence

A *probability space* is a totally finite measure space $(X, \mathbf{S}, \mu)$ for which $\mu(X) = 1$; the measure on a probability space is called a *probability measure.*

If $\mathbf{E}$ is a finite or infinite class of measurable sets in a probability space $(X, \mathbf{S}, \mu)$, the sets of the class $\mathbf{E}$ are (stochastically) *independent* (združene nezávislé) if

$$\mu\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} \mu(E_i)$$

for every finite class $\{E_i : i = 1, 2, \ldots, n\}$ of distinct sets in $\mathbf{E}$.

Pairwise independence, i.e. the independence of all pairs of distinct elements, does not imply (stochastic) independent. (**reformulate**)

If $\mathcal{E}$ is a finite or infinite set of real valued measurable functions on a probability space $(X, \mathbf{S}, \mu)$, the functions of the set $\mathcal{E}$ are (stochastically) *independent* if

$$\mu\left(\bigcap_{i=1}^{n} \{x : f_i(x) \in M_i\}\right) = \prod_{i=1}^{n} \mu(\{x : f_i(x) \in M_i\})$$

for every finite subset $\{f_i : i = 1, 2, \ldots, n\}$ of distinct functions in $\mathcal{E}$ and every finite class $\{M_i : i = 1, 2, \ldots, n\}$ of Borel sets on the real line.

Theorem A. *If f_1 and f_2 are independent functions, neither of which vanishes a.e., then a necessary and sufficient condition that both f_1 and f_2 be integrable is that their product $f_1 f_2$ be integrable; if this condition is satisfied, then*

$$\int f_1 f_2 d\mu = \int f_1 d\mu. \int f_2 d\mu.$$

Theorem B. *If $\{f_{ij} : i = 1, 2, \ldots, k; j = 1, 2, \ldots, n_i\}$ is a set of independent functions, if ϕ_i is a real valued, Borel measurable function of n_i real variables, $i = 1, 2, \ldots, k$, and if*

$$f_i(x) = \phi_i(f_{i1}(x), \ldots, f_{in_i}(x)),$$

then the functions $f_1, \ldots, f_k$ are independent.

Let f^2 be integrable function. If $\int f d\mu = \alpha$, then the *variance* of f, denoted by $\sigma^2(f)$, is defined by $\int (f-\alpha)^2 d\mu$.

$$\sigma^2(f) = \left(\int f^2 d\mu\right) - \left(\int f d\mu\right)^2, \quad \sigma^2(cf) = c^2\sigma^2(f)$$

Theorem C. *If f and g are independent functions with a finite variance, then*

$$\sigma^2(f+g) = \sigma^2(f) + \sigma^2(g).$$

If F is a measurable set of positive measure in a probability space $(X, \mathbf{S}, \mu)$, and if, for every measurable set E, $\mu_F(E) := \frac{\mu(F\cap E)}{\mu(F)}$, then μ_F is a probability measure of $\mathbf{S}$ such that $\mu_F(F) = 1$. The sets E and F are independent if and only if $\mu_F(E) = \mu(E)$. The number $\mu_F(E)$ is called the *conditional probability of E given F (podmienená pravdepodobnosť E vzhľadom na F).*

Multiplicative theorem for conditional probabilities:

If $\{E_i : i = 1, \cdots, n\}$ is a finite class of measurable sets of positive measure, then $\mu(E_1 \cap E_2 \cap \cdots \cap E_n) = \mu(E_1)\mu_{E_1}(E_2)\ldots\mu_{E_1\cap\cdots\cap E_{n-1}}(E_n)$.

The *B*ayes's theorem (1. Bayesova veta):
If $\{E_i : i = 1, \cdots, n\}$ is a finite, disjoint class of measurable sets of positive measure whose union is X (i.e. $\{E_i\}$ is a partition of X), then, for every measurable set F, $\mu(F) = \sum_{i=1}^n \mu(E_i)\mu_{E_i}(F)$, and, if F has positive measure,

$$\mu_F(E_j) = \frac{\mu(E_j)\mu_{E_j}(F)}{\sum_{i=1}^n \mu(E_i)\mu_{E_i}(F)}.$$

Let $X = \{x : 0 \le x \le 1\}$ be the unit interval with Lebesgue measure. For every positive integer n define a function f_n on X by setting $f_n(x) = +1$ or -1 according as the integer i for which $\frac{i-1}{2^n} \le x < \frac{i}{2^n}$ is odd or even. The functions f_n are called *Rademacher functions.* Any two of the functions f_1, f_2 and $f_1 f_2$ are independent, but the three together are not.

Coefficient of correlation:

$$r(f,g) = \frac{\int fg\, d\mu - \int f\, d\mu \cdot \int g\, d\mu}{\sigma(f)\sigma(g)},$$

where $\sigma(f) = \sqrt{\sigma^2(f)}$ is *standard deviation* of f. The functions f and g are *uncorrelated* if $r(f,g) = 0$. If f, g are independent functions, then f, g are uncorrelated. The converse is not true in general. A necessary and sufficient condition that $\sigma^2(f+g) = \sigma^2(f) + \sigma^2(g)$ is that f, g are uncorrelated.

9.46 Series of independent functions

Throughout this section we shall work with a fixed probability space $(X, \mathbf{S}, \mu)$.

