Prerequisites

**Topological spaces.** A set $E$ in a space $X$ is $\sigma$-compact if there exists a sequence of compact sets such that $E = \bigcup_{n=1}^{\infty} C_n$. A space $X$ is locally compact if every point of $X$ has a neighborhood whose closure is compact. A subset $E$ of a locally compact space is bounded if there exists a compact set $C$ such that $E \subseteq C$.

**Topological groups.** The set $x E$ [or $E x$] is called a left translation [or right translation]. If $Y$ is a subgroup of $X$, the sets $x Y$ and $Y x$ are called (left and right) cosets of $Y$.

A topological group is a group $X$ which is a Hausdorff space such that the transformation (from $X \times X$ onto $X$) which sends $(x, y)$ into $x^{-1} y$ is continuous.

A class $N$ of open sets containing $e$ in a topological group is a base at $e$ if

(a) for every $x$ different from $e$ there exists a set $U$ in $N$ such that $x \notin U$,
(b) for any two sets $U$ and $V$ in $N$ there exists a set $W$ in $N$ such that $W \subseteq U \cap V$,
(c) for any set $U \in N$ there exists a set $W \in N$ such that $V^{-1} \subseteq W \subseteq U$,
(d) for any set $U \in N$ and any element $x \in X$, there exists a set $V \in N$ such that $V \subseteq x U x^{-1}$, and
(e) for any set $U \in N$ there exists a set $V \in N$ such that $V x \subseteq U$.

If $N$ is a satisfies the conditions described above, and if the class of all translation of sets of $N$ is taken for a base, then, with respect to the topology so defined, $X$ becomes a topological group.

A subset $E$ of a topological group $X$ is bounded if, for every neighborhood $U$ of $e$, there exists a finite set $\{x_1, \ldots, x_n\}$ such that $E \subseteq \bigcup_{i=1}^{n} x_i U$; if $X$ is locally compact, then this definition of boundedness agrees with the one applicable in any locally compact space (i.e. the one that requires that the closure of $E$ be compact). If a continuous, real valued function $f$ on $X$ is such that the set $N(f) = \{ x : f(x) \neq 0 \}$ is bounded then $f$ is uniformly continuous in the sense that to every positive number $\varepsilon$ there corresponds a neighborhood $U$ of $e$ such that $|f(x_1) - f(x_2)| < \varepsilon$ whenever $x_1 x^{-1} \in U$.

A topological group is locally bounded if there exists in it a bounded neighborhood of $e$. To every locally bounded topological group $X$, there corresponds a locally compact topological group $X^*$, called the completion of $X$ (uniquely determined to within an isomorphism), such that $X$ is a dense subgroup of $X^*$. Every closed subgroup and every quotient group of a locally compact group is a locally compact group.

1 Sets and classes

1.3 Limits, complements, and differences

If $(E_n)$ is a sequence of subsets of $X$, the set $E^*$ of all those point of $X$ which belong to $E_n$ for infinitely many values of $n$ is called the superior limit of the sequence and is denoted by $E^* = \limsup_n E_n$. 

1
The set \( E_* \) of all those points of \( X \) which belong to \( E_n \) for all but a finite number of values of \( n \) is called the inferior limit of the sequence and is denoted by

\[
E_* = \liminf_n E_n.
\]

\[
\limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k, \quad \liminf_n E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k.
\]

If it so happens that \( E^* = E_* \), we shall use the notation \( \lim_n E_n \). If \( (E_n) \) is a monotone sequence, then \( \lim_n E_n \) exists and is equal to \( \bigcup_n E_n \) or \( \bigcap_n E_n \) according to as the sequence is increasing or decreasing.

The difference \( E - F \) is called proper if \( E \supset F \).

### 1.4 Ring and algebras

A ring (or Boolean ring) of sets is a non-empty class \( R \) of sets such that if \( E, F \in R \) then \( E \cup F \in R \) and \( E - F \in R \).

The empty set belongs to every ring \( R \). A non empty class of sets closed under the formation of unions and proper differences is a ring. If a non empty class of sets is closed under the formation of intersections, proper differences, and disjoint unions, then it is a ring.

An algebra (or Boolean algebra) of sets is a non empty class \( A \) of sets such that if \( E, F \in A \) then \( E \cup F \in A \), and if \( E \in A \), then \( E' \in A \).

Algebra = ring containing \( X \).

A semiring is a non empty class \( P \) of sets such that:

a) if \( E, F \in P \) then \( E \cap F \in P \),

b) if \( E, F \in P \) and \( E \subset F \) then there is a finite class \( \{C_0, C_1, \ldots, C_n\} \) of sets in \( P \) such that \( E = C_0 \subset C_1 \subset \ldots \subset C_n = F \) and \( D_i = C_i - C_{i-1} \in P \) for \( i = 1, 2, \ldots, n \).

### 1.5 Generated rings and \( \sigma \)-rings

**Theorem A.** If \( E \) is any class of sets, then there exists a unique ring \( R_0 \) such that \( R_0 \supset E \) and such that if \( R \) is any other ring containing \( E \) then \( R_0 \subset R \).

The ring \( R_0 \), the smallest ring containing \( E \), is called the ring generated by \( E \); it will be denoted by \( R(E) \).

**Theorem B.** If \( E \) is any class of sets, then every set in \( R(E) \) may be covered by a finite union of sets in \( E \).

**Theorem C.** If \( E \) is a countable class of sets, then \( R(E) \) is countable.

A \( \sigma \)-ring is a non empty class \( S \) of sets such that

(a) if \( E \in S \) and \( F \in S \), then \( E - F \in S \), and

(b) if \( E_i \in S \), \( i = 1, 2, \ldots \), then \( \bigcup_{i=1}^{\infty} E_i \in S \).

Equivalently a \( \sigma \)-ring is a ring closed under the formation of countable unions. A \( \sigma \)-ring is closed under the formation of countable intersections.

The \( \sigma \)-ring \( S(E) \) generated by a class \( E \) of sets is the smallest \( \sigma \)-ring containing \( E \).

**Theorem D.** If \( E \) is any class of sets and \( E \) is any set in \( S = S(E) \), then there exists a countable subclass \( D \) of \( E \) such that \( E \in S(D) \).