Theorem A (Kolmogoroff's inequality). *If f_i, $i = 1, 2, \ldots, n$ are independent functions such that $\int f_i d\mu = 0$, and $\int f_i^2 d\mu < \infty$, $i = 1, 2, \ldots, n$, and if $f(x) = \bigcup_{k=1}^n |\sum_{i=1}^k f_i(x)|$ (i.e. f is the maximum of the absolute values of the partial sums of the f_i's), then, for every positive number ε,*

$$\mu(\{x : |f(x)| \ge \varepsilon\}) \le \frac{1}{\varepsilon^2}\sum_{k=1}^n \sigma^2(f_k).$$

Theorem B. *If* (f_n) *is a sequence of independent functions such that* $\int f_n d\mu = 0$ *and* $\sum_{n=1}^{\infty} \sigma^2(f_n) < +\infty$, *then the series* $\sum_{n=1}^{\infty} f_n(x)$ *converges a.e.*

Theorem C. *If* (f_n) *is a sequence of independent functions and* c *is a positive constant such that* $\int f_n d\mu = 0$ *and* $|f_n(x)| \leq c$ *a.e.,* $n = 1, 2, \ldots$, *and if* $\sum_{n=1}^{\infty} f_n(x)$ *converges on a set of positive measure, then* $\sum_{n=1}^{\infty} \sigma^2(f_n) < \infty$.

Theorem D. *If* (f_n) *is a sequence of independent functions and* c *is a positive constant such that* $|f_n(x)| \leq c$ *a.e.,* $n = 1, 2, \ldots$, *then* $\sum_{n=1}^{\infty} f_n(x)$ *converges a.e. if and only if both the series* $\sum_{n=1}^{\infty} \int f_n d\mu$ *and* $\sum_{n=1}^{\infty} \sigma^2(f_n)$ *are convergent.*

Theorem E (Three series theorem). *If* (f_n) *is a sequence of independent functions and* c *is a positive constant, and if* $E_n = \{x : |f_n(x)| \leq c\}$, $n = 1, 2, \ldots$, *then a necessary and sufficient condition for the convergence a.e. of* $\sum_{n=1}^{\infty} f_n(x)$ *is the convergence of all the series*

$$\sum_{n=1}^{\infty} \mu(E_n'),$$

$$\sum_{n=1}^{\infty} \int_{E_n} f_n d\mu,$$

$$\sum_{n=1}^{\infty} \left(\int_{E_n} f_n^2 d\mu - \left(\int_{E_n} f_n d\mu \right)^2 \right).$$

Tchebycheff's inequality: If f is a measurable function with finite variance, then, for every positive number ε,

$$\mu(\{x : |f(x) - \int f \, d\mu| \geq \varepsilon\}) \leq \frac{1}{\varepsilon^2} \sigma^2(f).$$

Zero-one law: Suppose that the probability space X is the Cartesian product of a sequence (X_n) of probability spaces. If, for each positive integer n, $J_n = \{n+1, n+2, \cdots\}$, and if a measurable set E in X is a J_n-cylinder for every n, then $\mu(E) = 0$ or 1.
Borel-Cantelli lemma:

If (E_n) is a sequence of independent sets, then $\mu(\limsup_n E_n) = 0$ if and only if $\sum_{n=1}^{\infty} \mu(E_n) < +\infty$.

9.47 The law of large numbers

Theorem A (Bernoulli's theorem, Weak law of large numbers). *If* (f_n) *is a sequence of independent functions with finite variances, such that* $\int f_n d\mu = 0$, $n = 1, 2, \ldots$, *and* $\lim_n \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2(f_i) = 0$, *then the sequence* $\left(\frac{1}{n} \sum_{i=1}^{n} f_i\right)$ *of averages converges to 0 in measure.*

Two real valued measurable functions f and g on a probability space $(X, \mathbf{S}, \mu)$ have the *same distribution* if $\mu(f^{-1}(M)) = \mu(g^{-1}(M))$ for all real Borel sets M.

Theorem B. *If* (y_n) *is a sequence of real numbers which converges to a finite limit* y, *then* $\lim_n \frac{1}{n} \sum_{i=1}^{n} y_i = y$.

Theorem C. *If (y_n) is a sequence of real numbers such that the series $\sum_{n=1}^{\infty} \frac{1}{n} y_n$ is convergent, then $\lim_n \frac{1}{n} \sum_{i=1}^{n} y_i = 0$.*

Theorem D (Strong law of large numbers). *If (f_n) is a sequence of independent functions with finite variances, such that $\int f_n d\mu = 0$, $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} \frac{\sigma^2(f_n)}{n^2} < \infty$, then the sequence $\left(\frac{1}{n} \sum_{i=1}^{n} f_i\right)$ converges to 0 almost everywhere.*

9.48 Conditional probabilities and expectations

Let T be a measurable transformation from the probability space $(X, \mathbf{S}, \mu)$ into a measurable space $(Y, \mathbf{T})$. It follows from the Radon-Nikodym theorem that there exists an integrable function p_E on Y such that

$$\mu(E \cap T^{-1}(F)) = \int_F p_E(y) d\mu T^{-1}(y)$$

for every F in $\mathbf{T}$; the function p_E is uniquely determined modulo μT^{-1}. We shall call $p_E(y)$ the *conditional probability of E given y* or the conditional probability of E given that $T(x) = y$.

We shall generally write $p(E, y)$ for $p_E(y)$.