**Theorem E.** If \( E \) is any class of sets and if \( A \) is any subset of \( X \), then

\[
S(E) \cap A = S(E \cap A).
\]
1.6 Monotone classes

A non empty class $M$ of sets is monotone if, for every monotone sequence $(E_n)$ of sets in $M$, we have $\lim_n E_n \in M$.

The monotone class $M(E)$ generated by a class $E$ of sets is the smallest monotone class containing $E$.

**Theorem A.** A $\sigma$-ring is a monotone class; a monotone ring is a $\sigma$-ring.

**Theorem B.** If $R$ is a ring, then $M(R) = S(R)$. Hence if a monotone class contains a ring $R$, then it contains $S(R)$.

2 Measures and outer measures

2.7 Measure on rings

A set function is a function whose domain of definition is a class of sets. An extended real valued set function $\mu$ defined on a class $E$ is additive if, whenever $E \in E$, $F \in E$, $E \cup F \in E$, and $E \cap F = 0$,

$$\mu(E \cup F) = \mu(E) + \mu(F).$$

An extended real valued set function $\mu$ defined on a class $E$ is finitely additive if, for every finite, disjoint class $\{E_1, \ldots, E_n\}$ of sets in $E$ whose union is also in $E$, we have

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i).$$

An extended real valued set function $\mu$ defined on a class $E$ is countably additive if, for every disjoint sequence $(E_n)$ of sets in $E$ whose union is also in $E$, we have

$$\mu\left(\bigcup_{n=1}^\infty E_n\right) = \sum_{n=1}^\infty \mu(E_n).$$

A measure is an extended real valued, non negative, and countably additive set function $\mu$, defined on a ring $R$, and such that $\mu(0) = 0$.

If $\mu$ is a measure on a ring $R$, a set $E$ in $R$ is said to have finite measure if $\mu(E) < +\infty$; the measure of $R$ is $\sigma$-finite if there exists a sequence $(E_n)$ of sets in $E$ such that

$$E \subset \bigcup_{n=1}^\infty E_n \quad \text{and} \quad \mu(E_n) < +\infty, \quad n = 1, 2, \ldots$$

If the measure of every set $E$ in $R$ is finite [or $\sigma$-finite], the measure $\mu$ is called finite [or $\sigma$-finite] on $R$. If $X \in R$ (i.e. if $R$ is an algebra) and $\mu(X)$ is finite or $\sigma$-finite, then $\mu$ is called totally finite or totally $\sigma$-finite respectively. The measure $\mu$ is called complete if the conditions

$$E \in R, \quad F \subset E, \quad \text{and} \quad \mu(E) = 0$$

imply that $F \in R$. 

3
2.8 Measure on intervals

\( P = \) the class of all bounded, left closed and right opened intervals

\( R = \) all finite, disjoint unions of sets of \( P \)

On the class \( P \) we define a set function \( \mu \) by \( \mu((a,b)) = b - a \).

**Theorem A.** If \( \{E_1, \ldots, E_n\} \) is a finite, disjoint class of sets in \( P \), each contained in a given set \( E_0 \) in \( P \), then

\[
\sum_{i=1}^{n} \mu(E_i) \leq \mu(E_0).
\]

**Theorem B.** If a closed interval \( F_0 = (a_0, b_0) \), is contained in the union of a finite number of bounded, open intervals, \( U_1, \ldots, U_n \), \( U_i = (a_i, b_i) \), then

\[
b_0 - a_0 < \sum_{i=1}^{n} (b_i - a_i).
\]

**Theorem C.** If \( (E_0, E_1, E_2, \ldots) \) is a sequence of sets in \( P \), such that

\[
E_0 \subset \bigcup_{i=1}^{\infty} E_i,
\]

then

\[
\mu(E_0) \leq \sum_{i=1}^{\infty} \mu(E_i).
\]

**Theorem D.** The set function \( \mu \) is countably additive on \( P \).

**Theorem E.** There exists a unique, finite measure \( \overline{\mu} \) on the ring \( R \) such that, whenever \( E \in P \), \( \overline{\mu}(E) = \mu(E) \).

2.9 Properties of measures

An extended real valued set function \( \mu \) on a class \( E \) is monotone if, whenever \( E, F \in E \) and \( E \subset F \), then \( \mu(E) \leq \mu(F) \). An extended real valued set function \( \mu \) on a class \( E \) is subtractive if, whenever \( E, F \in E \), \( E \subset F \), \( F - E \in E \) and \( |\mu(E)| < +\infty \), then

\[
\mu(F - E) = \mu(F) - \mu(E).
\]

**Theorem A.** If \( \mu \) is a measure on a ring \( R \), then \( \mu \) is monotone and subtractive.

**Theorem B.** If \( \mu \) is a measure on a ring \( R \), if \( E \in R \), and if \( (E_i) \) is a finite or infinite sequence of sets in \( R \) such that \( E \subset \bigcup_{i} E_i \), then \( \mu(E) \leq \sum_{i} \mu(E_i) \).

**Theorem C.** If \( \mu \) is a measure on a ring \( R \), if \( E \in E \), and if \( (E_i) \) is a finite or infinite disjoint sequence of sets in \( R \) such that \( \bigcup_{i} E_i \subset E \), then

\[
\sum_{i} \mu(E_i) \leq \mu(E).
\]

**Theorem D.** If \( \mu \) is a measure on a ring \( R \) and if \( (E_n) \) is an increasing sequence of sets in \( R \) for which \( \lim_{n} E_n \in R \), then \( \mu(\lim_{n} E_n) = \lim_{n} \mu(E_n) \).

**Theorem E.** If \( \mu \) is a measure on a ring \( R \), and if \( (E_n) \) is a decreasing sequence of sets in \( R \) of which at least one has finite measure and for which \( \lim_{n} E_n \in R \), then \( \mu(\lim_{n} E_n) = \lim_{n} \mu(E_n) \).
We shall say that an extended real valued set function on a class $E$ is continuous from below at a set $E$ (in $E$) if, for every increasing sequence $(E_n)$ of sets in $E$ for which $\lim_n E_n = E$, we have $\lim_n \mu(E_n) = \mu(E)$. Similarly $\mu$ is continuous from above at $E$ if, for every decreasing sequence $(E_n)$ of sets in $E$ for which $|\mu(E_m)| < +\infty$ for at least one value of $m$ and for which $\lim_n E_n = E$, we have $\lim_n \mu(E_n) = \mu(E)$.

**Theorem F.** Let $\mu$ be a finite, non negative, and additive set function on a ring $\mathbb{R}$. If $\mu$ is either continuous from below at every $E$ in $\mathbb{R}$, or continuous from above at $0$, then $\mu$ is a measure on $\mathbb{R}$.

### 2.10 Outer measures

A non empty class $E$ of sets is hereditary if, whenever $E \in E$ and $F \subset E$, then $F \in E$.

If $E$ is any class of sets, we shall denote the hereditary $\sigma$-ring generated by $E$, i.e. the smallest hereditary $\sigma$-ring containing $E$, by $H(E)$.

An extended real valued set function $\mu^*$ defined on a class $E$ of sets is subadditive if, whenever $E, F \in E$, and $E \cup F \in E$, then $\mu^*(E \cup F) \leq \mu^*(E) + \mu^*(F)$.

An extended real valued set function $\mu^*$ defined on a class $E$ of sets is finitely subadditive if, for every finite class $\{E_1, \ldots, E_n\}$ of sets in $E$ whose union is also in $E$, we have

$$\mu^*\left( \bigcup_{i=1}^n E_i \right) \leq \sum_{i=1}^n \mu^*(E_i).$$

An extended real valued set function $\mu^*$ defined on a class $E$ of sets is countably subadditive if, for sequence $(E_i)$ of sets in $E$ whose union is also in $E$, we have

$$\mu^*\left( \bigcup_{i=1}^\infty E_i \right) \leq \sum_{i=1}^\infty \mu^*(E_i).$$

An outer measure is an extended real valued, non negative, monotone, and countably subadditive set function $\mu^*$, defined on a hereditary $\sigma$-ring $H$, and such that $\mu^*(0) = 0$.

**Theorem A.** If $\mu$ is a measure on a ring $\mathbb{R}$ and if, for every $E$ in $H(\mathbb{R})$,

$$\mu^*(E) = \inf\{\sum_{n=1}^\infty \mu(E_n) : E_n \in \mathbb{R}, n = 1, 2, \ldots, E \subset \bigcup_{n=1}^\infty E_n\},$$

then $\mu^*$ is an extension of $\mu$ to an outer measure on $H(\mathbb{R})$; if $\mu$ is [totally] $\sigma$-finite, then so is $\mu^*$.

### 2.11 Measurable sets

Let $\mu^*$ be an outer measure on a hereditary $\sigma$-ring $H$. A set $E$ in $H$ is $\mu^*$-measurable if, for every set $A$ in $H$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$
Theorem A. If $\mu^*$ is an outer measure on a hereditary $\sigma$-ring $H$ and if $S$ is the class of all $\mu^*$-measurable sets, then $S$ is a ring.

Theorem B. If $\mu^*$ is an outer measure on a hereditary $\sigma$-ring $H$ and if $S$ is the class of all $\mu^*$-measurable sets, then $S$ is a $\sigma$-ring. If $A \in H$ and if $(E_n)$ is a disjoint sequence of set in $S$ with $\bigcup_{n=1}^{\infty} = E$, then

$$\mu^*(A \cap E) = \sum_{n=1}^{\infty} \mu^*(A \cap E_n).$$

Theorem C. If $\mu^*$ is an outer measure on a hereditary $\sigma$-ring $H$ and if $S$ is the class of all $\mu^*$-measurable sets, then every set of outer measure zero belongs to $S$ and the set function $\mu$, defined for $E$ in $S$ by $\mu(E) = \mu^*(E)$, is a complete measure on $S$.

The measure $\mu$ is called the measure induced by the outer measure $\mu^*$.

An outer measure $\mu^*$ on the class $H$ of all subsets of a metric space is called a metric outer measure if

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$$

whenever $\varrho(E,F) > 0$, where $\varrho$ is the metric on $X$.

If $\mu^*$ is a metric outer measure, then every open set (and therefore every Borel set) is $\mu^*$-measurable. If $\mu^*$ is an outer measure on the class of all subsets of a metric space $X$ such that every open set is $\mu^*$-measurable, then $\mu^*$ is a metric outer measure.

3 Extension of measures

3.12 Properties of induced measures

Throughout this section we shall assume that $\mu$ is a measure on a ring $R$, $\mu^*$ is the induced outer measure on $H(R)$, and $\overline{\mu}$ is the measure induced by $\mu^*$ on the $\sigma$-ring $\overline{S}$ of all $\mu^*$-measurable sets.

Theorem A. Every set in $S(R)$ is $\mu^*$-measurable.

Theorem B. If $E \in H(R)$, then

$$\mu^*(E) = \inf \{ \overline{\mu}(F) : E \subset F \in \overline{S} \} = \inf \{ \overline{\mu}(F) : E \subset F \in S(R) \}.$$
Theorem E. If $\mu$ on $\mathbb{R}$ is $\sigma$-finite, then so are the measures $\overline{\mu}$ on $\mathcal{S}(\mathbb{R})$ and $\overline{\mu}$ on $\mathcal{S}$.

Suppose that we start with an outer measure $\mu^*$, form the induced measure $\mu$, and then form the outer measure $\bar{\mu}^*$ induced by $\overline{\mu}$. In general these two set functions are not the same; if, however, the induced outer measure $\{\overline{\mu}\}^*$ does coincide with the original outer measure $\mu^*$, then $\mu^*$ is called regular.

If $X$ is a metric space, $p$ is a positive real number, and $E$ is a subset of $X$, then the $p$-dimensional Hausdorff (outer) measure of $E$ is defined to be the number

$$\mu_p^*(E) = \sup_{\varepsilon > 0} \{ \sum_{i=1}^{\infty} (\delta(E_i))^p : E = \bigcup_{i=1}^{\infty} E_i, \delta(E_i) < \varepsilon, i = 1, 2, \ldots \},$$

where $\delta(E)$ denotes the diameter of $E$.

The set function $\mu_p^*$ is a metric outer measure. The outer measure $\mu_p^*$ is regular.

3.13 Extension, completion and approximation

Theorem A. If $\mu$ is a $\sigma$-finite measure on a ring $\mathcal{R}$, then there is a unique measure $\overline{\mu}$ on the $\sigma$-ring $\mathcal{S}(\mathcal{R})$ such that, for $E$ in $\mathcal{R}$, $\overline{\mu}(E) = \mu(E)$; the measure $\overline{\mu}$ is $\sigma$-finite.