Theorem A. *For each fixed measurable set E in X,*

$$0 \leq p(E, y) \leq 1 \, [\mu T^{-1}];$$

for each fixed disjoint sequence (E_n) of measurable sets in X,

$$p\left(\bigcup_{n=1}^{\infty} E_n, y\right) = \sum_{n=1}^{\infty} p(E_n, y) \, [\mu T^{-1}].$$

If f is any integrable function X, then we may consider its indefinite integral ν, defined by

$$\nu(F) = \int_{T^{-1}(F)} f(x) d\mu(x),$$

as a signed measure on $\mathbf{T}$. Since clearly $\nu \ll \mu T^{-1}$, it follows form the Radon-Nikodym theorem that there exists an integrable function e_f on Y such that

$$\int_{T^{-1}(F)} f(x) d\mu(x) = \int_F e_f(y) d\mu T^{-1}(y)$$

for every F in $\mathbf{T}$. The function e_f is uniquely determined modulo μT^{-1}. We shall call $e_f(y)$ the *conditional expectation of f given y;* we shall also write $e(f, y)$ instead of $e_f(y)$.

Theorem B. *If f is an integrable function on Y, then fT is an integrable function on X and $e(fT, y) = f(y) \, [\mu T^{-1}]$.*

9.49 Measures on product spaces

Suppose that, for each positive integer n, X_n is the closed interval and $\mathbf{S}_n$ is the class of all Borel sets in X_n, and write $(X, \mathbf{S}) = \times_{n=1}^{\infty}(X_n, \mathbf{S}_n)$. Let $\mathbf{F}_n$ be the σ-ring of all measurable $\{1, \cdots, n\}$-cylinders in X and let $\mathbf{F}$ ($= \bigcup_{n=1}^{\infty} \mathbf{F}_n$) be the ring of all measurable, finite dimensional subsets of X.

Theorem A. *If μ is a set function on $\mathbf{F}$ such that, for each positive integer n, μ is a probability measure on $\mathbf{F}_n$, then μ has a unique extension to a probability measure on $\mathbf{S}$.*

Theorem B. *For every measurable set E in X,*

$$\lim_{n\to\infty} p(E, T_n(X)) = \chi_E(x)\,[\mu];$$

in other words, the condition probabilities of E, for given values of the first n coordinates of a point x, converge (except perhaps on a set of x's of measure zero) to 0 or 1 according as $x \in E$ or $x \notin E$.

10 Locally compact spaces

10.50 Topological lemmas

Throughout this chapter, unless we explicitly say otherwise, we shall assume that X is a locally compact Hausdorff spaces. We shall use the symbol $\mathcal{F}$ for the class of all real valued, continuous functions f on X such that $0 \le f(x) \le 1$ for all x in X.

Theorem A. *If C is a compact set and U and V are open sets such that $C \subset U \cup V$, then there exists compact sets D and E such that $D \subset U$, $E \subset V$, and $C = D \cup E$.*

Theorem B. *If C is a compact set, F is a closed set, and $C \cap F = 0$, then there exist a function f in $\mathcal{F}$ such that $f(x) = 0$ for x in C and $f(x) = 1$ for x in F.*

Theorem C. *If f is a real valued continuous function on X and c is a real number, then each of the three sets*

$$\{x : f(x) \ge c\}, \qquad \{x : f(x) \le c\}, \qquad \text{and} \qquad \{x : f(x) = c\}$$

is a closed G_δ. If, conversely, C is a compact G_δ, then there exists a function f in $\mathcal{F}$ such that $C = \{x : f(x) = 0\}$.

Theorem D. *If C is compact, U is open, and $C \subset U$, then there exist sets C_0 and U_0 such that C_0 is a compact G_δ, U_0 is a σ-compact open set and*

$$C \subset U_0 \subset C_0 \subset U.$$

Theorem E. *If X is separable, then every compact subset C of X is a G_δ.*

10.51 Borel sets and Baire sets

We continue with our study of a fixed locally compact Hausdorff space X.

We shall denote by $\mathbf{C}$ the class of all compact subsets of X, by $\mathbf{S}$ the σ-ring generated by $\mathbf{C}$, and by $\mathbf{U}$ the class of all open sets belonging to $\mathbf{S}$. We shall call the sets of $\mathbf{S}$ the *Borel sets* of X. A real valued function is *Borel measurable* (or simply a *Borel function*) if it is measurable with respect to the σ-ring $\mathbf{S}$.

Theorem A. *Every Borel set is σ-bounded; every σ-bounded open set is a Borel set.*

We shall denote by $\mathbf{C}_0$ the class of all compact subsets of X, by $\mathbf{S}_0$ the σ-ring generated by $\mathbf{C}_0$, and by $\mathbf{U}_0$ the class of all open sets belonging to $\mathbf{S}_0$. We shall call the sets of $\mathbf{S}_0$ the *Baire sets* of X. A real valued function is *Baire measurable* (or simply a *Baire function*) if it is measurable with respect to the σ-ring $\mathbf{S}_0$.

Theorem B. *If a real valued continuous function f on X is such that the set $N(f) = \{x : f(x) \neq 0\}$ is σ-bounded, then f is Baire measurable.*

Theorem C. *If $\mathbf{B}$ is a subbase and if $\hat{\mathbf{S}}$ is a σ-ring containing $\mathbf{B}$, then $\hat{\mathbf{S}} \supset \mathbf{S}_0$.*

Theorem D. *Every compact Baire set is a G_δ.*

Theorem E. *If X and Y are locally compact Hausdorff spaces, and if $\mathbf{A}_0$, $\mathbf{B}_0$ and $\mathbf{S}_0$ are the σ-rings of Baire sets in X, Y and $X \times Y$ respectively, then $\mathbf{S}_0 = \mathbf{A}_0 \times \mathbf{B}_0$.*

Theorem F. *The class of all finite, disjoint unions of proper differences of sets of $\mathbf{C}$ [or of $\mathbf{C}_0$] is a ring; the σ-ring it generates coincides with $\mathbf{S}$ [or, respectively, with $\mathbf{S}_0$].*

10.52 Regular measures

A *Borel measure* is a measure μ defined on the class $\mathbf{S}$ of all Borel sets and such that $\mu(C) < \infty$ for every C in $\mathbf{C}$; a *Baire measure* is a measure μ defined on the class $\mathbf{S}_0$ of all Baire sets and such that $\mu(C_0) < \infty$ for every C_0 in $\mathbf{C}_0$.

Throughout this section we shall use $\hat{\mathbf{C}}$, $\hat{\mathbf{U}}$ and $\hat{\mathbf{S}}$ to stand either for $\mathbf{C}$, $\mathbf{U}$ and $\mathbf{S}$ or else for $\mathbf{C}_0$, $\mathbf{U}_0$ and $\mathbf{S}_0$, respectively, and we shall study a measure $\hat{\mu}$ which is a Borel measure if $\hat{\mathbf{S}} = \mathbf{S}$ and a Baire measure if $\hat{\mathbf{S}} = \mathbf{S}_0$.

A set E in $\hat{\mathbf{S}}$ is *outer regular* (with respect to the measure $\hat{\mu}$) if

$$\hat{\mu}(E) = \inf\{\hat{\mu}(U) : E \subset U \in \hat{\mathbf{U}}\};$$

a set E in $\hat{\mathbf{S}}$ is *inner regular* (with respect to the measure $\hat{\mu}$) if

$$\hat{\mu}(E) = \sup\{\hat{\mu}(C) : E \supset C \in \hat{\mathbf{C}}\}.$$

A set E in $\hat{\mathbf{S}}$ is *regular* if it is both inner regular and outer regular; a measure $\hat{\mu}$ is regular if every set E in $\hat{\mathbf{S}}$ is regular.

Theorem A. *If every set in $\hat{\mathbf{C}}$ is outer regular, then so i s every proper difference of two sets of $\hat{\mathbf{C}}$; if every bounded set in $\hat{\mathbf{U}}$ is inner regular, then so is every proper difference of two sets of $\hat{\mathbf{C}}$.*

Theorem B. *A finite, disjoint union of inner regular sets of finite measure is inner regular.*

Theorem C. *The union of a sequence of outer regular sets is outer regular; the union of an increasing sequence of inner regular sets is inner regular.*

Theorem D. *The intersection of a sequence of inner regular sets of finite measure is inner regular; the intersection of a decreasing sequence of outer regular sets of finite measure is outer regular.*

Theorem E. *A necessary and sufficient condition that every set in $\hat{\mathbf{C}}$ be outer regular is that every bounded set in $\hat{\mathbf{U}}$ be inner regular.*

Theorem F. *Either the outer regularity of every set in $\hat{\mathbf{C}}$ or the inner regularity of every bounded set in $\hat{\mathbf{U}}$ is a necessary and sufficient condition for the regularity of the measure $\hat{\mu}$.*

Theorem G. *Every Baire measure ν is regular; if $C \in \mathbf{C}$, then*

$$\nu^*(C) = \inf\{\nu(U_0) : C \subset U_0 \in \mathbf{U}_0\},$$

and, if $U \in \mathbf{U}$, then

$$\nu_*(U) = \sup\{\nu(C_0) : U \supset C_0 \in \mathbf{C}_0\}$$

Theorem H. *Let μ be a Borel measure and let ν be the Baire contraction of μ (defined for every Baire set E by $\nu(E) = \mu(E)$). Either of the two conditions,*

$$\begin{aligned} \mu(C) &= \nu^*(C) \qquad \textit{for all } C \textit{ in } \mathbf{C}, \\ \mu(U) &= \nu_*(U) \qquad \textit{for all } U \textit{ in } \mathbf{U}, \end{aligned}$$

is necessary and sufficient for the regularity of μ. If two regular Borel measures agree on all Baire sets, then they agree on all Borel sets.

If μ is any Borel measure, its Baire contraction μ_0, defined for all E in $\mathbf{S}_0$ by $\mu_0(E) = \mu(E)$, is a Baire measure associated with μ in a natural way. If it happens that every set in $\mathbf{C}$ or every bounded set in $\mathbf{U}$, and therefore in either case, every set in $\mathbf{S}$ is μ_0^*-measurable (i.e. if all compact sets, and therefore all Borel sets, belong to the domain of definition of the completion of μ_0), then we shall say that the Borel measure μ is *completion regular.*

10.53 Generation of Borel measures

We define a *content* as a non negative, finite, monotone, additive, and subadditive set function on the class $\mathbf{C}$ of all compact sets. In other words, a content is a set function λ on $\mathbf{C}$ which is such that

(a) $0 \leq \lambda(C) < \infty$ for all $C \in \mathbf{C}$,

(b) if C and D are compact sets for which $C \subset D$, then $\lambda(C) \leq \lambda(D)$

(c) if C and D are disjoint compact sets, then $\lambda(C \cup D) = \lambda(C) + \lambda(D)$, and

(d) if C and D are any two compact sets, then $\lambda(C \cup D) \leq \lambda(C) + \lambda(D)$.

The *inner content* λ_* induced by a content λ, is the set function defined for every $U \in \mathbf{U}$ by

$$\lambda_*(U) = \sup\{\lambda(C) : U \supset C \in \mathbf{C}\}.$$

Theorem A. *The inner content λ_* induced by a content λ vanishes at 0, and is monotone, countably subadditive, and countably additive.*

If λ is a content and λ_* is the inner content induced by λ, we define a set function μ^* on the hereditary ring of all σ-bounded sets by

$$\mu^*(E) = \inf\{\lambda_*(U) : E \subset U \in \mathbf{U}\}.$$

The set function μ^* is called the *outer measure induced* by λ.