The measure $\overline{\mu}$ is called the extension of $\mu$; except when it is likely to lead to confusion, we shall write $\mu(E)$ instead of $\overline{\mu}(E)$ even for sets $E$ in $\mathcal{S}(R)$.

Theorem B. If $\mu$ is a measure on a $\sigma$-ring $\mathcal{S}$, then the class $\mathcal{S}$ of all sets of the form $E \triangle N$, where $E \in \mathcal{S}$ and $N$ is a subset of a set of measure zero in $\mathcal{S}$, is a $\sigma$-ring, and the set function $\overline{\mu}$ defined by $\overline{\mu}(E \triangle N) = \mu(E)$ is a complete measure on $\mathcal{S}$.

The measure $\overline{\mu}$ is called the completion of $\mu$.

Theorem C. If $\mu$ is a $\sigma$-finite measure on a ring $\mathcal{R}$, and if $\mu^*$ is the outer measure induced by $\mu$, then the completion of the extension of $\mu$ to $\mathcal{S}(\mathcal{R})$ is identical with $\mu^*$ on the class of all $\mu^*$-measurable sets.

Theorem D. If $\mu$ is a $\sigma$-finite measure on a ring $\mathcal{R}$, then, for every set $E$ of finite measure in $\mathcal{S}(\mathcal{R})$ and for every positive number $\varepsilon$, there exists a set $E_0$ in $\mathcal{R}$ such that $\mu(E \triangle E_0) \leq \varepsilon$.

3.14 Inner measures

For every $E$ in $\mathcal{H}(\mathcal{S})$ we write

$$\mu_*(E) = \sup\{ \mu(F) : E \supset F \in \mathcal{S} \}.$$

$\mu_*$ = inner measure $\mu_*$ induced by $\mu$.

Throughout this section we shall assume that $\mu$ is a $\sigma$-finite measure on a $\sigma$-ring $\mathcal{S}$, $\mu^*$ and $\mu_*$ are the outer and the inner measure induced by $\mu$, respectively, and $\overline{\mu}$ on $\mathcal{S}$ is the completion of $\mu$. 

7
Theorem A. If $E \in H(S)$, then
\[ \mu_*(E) = \sup \{ \pi(F) : E \supset F \in S \}. \]

If $E \in H(S)$ and $F \in S$, we shall say that $F$ is a measurable kernel of $E$, if $F \subset E$ and if, for every set $G$ in $S$ for which $G \subset E - F$, we have $\mu(G) = 0$. Loosely speaking a measurable kernel of a set $E$ in $H(S)$ is a maximal set in $S$ which is contained in $E$.

Theorem B. Every set $E$ in $H(S)$ has a measurable kernel.

Theorem C. If $E \in H(S)$ and $F \in S$, we shall say that $F$ is a measurable kernel of $E$, if $F \subset E$ and if, for every set $G$ in $S$ for which $G \subset E - F$, we have $\mu(G) = 0$.

Theorem D. If $(E_n)$ is a disjoint sequence of sets in $H(S)$, then
\[ \mu_*(\bigcup_{n=1}^{\infty} E_n) \geq \sum_{n=1}^{\infty} \mu_*(E_n). \]

Theorem E. If $A \in H(S)$ and if $(E_n)$ is a disjoint sequence of sets in $\bar{S}$ with $\bigcup_{n=1}^{\infty} E_n = E$, then
\[ \mu_*(A \cap E) = \sum_{n=1}^{\infty} \mu_*(A \cap E_n). \]

Theorem F. If $E \in S$, then
\[ \mu^*(E) = \mu_*(E) = \overline{\mu}(E), \]
and conversely, if $E \in H(S)$ and
\[ \mu^*(E) = \mu_*(E) < \infty, \]
then $E \in S$.

Theorem G. If $E$ and $F$ are disjoint sets in $H(S)$, then
\[ \mu_*(E \cup F) \leq \mu_*(E) + \mu^*(F) \leq \mu^*(E \cup F). \]

Theorem H. If $E \in S$, then, for every subset $A$ of $X$,
\[ \mu_*(A \cap E) + \mu^*(A' \cap E) = \overline{\pi}(E). \]

3.15 Lebesgue measure

Throughout this section we will assume that $X$ is the real line, $P$ is the class of all bounded, semiclosed intervals of the form $\langle a, b \rangle$, $S$ is the $\sigma$-ring generated by $P$, and $\mu$ is the set function on $P$ defined by $\mu(\langle a, b \rangle) = b - a$.

The sets of the $\sigma$-ring $S$ are called the Borel sets of the line; according to the extension theorems 8.E and 13.A we may assume that $\mu$ is defined for all Borel sets. If $\pi$ on $\bar{S}$ is the completion of $\mu$ on $S$, the sets of $\bar{S}$ are the Lebesgue measurable sets of the line; the measure $\overline{\pi}$ is Lebesgue measure.

Measures $\mu$ on $S$ and $\overline{\pi}$ on $\bar{S}$ are totally $\sigma$-finite.
Theorem A. Every countable set is a Borel set of measure zero.

Theorem B. The class $S$ of all Borel sets coincides with the $\sigma$-ring generated by the class $U$ of all open sets.

Theorem C. If $U$ is the class of all open sets, then, for every $E$ in $X$,
$$
\mu^*(E) = \inf\{\mu(U) : E \subset U \in U\}.
$$

Theorem D. Let $T$ be the one to one transformation of the entire real line onto itself, defined by $T(x) = \alpha x + \beta$, where $\alpha$ and $\beta$ are real numbers and $\alpha \neq 0$. If, for every subset $E$ of $X$, $T(E)$ denotes the set of all points of the form $T(x)$ with $x$ in $E$, $T(E)$ denotes the set of all points of the form $T(x)$ with $x$ in $E$, i.e. $T(E) = \{\alpha x + \beta ; x \in E\}$, then
$$
\mu^*(T(E)) = |\alpha| \mu^*(E) \quad \text{and} \quad \mu_*(T(E)) = |\alpha| \mu_*(E).
$$

The set $T(E)$ is a Borel set or a Lebesgue measurable set if and only if $E$ is a Borel set or a Lebesgue measurable set, respectively.

$C$ = the set of all those numbers $x$ in whose ternary expansion the digit 1 is not needed. The set $C$ is called the Cantor set. $\mu(C) = 0$. $C$ is nowhere dense, perfect. $C$ has the cardinal number of continuum.