Theorem B. *The outer measure μ^* induced by a content λ is an outer measure.*

Theorem C. *If λ_* is the inner content and μ^* is the outer measure induced by a content λ, then $\mu^*(U) = \lambda_*(U)$ for every U in $\mathbf{U}$ and $\mu^*(C^0) \leq \lambda(C) \leq \mu^*(C)$ for every C in $\mathbf{C}$.*

(We recall that C^0 denotes the interior of the set C.)

Theorem D. *If μ^* is the outer measure induced by a content λ, then a σ-bounded set E is μ^*-measurable if and only if*

$$\mu^*(U) \geq \mu^*(U \cap E) + \mu^*(U \cap E')$$

for every U in $\mathbf{U}$.

Theorem E. *If μ^* is the outer measure induced by a content λ, then the set function μ, defined for every Borel set E by $\mu(E) = \mu^*(E)$, is a regular Borel measure.*

We shall call μ the Borel measure *induced* by the content λ.

Theorem F. *Suppose that T is a homeomorphism of X onto itself and that λ is a content. If, for every C in $\mathbf{C}$, $\hat{\lambda}(C) = \lambda(T(C))$, and if μ and $\hat{\mu}$ are the Borel measures induced by λ and $\hat{\lambda}$ respectively, then $\hat{\mu}(E) = \mu(T(E))$ for every Borel set E. If, in particular, λ is invariant under T, then the same is true of μ.*

10.54 Regular contents

A content λ is *regular* if, for every C in $\mathbf{C}$,

$$\lambda(C) = \inf\{\lambda(D) : C \subset D^0 \subset D \in \mathbf{C}\}.$$

Theorem A. *If μ is the Borel measure induced by a regular content λ, then $\mu(C) = \lambda(C)$ for every C in $\mathbf{C}$.*

Theorem B. *If μ is a regular Borel measure and if, for every C in $\mathbf{C}$, $\lambda(C) = \mu(C)$, then λ is a regular content and the Borel measure induced by λ coincides with μ.*

Theorem C. *If μ_0 is a Baire measure and if, for every C in $\mathbf{C}$,*

$$\lambda(C) = \inf\{\mu_0(U_0) : C \subset U_0 \in \mathbf{U}_0\},$$

then λ is a regular content.

Theorem D. *If μ_0 is a Baire measure, then there exists a unique, regular Borel measure μ such that $\mu(E) = \mu_0(E)$ for every Baire set E.*

10.55 Classes of continuous functions

If X is, as usual, a locally compact Hausdorff space, we shall denote by $\mathcal{L}(X)$ or simply by $\mathcal{L}$ the class of all those real valued, continuous functions on X which vanish outside a compact set.

If X is not compact and if X^* is the one-point compactification of X by x^*, then the point x^* is frequently called the *point at infinity.*

We shall denote by $\mathcal{L}_+(X)$ or simply by $\mathcal{L}_+$ the subclass of all nonnegative functions in $\mathcal{L}$.

Theorem A. *If C is any compact Baire set, then there exists a decreasing sequence (f_n) of functions in $\mathcal{L}_+$ such that*

$$\lim_{n\to\infty} f_n(x) = \chi_C(x)$$

for every x in X.

Theorem B. *If a Baire measure μ is such that the measure of every non empty Baire open set is positive, and if $f \in \mathcal{L}_+$, then a necessary and sufficient condition that $\int f d\mu = 0$ is that $f(x) = 0$ for every x in X.*

Theorem C. *If μ_0 is a Baire measure and $\varepsilon > 0$, then, corresponding to every integrable simple Baire function f, there exists an integrable simple function g,*

$$g = \sum_{i=1}^{\infty} \alpha_i \chi_{C_i},$$

such that C_i is a compact Baire set, $i = 1, \ldots, n$, and

$$\int |f - g| d\mu_0 \leq \varepsilon.$$

Theorem D. *If μ_0 is a Baire measure, if $\varepsilon > 0$ and if $g = \sum\limits_{i=1}^{\infty} \alpha_i \chi_{C_i}$ is a simple function such that C_i is a compact Baire set, $i = 1, \ldots, n$, then there exists a function h in $\mathcal{L}$ such that*

$$\int |g - h| d\mu_0 \leq \varepsilon.$$

Lusin's theorem: If μ is a regular Borel measure, E is a Borel set of finite measure, and f is a Borel measurable function on E, then, for every $\varepsilon > 0$, there exists a compact set C in E such that $\mu(E - C) \leq \varepsilon$ and such that f is continuous on C.

10.56 Linear functionals

A *linear functional* on $\mathcal{L}$ is a real valued function Λ of the functions in $\mathcal{L}$ such that

$$\Lambda(\alpha f + \beta g) = \alpha\Lambda(f) + \beta\Lambda(g)$$

for every $f, g \in \mathcal{L}$ and $\alpha, \beta \in \mathbb{R}$. A linear functional Λ on $\mathcal{L}$ is *positive* if $\Lambda(f) \geq 0$ for every f in $\mathcal{L}_+$.

If E is any subset of X and f is any real valued function on X, then we shall write $E \subset f$ [or $E \supset f$] if $\chi_E(x) \leq f(x)$ [$\chi_E(x) \geq f(x)$] for every x in X.

Theorem A. *If Λ is a positive linear functional on $\mathcal{L}$ and if, for every C in* **C**,

$$\lambda(C) = \inf\{\lambda(f) : C \subset f \in \mathcal{L}_+\},$$

then λ is a regular content. If μ is the Borel measure induced by λ, then

$$\mu(U) \leq \Lambda(f)$$

for every bounded open set U and for every f in $\mathcal{L}_+$ for which $U \subset f$.

Theorem B. *If Λ is a positive linear functional on $\mathcal{L}$, if, for every C in* **C**,

$$\lambda(C) = \inf\{\Lambda(f) : C \subset f \in \mathcal{L}_+\},$$

and if μ is the Borel measure induced by the content λ, then

$$\int f d\mu \leq \Lambda(f)$$

for every f in $\mathcal{L}_+$.

Theorem C. *If Λ is a positive linear functional on $\mathcal{L}$, if, for every C in $\mathcal{C}$,*

$$\lambda(C) = \inf\{\Lambda(f) : C \subset f \in \mathcal{L}_+\},$$

and if μ is the Borel measure induced by the content λ, then, corresponding to every compact set C and every positive number ε, there exists a function f_0 in $\mathcal{L}_+$ such that $C \subset f_0$, $f_0 \leq 1$, and

$$\Lambda(f_0) \leq \int f_0 d\mu + \varepsilon.$$

Theorem D. *If Λ is a positive linear functional on $\mathcal{L}$, then there exists a Borel measure μ such that, for every f in $\mathcal{L}$,*

$$\Lambda(f) = \int f d\mu.$$

Theorem E. *If μ is a regular Borel measure, if, for every f in $\mathcal{L}$, $\Lambda(f) = \int f d\mu$, and if, for every C in* **C**

$$\lambda(C) = \inf\{\Lambda(f) : C \subset f \in \mathcal{L}_+\},$$

then $\mu(C) = \lambda(C)$ for every C in **C**. *Hence, in particular, the representation of a positive linear functional as an integral with respect to a regular Borel measure is unique.*

11 Haar measure

11.57 Full subgroups

A subgroup Z of a topological group X is *full* if it has a non empty interior.

Theorem A. *If Z is a full subgroup of a topological group X, then every union of left cosets of Z is both open and closed in X.*

Theorem B. *If E is any Borel set in a locally compact topological group X, then there exists a σ-compact full subgroup Z of X such that $E \subset Z$.*

11.58 Existence

A *Haar measure* is a Borel measure μ in a locally compact topological group X, such that $\mu(U) > 0$ for every non empty Borel open set U, and $\mu(xE) = \mu(E)$ for every Borel set E.

If E is any bounded set and F is any set with a non empty interior, we define the "ratio" $E : F$ as the least non negative integer n with the property that E may be covered by n left translations of F. It is easy to verify that (since E is bounded and F^0 is non empty) $E : F$ is always finite, and that, if A is a bounded set with a non empty interior, then

$$E : F \leq (E : A)(A : F).$$

Theorem A. *For each fixed, non empty open set U and compact set A with a non empty interior, the set function λ_U, defined for all compact sets C by*

$$\lambda_U(C) = \frac{C : U}{A : U}$$

is non negative, finite, monotone, subadditive, and left invariant; it is additive in the restricted sense that if C and D are compact sets for which $CU^{-1} \cap DU^{-1} = 0$, then

$$\lambda_U(C \cup D) = \lambda_U(C) + \lambda_U(D).$$

Theorem B. *In every locally compact topological group X there exists at least one regular Haar measure.*

11.59 Measurable groups

A *measurable group* is a σ-finite measure space $(X, \mathbf{S}, \mu)$ such that

(a) μ is not identically zero,

(b) X is a group,

(c) the σ-ring $\mathbf{S}$ and the measure μ are invariant under left translations, and

(d) the transformation S of $X \times X$ onto itself, defined by $S(x, y) = (x, xy)$, is measurability preserving.

(To say that $\mathbf{S}$ is invariant under left translations means that $xE \in \mathbf{S}$ for every x in X and E in $\mathbf{S}$; by a measurable subset of $X \times X$ we mean, as always, a set in the σ-ring $\mathbf{S} \times \mathbf{S}$.)

If X is any measurable space (and hence, in particular, if X is any measurable group), then the one to one transformation R of $X \times X$ onto itself, defined by $R(x, y) = (y, x)$, is measurability preserving. We shall frequently use the transformation $T = R^{-1}SR$; we observe that $T(x, y) = (yx, y)$.

Theorem A. *If E is any subset of $X \times X$, then*

$$(S(E))_x = xE_x \qquad \text{and} \qquad (T(E))^y = yE^y$$

for every $x, y \in X$.

Theorem B. *The transformations S and T are measure preserving transformation of the measure space $(X \times X, \mathbf{S} \times \mathbf{S}, \mu \times \nu)$ onto itself.*

Theorem C. *If* $Q = S^{-1}RS$ *then*

$$(Q(A \times B))_{x^{-1}} = xA \cap B^{-1},$$

and

$$(Q(A \times B))^{y^{-1}} = \begin{cases} Ay & \text{if } y \in B, \\ 0 & \text{if } y \notin B. \end{cases}$$

Theorem D. *If* A *is a measurable subset of* X [*of positive measure*], *and* $y \in X$, *then* Ay *is a measurable set* [*of positive measure*] *and* A^{-1} *is a measurable set* [*of positive measure*]. *If* f *is a measurable function,* A *is a measurable set of positive measure, and, for every* x *in* X, $g(x) = \frac{f(x^{-1})}{\mu(Ax)}$, *then* g *is measurable.*

Theorem E. *If* A *and* B *are measurable sets of positive measure, then there exist measurable sets* C_1 *and* C_2 *of positive measure and elements* x_1, y_1, x_2, *and* y_2, *such that*

$$x_1C_1 \subset A, \; y_1C_1 \subset B, \; C_2x_2 \subset A, \; C_2y_2 \subset B.$$