3.16 Non measurable sets

The symbol $D(E)$ will be used to denote the difference set of $E$, i.e. the set of all numbers of the form $x - y$ with $x, y \in E$.

Theorem A. If $E$ is a Lebesgue measurable set of positive finite measure, and if $0 \leq \alpha < 1$, then there exists an open interval $U$ such that $\mu(E \cap U) \geq \alpha \mu(U)$.

Theorem B. If $E$ is a Lebesgue measurable set of positive measure, then there exists an open interval containing the origin and entirely contained in the difference set $D(E)$.

Theorem C. If $\xi$ is an irrational number, then the set $A$ of all numbers of the form $n + m \xi$, where $n$ and $m$ are arbitrary integers, is everywhere dense on the line; the same is true of the subset $B$ of all numbers of the form $n + m \xi$ with $n$ even, and the subset $C$ of numbers of the form $n + m \xi$ with $n$ odd.

Theorem D. There exists at least one set $E_0$ which is not Lebesgue measurable.

Theorem E. There exists a subset $M$ of the real line such that, for every Lebesgue measurable set $E$,
$$
\mu_*(M \cap E) = 0 \quad \text{and} \quad \mu^*(M \cap E) = \mu(E).
$$

4 Measurable functions

4.17 Measure spaces

A measurable space is a set $X$ and a $\sigma$-ring $S$ of subsets of $X$ with the property that $\bigcup S = X$. A subset $E$ of $X$ is called measurable if and only if it belongs to the $\sigma$-ring $S$. 

9
A measure space is a measurable space \((X, \mathcal{S}, \mu)\) and a measure \(\mu\) on \(\mathcal{S}\). The measure space \(X\) is called [totally] finite, \(\sigma\)-finite, or complete, according as the measure \(\mu\) is [totally] finite, \(\sigma\)-finite or complete.

A subset \(X_0\) of a measure space \((X, \mathcal{S}, \mu)\) is thick if \(\mu^*(E - X_0) = 0\) for every measurable set \(E\).

**Theorem A.** If \(X_0\) is a thick subset of a measure space \((X, \mathcal{S}, \mu)\), if \(\mathcal{S}_0 = \mathcal{S} \cap X_0\), and if, for \(E \in \mathcal{S}\), \(\mu_0(E \cap X_0) = \mu(E)\), then \((X_0, \mathcal{S}_0, \mu_0)\) is a measure space.

### 4.18 Measurable functions

\((X, \mathcal{S})\) is a measurable space. We put \(N(f) = \{x : f(x) \neq 0\}\). If a real valued function \(f\) is such that, for every Borel subset \(M\) of the real line the set \(N(f) \cap f^{-1}(M)\) is measurable, then \(f\) is called a measurable function.

- **Borel measurable functions** - \(\mathcal{S} = \)Borel sets
- **Lebesgue measurable functions** - \(\mathcal{S} = \)Lebesgue measurable sets

**Theorem A.** A real function \(f\) on a measurable space \((X, \mathcal{S})\) is measurable if and only if, for every real number \(c\), the set \(N(f) \cap \{x : f(x) < c\}\) is measurable.

### 4.19 Combination of measurable functions

**Theorem A.** If \(f\) and \(g\) are extended real valued measurable functions on a measurable space \((X, \mathcal{S})\), and if \(c\) is any real number, then each of the three sets

- \(A = \{x : f(x) < g(x) + c\}\),
- \(B = \{x : f(x) \leq g(x) + c\}\),
- \(C = \{x : f(x) = g(x) + c\}\),

has a measurable intersection with every measurable set.

**Theorem B.** If \(\phi\) is an extended real valued Borel measurable function on the extended real line such that \(\phi(0) = 0\), and if \(f\) is an extended real valued measurable function on a measurable space \(X\), then the function \(\bar{f}\), defined by \(\bar{f} = \phi(f(x))\), is a measurable function on \(X\).

**Theorem C.** If \(f\) and \(g\) are extended real valued measurable functions on a measurable space \(X\), then so also are \(f + g\) and \(fg\).

\[
\begin{align*}
f^+ &= \max(f, 0) \quad \text{and} \quad f^- = -\min(f, 0) = \max(-f, 0) \\
f &= f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^- \\
f^+ &= \text{positive part}, \quad f^- = \text{negative part}
\end{align*}
\]

Cantor function \(\psi\):
- \(x \notin C \Rightarrow \psi(x) = (0, \alpha_1, \alpha_2, \alpha_3, \ldots)_2\),
- \(x \in C \Rightarrow \psi(x) = (0, \alpha_1, \alpha_2, \alpha_3, \ldots)_2\).
- \(\psi\) is nondecreasing, continuous.
4.20 Sequences of measurable functions

**Theorem A.** If \((f_n)\) is a sequence of extended real valued, measurable functions on a measurable space \(X\), then each of the four functions \(h, g, f^*\) and \(f_*\), defined by

\[
\begin{align*}
  h(x) &= \sup\{f_n(x) : n = 1, 2, \ldots\}, \\
  g(x) &= \inf\{f_n(x) : n = 1, 2, \ldots\}, \\
  f^*(x) &= \limsup f_n(x), \\
  f_*(x) &= \liminf f_n(x),
\end{align*}
\]

is measurable.

A function \(f\), defined on a measurable space \(X\), is called simple if there is a finite, disjoint class \(\{E_1, \ldots, E_n\}\) of measurable sets and a finite set \(\{\alpha_1, \ldots, \alpha_n\}\) of real numbers such that

\[
f(x) = \begin{cases} 
  \alpha_i & \text{if } x \in E_i, i = 1, \ldots, n, \\ 
  0 & \text{if } x \notin E_1 \cup \ldots \cup E_n.
\end{cases}
\]

**Theorem B.** Every extended real valued measurable function \(f\) is the limit of a sequence \((f_n)\) of simple functions; if \(f\) is non negative, then each \(f_n\) may be taken non negative and the sequence \((f_n)\) may be assumed increasing.

4.21 Pointwise convergence

If a certain proposition is true for almost every point (i.e. with the exception at most of a measurable set of measure zero) we say that it is true almost everywhere.

A function \(f\) is called essentially bounded if it is bounded a.e., i.e. if there exists a positive, finite constant \(c\) such that \(\{x : |f(x)| > c\}\) is a set of measure zero. The infimum of the values of \(c\) for which this statement is true is called the essential supremum of \(|f|\), abbreviated to ess. sup. \(|f|\).

The sequence \((f_n)\) converges to \(f\) uniformly a.e. if there exists a set \(E_0\) of measure zero, such that, for every \(\varepsilon > 0\), an integer \(n_0 = n_0(\varepsilon)\) can be found with the property that

\[
|f_n(x) - f(x)| < \varepsilon, \text{ if } n \geq n_0 \text{ and } x \notin E_0,
\]

in other words if the sequence of functions converges uniformly to \(f\) (in the ordinary sense of the phrase) on the set \(X - E_0\).

**Theorem A (Egoroff’s theorem).** If \(E\) is a measurable set of finite measure, and if \((f_n)\) is a sequence of a.e. finite valued measurable functions which converges a.e. on \(E\) to a finite valued measurable function \(f\), then, for every \(\varepsilon > 0\) there exists a measurable subset \(F\) of \(E\) such that \(\mu(F) < \varepsilon\) and such that the sequence \((f_n)\) converges to \(f\) uniformly on \(E - F\).

A sequence \((f_n)\) of a.e. finite valued measurable functions will be said to converge to the measurable function \(f\) almost uniformly if, for every \(\varepsilon > 0\), there exists a measurable set \(F\) such that \(\mu(F) < \varepsilon\) and such that the sequence \((f_n)\) converges to \(f\) uniformly on \(F'\). In this language Egoroff’s theorem asserts that on a set of finite measure convergence a.e. implies almost uniform convergence.
Theorem B. If \((f_n)\) is a sequence of measurable functions which converges to \(f\) almost uniformly, then \((f_n)\) converges to \(f\) a.e.

4.22 Convergence in measure

Theorem A. Suppose that \(f\) and \(f_n, n = 1, 2, \ldots\), are real valued functions on a set \(E\) of finite measure, and write, for every \(\varepsilon > 0\),

\[ E_n(\varepsilon) = \{ x : |f_n(x) - f(x)| \geq \varepsilon \}, \quad n = 1, 2, \ldots \]

The sequence \((f_n)\) converges to \(f\) a.e. if and only if

\[ \lim_{n \to \infty} \mu(E \cap \bigcup_{m=n}^{\infty} E_m(\varepsilon)) = 0 \]

for every \(\varepsilon > 0\).

A sequence \((f_n)\) of a.e. finite valued, measurable function converges in measure to the measurable function \(f\) if, for every \(\varepsilon > 0\),

\[ \lim_{n \to \infty} \mu(\{ x : |f_n(x) - f(x)| \geq \varepsilon \}) = 0 \]

A sequence \((f_n)\) of a.e. finite valued, measurable function is fundamental in measure if, for every \(\varepsilon > 0\),

\[ \mu(\{ x : |f_n(x) - f_m(x)| \}) \to 0 \quad \text{as} \quad n \text{ and } m \to \infty. \]

If a sequence of finite valued measurable functions converges a.e. to a finite limit [or is fundamental a.e.] on a set \(E\) of finite measure, then it converges in measure [or is fundamental in measure] on \(E\).

Theorem B. Almost uniform convergence implies convergence in measure.

Theorem C. If \((f_n)\) converges in measure to \(f\), then \((f_n)\) is fundamental in measure. If also \((f_n)\) converges in measure to \(g\), then \(f = g\) a.e.

Theorem D. If \((f_n)\) is a sequence of measurable functions which is fundamental in measure, then some subsequence \((f_{n_k})\) is almost uniformly fundamental.

Theorem E. If \((f_n)\) is a sequence of measurable functions which is fundamental in measure, then there exists a measurable function \(f\) such that \((f_n)\) converges in measure to \(f\).

5 Integration

5.23 Integrable simple functions

A simple function \(f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}\) on a measure space \((X, \mathcal{S}, \mu)\) is integrable if \(\mu(E_i) < \infty\) for every index \(i\) for which \(\alpha_i \neq 0\). The integral of \(f\), in symbols

\[ \int f(x) d\mu(x) \quad \text{or} \quad \int f d\mu \]

is defined by \(\int f d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i)\).

If \(E\) is a measurable set and \(f\) is an integrable simple function we define the integral of \(f\) over \(E\) by

\[ \int_{E} f d\mu = \int \chi_E f d\mu. \]
**Theorem A.** If $f$ and $g$ are integrable functions and $\alpha$ and $\beta$ are real numbers, then
\[ \int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu. \]

**Theorem B.** If an integrable function $f$ is non-negative a.e., then $\int f \, d\mu \geq 0$.

**Theorem C.** If $f$ and $g$ are integrable functions such that $f \geq g$ a.e., then
\[ \int f \, d\mu \geq \int g \, d\mu. \]

**Theorem D.** If $f$ and $g$ are integrable functions, then
\[ \int |f + g| \, d\mu \leq \int |f| \, d\mu + \int |g| \, d\mu. \]

**Theorem E.** If $f$ is an integrable function, then
\[ \left| \int f \, d\mu \right| \leq \int |f| \, d\mu. \]

**Theorem F.** If $f$ is an integrable function, $\alpha$ and $\beta$ are real numbers, and $E$ is a measurable set such that, for $x$ in $E$, $\alpha \leq f(x) \leq \beta$, then
\[ \alpha \mu(E) \leq \int_E f \, d\mu \leq \beta \mu(E). \]

The indefinite integral of an integrable function $f$ is the set function $\nu$, defined for every measurable set $E$ by $\nu(E) = \int_E f \, d\mu$.

**Theorem G.** If an integrable function $f$ is non-negative a.e., then its indefinite integral is monotone.

A finite valued set function of the class of all measurable sets of a measure space $(X, S, \mu)$ is absolutely continuous if for every positive number $\varepsilon$ there exists a positive number $\delta$ such that $|\nu(E)| < \varepsilon$ for every measurable set $E$ for which $\mu(E) < \delta$.