Theorem F (average theorem). *If* A *and* B *are measurable sets and if* $f(x) = \mu(x^{-1}A \cap B)$, *then* f *is a measurable function and*

$$\int f d\mu = \mu(A)\mu(B^{-1}).$$

If $g(x) = \mu(xA \triangle B)$, *and if* $\varepsilon < \mu(A) + \mu(B)$, *then the set* $\{x : g(x) < \varepsilon\}$ *is measurable.*

11.60 Uniqueness

Theorem A. *If* μ *and* ν *are two measures such that* $(X, \mathbf{S}, \mu)$ *and* $(X, \mathbf{S}, \nu)$ *are measurable groups, and if* E *in* $\mathbf{S}$ *is such that* $0 < \nu(E) < \infty$, *then, for every non negative measurable function* f *on* X,

$$\int f(x) d\mu(x) = \mu(E) \int \frac{f(y^{-1})}{\nu(Ey)} d\nu(y).$$

Theorem B. *If* μ *and* ν *are two measures such that* $(X, \mathbf{S}, \mu)$ *and* $(X, \mathbf{S}, \nu)$ *are measurable groups, and if* E *in* $\mathbf{S}$ *is such that* $0 < \nu(E) < \infty$, *then, for every* F *in* $\mathbf{S}$, $\mu(E)\nu(F) = \nu(E)\mu(F)$.

Theorem C. *If* μ *and* ν *are regular Haar measures on a locally compact topological group* X, *then there exists a positive finite constant* c *such that* $\mu(E) = c\nu(E)$ *for every Borel set* E.

12 Measure and topology in group

12.61 Topology in terms of measure

Throughout this section we shall assume that X is a locally compact topological group, μ is a regular haar measure on X, and $\varrho(E, F) = \mu(E \triangle F)$ for any two Borel sets E and F.

Theorem A. *If E is a Borel set of finite measure and if $f(x) = \varrho(xE, E)$, for every x in X, then f is continuous.*

Theorem B. *If U is any neighborhood of e, then there exist a Baire set E of positive, finite measure and a positive number ε such that $\{x : \varrho(xE, E) < \varepsilon\} \subset U$.*

Theorem C. *A necessary and sufficient condition that a set A be bounded is that there exist a Baire set E of positive, finite measure and a number ε, $0 \leq \varepsilon < 2\mu(E)$, such that*

$$A \subset \{x : \varrho(xE, E) \leq \varepsilon\}.$$

Density theorem for topological groups: If E is any bounded Borel set, and if, for every $x \in X$ and every bounded neighborhood U of e,

$$f_U(x) = \frac{\mu(E \cap Ux)}{\mu(Ux)},$$

then f_U converges in the mean (and therefore in measure) to χ_E as $U \to e$. In other words, for every positive number ε there exists a bounded neighborhood V of e such that if $U \subset V$, then $\int |f_U - \chi_E| d\mu < \varepsilon$.

12.62 Weil topology

Throughout this section we shall work with a fixed measurable group $(X, \mathbf{S}, \mu)$; we shall write $\varrho(E, F) = \mu(E \triangle F)$ for any two measurable sets E and F. We shall denote by $\mathbf{A}$ the class of all sets of the form EE^{-1}, where E is a measurable set of positive, finite measure, and by $\mathbf{N}$ the class of all sets of the form $\{x : \varrho(xE, E) < \varepsilon\}$, where E is a measurable set of positive, finite measure and ε is a real number such that $0 < \varepsilon < 2\mu(E)$.

Theorem A. *If $N = \{x : \varrho(xE, E) < \varepsilon\} \in \mathbf{N}$, then every measurable set F of positive measure contains a measurable subset G of positive, finite measure such that $GG^{-1} \subset N$.*

Theorem B. *If $A = EE^{-1} \in \mathbf{A}$ and $0 < \varepsilon < 2\mu(E)$, and if*

$$N = \{x : \varrho(xE, E) < \varepsilon\},$$

then $N \in \mathbf{N}$ and $N \subset A$.

Theorem C. *If $N = \{x : \varrho(xE, E) < \varepsilon\} \in \mathbf{N}$, then N is a measurable set of positive measure. If $\mu(E^{-1}) < \infty$, then $\mu(N) < \infty$.*

Theorem D. *If A and B are any two sets of $\mathbf{A}$, then there exists a set C in $\mathbf{A}$ such that $C \subset A \cap B$.*

We shall say that a measurable group X is *separated* if whenever an element x of the group is different from e, then there exists a measurable set E of positive, finite measure such that $\varrho(xE, E) > 0$.

Theorem E. *If X is separated, and if the class $\mathbf{N}$ is taken for a base at e, then, with respect to the induced topology, X is a topological group.*

We shall refer to this topology of the measurable group as the *Weil topology.*

Theorem F. *If X is a separated, measurable group, then X is locally bounded with respect to its Weil topology. If a measurable set E has a non empty interior, then $\mu(E) > 0$; if a measurable set E is bounded, then $\mu(E) < \infty$.*

Theorem G. *If μ is any Baire measure in a locally compact topological group X, and if Y is the set of all those elements y for which $\mu(yE) = \mu(E)$ for all Baire sets E, then Y is a closed subgroup of X.*

By a *thick subgroup* of a measurable group we mean a subgroup which is a thick set.