**Theorem H.** The indefinite integral of an integrable function is absolutely continuous.

**Theorem I.** The indefinite integral of an integrable function is countably additive.

If $f$ and $g$ are integrable functions, we define the distance, $\varrho(f, g)$, between them by the equation
\[ \varrho(f, g) = \int |f - g| \, d\mu. \]

5.24 **Sequences of integrable simple functions**

A sequence $(f_n)$ of integrable functions is fundamental in the mean, or mean fundamental, if
\[ \varrho(f_n, f_m) \to 0 \quad \text{as} \quad n \text{ and } m \to \infty. \]
Theorem A. A mean fundamental sequence \((f_n)\) of integrable functions is fundamental in measure.

Theorem B. If \((f_n)\) is a mean fundamental sequence of integrable functions, and if the indefinite integral of \(f_n\) is \(\nu_n, n = 1, 2, \ldots\), then

\[
\nu(E) = \lim_{n \to \infty} \nu_n(E)
\]

exists for every measurable set \(E\), and the set function \(\nu\) is finite valued and countably additive.

If \((\nu_n)\) is a sequence of finite valued set functions defined for all measurable sets, we say that the terms of the sequence are uniformly absolutely continuous whenever for every positive number \(\varepsilon\) there exists a positive number \(\delta\) such that \(|\nu_n(E)| < \varepsilon\) for every measurable set \(E\) for which \(\mu(E) < \delta\), and for every positive integer \(n\).

Theorem C. If \((f_n)\) is a mean fundamental sequence of integrable functions, and if the indefinite integral of \(f_n\) is \(\nu_n, n = 1, 2, \ldots\), then the set functions \(\nu_n\) are uniformly absolutely continuous.

Theorem D. If \((f_n)\) and \((g_n)\) are mean fundamental sequences of integrable simple functions which converge in measure to the same measurable function \(f\), if the indefinite integrals of \(f_n\) and \(g_n\) are \(\nu_n\) and \(\lambda_n\) respectively, and if, for every measurable set \(E\),

\[
\nu(E) = \lim_{n \to \infty} \nu_n(E) \quad \text{and} \quad \lambda(E) = \lim_{n \to \infty} \lambda_n(E),
\]

then the set function \(\nu\) and \(\lambda\) are identical.

5.25 Integrable functions

An a.e. finite valued, measurable function \(f\) on a measure space \((X, \mathcal{S}, \mu)\) is integrable if there exists a mean fundamental sequence \((f_n)\) of integrable simple functions which converges in measure to \(f\). The integral of \(f\), in symbols

\[
\int f(x) d\mu(x) \quad \text{or} \quad \int f d\mu
\]

is defined by \(\int f d\mu = \lim_{n \to \infty} f_n d\mu\).

We define the integral of \(f\) over \(E\) by

\[
\int_E f d\mu = \int \chi_E f d\mu.
\]

We shall say that a sequence \((f_n)\) of integrable functions converges in the mean, or mean converges, to an integrable function \(f\) if

\[
\nu(f_n, f) = \int |f_n - f| d\mu \to 0 \quad \text{as} \quad n \to \infty.
\]

Theorem A. If \((f_n)\) is a sequence of integrable functions which converges in the mean to \(f\), then \((f_n)\) converges to \(f\) in measure.
Theorem B. If $f$ is an a.e. non negative integrable function, then a necessary and sufficient condition that $\int f \, d\mu = 0$ is that $f = 0$ a.e.

Theorem C. If $f$ is an integrable function and $E$ is a set of measure zero, then

$$\int_E f \, d\mu = 0.$$  

Theorem D. If $f$ is an integrable function which is positive a.e. on a measurable set $E$, and if $\int_E f \, d\mu = 0$, then $\mu(E) = 0$.

Theorem E. If $f$ is an integrable function such that $\int_F f \, d\mu = 0$ for every measurable set $F$, then $f = 0$ a.e.

Theorem F. If $f$ is an integrable function, then the set $N(f) = \{x : f(x) \neq 0\}$ has $\sigma$-finite measure.

5.26 Sequences of integrable functions

Theorem A. If $(f_n)$ is a mean fundamental sequence of integrable simple functions which converges in measure to the integrable function $f$, then

$$\varrho(f, f_n) = \int |f - f_n| \, d\mu \to 0 \quad \text{as} \quad n \to \infty;$$

hence, to every integrable function $f$ and to every positive number $\varepsilon$, there corresponds an integrable simple function $g$ such that $\varrho(f, g) < \varepsilon$.

Theorem B. If $(f_n)$ is a mean fundamental sequence of integrable functions, then there exists an integrable function $f$ such that $\varrho(f_n, f) \to 0$ (and consequently $\int f_n \, d\mu \to \int f \, d\mu$) as $n \to \infty$.

If $(\nu_n)$ is a sequence of finite valued set functions on a class $E$, then the terms of the sequence are equicontinuous from above at 0 if, for every decreasing sequence $(E_n)$ of sets in $E$ for which $\lim E_n = 0$, and for every positive number $\varepsilon$, there exists a positive integer $m_0$ such that if $m \geq m_0$, then $|\nu_n(E_m)| \leq \varepsilon$, $n = 1, 2, \ldots$.

Theorem C. A sequence $(f_n)$ of integrable functions converges in the mean to the integrable function $f$ if and only if $(f_n)$ converges in measure to $f$ and the indefinite integrals of $|f_n|$, $n = 1, 2, \ldots$, are uniformly absolutely continuous and equicontinuous from above at 0.

Theorem D (Lebesgue’s bounded convergence theorem). If $(f_n)$ is a sequence of integrable functions which converges in measure to $f$ [or else converges to $f$ a.e.], and if $g$ is an integrable function such that $|f_n(x)| \leq |g(x)|$ a.e., $n = 1, 2, \ldots$, then $f$ is integrable and the sequence $(f_n)$ converges to $f$ in the mean.

5.27 Properties of integrals

Theorem A. If $f$ is measurable, $g$ is integrable, and $|f| \leq |g|$ a.e., then $f$ is integrable.
Theorem B. If \((f_n)\) is an increasing sequence of extended real valued nonnegative measurable functions and if \(\lim_{n \to \infty} f_n(x) = f(x)\) a.e., then \(\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu\).

Theorem C. A measurable function is integrable if and only if its absolute value is integrable.

Theorem D. If \(f\) is integrable and \(g\) is an essentially bounded measurable function, then \(fg\) is integrable.

Theorem E. If \(f\) is an essentially bounded measurable function and \(E\) is a measurable set of finite measure, then \(f\) is integrable over \(E\).