Theorem H. *If $(X, \mathbf{S}, \mu)$ is a separated, measurable group, then there exists a locally compact topological group $\hat{X}$ with a Haar measure $\hat{\mu}$ on the class $\hat{\mathbf{S}}$ of all Baire sets, such that X is a thick subgroup of $\hat{X}$, $\mathbf{S} \supset \hat{\mathbf{S}} \cap X$, and $\mu(E) = \hat{\mu}(E)$ whenever $\hat{E} \in \hat{\mathbf{S}}$ and $E = \hat{E} \cap X$.*

12.63 Quotient groups

Throughout this section we shall assume that X is a locally compact topological group and μ is a Haar measure in X; Y is a compact invariant subgroup of X, ν is a Haar measure in Y such that $\nu(Y) = 1$, and π is the projection from X onto the quotient group $\hat{X} = X/Y$.

Theorem A. *If a compact set C is a union of cosets of Y and if U is an open set containing C, then there exists an open set $\hat{V}$ in $\hat{X}$ such that*

$$C \subset \pi^{-1}(\hat{V}) \subset U.$$

Theorem B. *If $\hat{C}$ is a compact subset of $\hat{X}$, then $\pi^{-1}(\hat{C})$ is a compact subset of X; if $\hat{E}$ is a Baire set [or a Borel set] in $\hat{X}$, then $\pi^{-1}(\hat{E})$ is a Baire set [or a Borel set] in X.*

Theorem C. *If $\hat{\mu} = \mu\pi^{-1}$, then $\hat{\mu}$ is a Haar measure in $\hat{X}$.*

Theorem D. *If $f \in \mathcal{L}_+(X)$ and if*

$$g(x) = \int_Y f(xy) d\nu(y),$$

then $g \in \mathcal{L}_+(X)$ and there exists a (uniquely determined) function $\hat{g}$ in $\mathcal{L}_+(X)$ such that $g = \hat{g}\pi$.

Theorem E. *If C is a compact Baire set in X and if $g(x) = \nu(x^{-1}C \cap Y)$, then there exists a (uniquely determined) Baire measurable and integrable function $\hat{g}$ on $\hat{X}$ such that $g = \hat{g}\pi$. If C is a union of cosets ofY, then $\int \hat{g} d\hat{\mu} = \mu(C)$.*

Theorem F. *If, for each Baire set E in X,*

$$g_E(x) = \nu(x^{-1}E \cap Y),$$

then there exists a (uniquely determined) Baire measurable function $\hat{g}_E$ on $\hat{X}$ such that $g_E = \hat{g}_E\pi$.

Theorem G. *If, for each Baire set E in X, $\hat{g}_E$ is the unique Baire measurable function on $\hat{X}$ for which*

$$\hat{g}_E(\pi(x)) = \nu(x^{-1}E \cap Y) = g_E(x)$$

for every $x \in X$, then

$$\int \hat{g}_E d\hat{\mu} = \mu(E)$$

for every Baire set E.

12.64 The regularity of Haar measure

Throughout this section, up to the statement of the final, general result, we shall assume that X is a locally compact and σ-compact topological group, and μ is a left invariant Baire measure in X which is not identically zero (and which, therefore, is positive on all non empty open Baire sets).

By an *invariant σ-ring* we shall mean a σ-ring $\mathbf{T}$ of Baire sets such that if $E \in \mathbf{T}$ and $x \in X$, then $xE \in \mathbf{T}$. Since the class of all Baire sets is an invariant σ-ring, and since the intersection of any collection of invariant σ-rings is itself an invariant σ-ring, we may define the invariant σ-ring *generated* by any class $\mathbf{E}$ of Baire sets as the intersection of all invariant σ-rings containing $\mathbf{E}$.

Theorem A. *If $\mathbf{E}$ is a class of Baire sets and $\mathbf{T}$ is the invariant σ-ring generated by $\mathbf{E}$, then $\mathbf{T}$ coincides with the σ-ring $\mathbf{T}_0$ generated by the class $\{xE : x \in X, E \in \mathbf{E}\}$.*

Theorem B. *If $\mathbf{E}$ is a countable class of Baire sets of finite measure and $\mathbf{T}$ is the invariant σ-ring generated by $\mathbf{E}$, then the metric space $\mathcal{J}$ of all sets of finite measure in $\mathbf{T}$ (with the metric defined by $\varrho(E, F) = \mu(E \triangle F)$) is separable.*

Theorem C. *If $\mathbf{T}$ is an invariant σ-ring, if f is a function in $\mathcal{L}$ which is measurable $(\mathbf{T})$, and if y in X is such that $\varrho(yE, E) = 0$ for every E in $\mathbf{T}$, then $f(y^{-1}x) = f(x)$ for every x in X.*

Theorem D. *If $\mathbf{T}$ is an invariant σ-ring generated by its sets of finite measure and containing at least one bounded set of positive measure, if $\mathbf{E}$ is a class of sets dense in the metric space of sets of finite measure in $\mathbf{T}$, and if*

$$Y = \{y : \varrho(yE, E) = 0, E \in \mathbf{E}\},$$

then Y is a compact, invariant subgroup of X.

Theorem E. *If E is any Baire set in X, then there exists a compact, invariant Baire subgroup Y if X such that E is a union of cosets of Y.*

Theorem F. *If $\{e\}$ is a Baire set, then X is separable.*

Theorem G. *If E is any Baire set in X, then there exists a compact, invariant subgroup Y of X such that E is a union of cosets of Y, and such that the quotient group X/Y is separable.*

Theorem H. *Every Haar measure in X is completion regular.*

Theorem I. *If X is an arbitrary (not necessarily σ-compact) locally compact topological group, and if μ is a left invariant Borel measure in X, then μ is completion regular.*

Index

Contents