Theorem F (Fatou’s lemma). If \((f_n)\) is a sequence of non negative integrable functions for which
\[\lim \inf \int f_n \, d\mu < \infty,\]
then the function \(f\), defined by
\[f(x) = \lim \inf f_n(x),\]
is integrable and
\[\int f \, d\mu \leq \lim \inf \int f_n \, d\mu.\]

6 General set functions

6.28 Signed measures

We define a signed measure as an extended real valued, countably additive set function \(\mu\) on the class of all measurable sets of a measurable space \((X, S)\), such that \(\mu(0) = 0\), and such that \(\mu\) assumes at most one of the values \(+\infty\) and \(−\infty\).

The words “[totally] finite” and “[totally] \(\sigma\)-finite” will be used for signed measures just as for measures, except the \(\mu(E)\) has to be replaced by \(|\mu(E)|\), or, equivalently, \(\mu(E) < \infty\) has to be replaced by \(−\infty < \mu(E) < \infty\).

Theorem A. If \(E\) and \(F\) are measurable sets and \(\mu\) is a signed measure such that \(E \subset F\) and \(|\mu(F)| < \infty\), then \(|\mu(E)| < \infty\).

Theorem B. If \(\mu\) is a signed measure and \(\{E_n\}\) is a disjoint sequence of measurable sets such that \(|\mu(\bigcup_{n=1}^{\infty} E_n)| < \infty\), then the series \(\sum_{n=1}^{\infty} \mu(E_n)\) is absolutely convergent.

Theorem C. If \(\mu\) is a signed measure, if \(\{E_n\}\) is a monotone sequence of measurable sets, and if, in case \(\{E_n\}\) is a decreasing sequence, \(|\mu(E_n)| < \infty\) for at least one value of \(n\), then
\[\mu(\lim_n E_n) = \lim_n \mu(E_n).\]
6.29 Hahn and Jordan decomposition

If $\mu$ is a signed measure on the class of all measurable sets of a measurable space $(X, S)$, we shall call a set $E$ positive if, for every measurable set $F$, $E \cap F$ is measurable and $\mu(E \cap F) \geq 0$; similarly we call $E$ negative if, for every measurable set $F$, $E \cap F$ is measurable and $\mu(E \cap F) \leq 0$.

**Theorem A.** If $\mu$ is a signed measure, then there exist two disjoint sets $A$ and $B$ whose union is $X$, such that $A$ is positive and $B$ is negative with respect to $\mu$.

The sets $A$ and $B$ are said to form a Hahn decomposition of $X$ with respect to $\mu$. The equations

$$\mu^+(E) = \mu(E \cap A) \quad \text{and} \quad \mu^-(E) = -\mu(E \cap B)$$

unambiguously define two set functions $\mu^+$ and $\mu^-$ on the class of all measurable sets, called, respectively, the upper variation and the lower variation of $\mu$. The set $|\mu|$, defined for every measurable set $E$ by $|\mu|(E) = \mu^+(E) + \mu^-(E)$, is the total variation of $\mu$.

**Theorem B.** The upper, lower, and total variations of a signed measure $\mu$ are measures and $\mu(E) = \mu^+(E) - \mu^-(E)$ for every measurable set $E$. If $\mu$ is [totally] finite or $\sigma$-finite, then so also are $\mu^+$ and $\mu^-$; at least one of the measures $\mu^+$ and $\mu^-$ is always finite.

The representation of $\mu$ as the difference of its upper and lower variations is called the Jordan decomposition of $\mu$.

If $(X, S, \mu)$ is a measure space and $f$ is an integrable function on $X$, then the set function $\nu$, defined by $\nu(E) = \int_E f(x) d\mu(x)$, is a finite signed measure, and $\nu^+(E) = \int_E f^+ d\mu$, $\nu^-(E) = \int_E f^- d\mu$.

6.30 Absolute continuity

If $(X, S)$ is a measurable space and $\mu$ and $\nu$ are signed measures on $S$, we say that $\nu$ is absolutely continuous with respect to $\mu$, in symbols $\nu \ll \mu$, if $\nu(E) = 0$ for every measurable set $E$ for which $|\mu|(E) = 0$.

**Theorem A.** If $\mu$ and $\nu$ are signed measures, then the conditions

(a) $\nu \ll \mu$,

(b) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$,

(c) $|\nu| \ll |\mu|$,

are mutually equivalent.

**Theorem B.** If $\nu$ is a finite signed measure and if $\mu$ is a signed measure such that $\nu \ll \mu$, then, corresponding to every positive number $\varepsilon$, there is a positive number $\delta$ such that $|\nu|(E) < \varepsilon$ for every measurable set $E$ for which $|\mu|(E) < \delta$.

Two signed measures $\mu$ and $\nu$ for which both $\nu \ll \mu$ and $\mu \ll \nu$ are called equivalent, in symbols $\mu \equiv \nu$. 

17
If \((X, S)\) is a measurable space and \(\mu\) and \(\nu\) are signed measures on \(S\), we say that \(\mu\) and \(\nu\) are mutually singular, or more simply that \(\mu\) and \(\nu\) are singular, in symbols \(\mu \perp \nu\), if there exist two disjoint sets \(A\) and \(B\) whose union is \(X\) such that, for every measurable set \(E\), \(A \cap E\) and \(B \cap E\) are measurable and \(|\mu|(A \cap E) = |\nu|(B \cap E) = 0\).

If, for each point \(x\) of a measurable space \((X, S)\), \(\pi(x)\) is a proposition concerning \(x\), and \(\mu\) is a signed measure on \(S\), then the symbol \(\pi(x)[\mu]\) or \(\pi[\mu]\) shall mean that \(\pi(x)\) is true for almost every \(x\) with respect to the measure \(|\mu|\).

The symbol \([\mu]\) may be read as “modulo” \(\mu\).

### 6.31 The Radon-Nikodym theorem

**Theorem A.** If \(\mu\) and \(\nu\) are totally finite measures such that \(\nu \ll \mu\) and \(\nu\) is not identically zero, then there exists a positive number \(\varepsilon\) and a measurable set \(A\) such that \(\mu(A) > 0\) and such that \(A\) is a positive set for the signed measure \(\nu - \varepsilon \mu\).

**Theorem B (Radon-Nikodym theorem).** If \((X, S, \mu)\) is a totally \(\sigma\)-finite measure space and if a \(\sigma\)-finite measure \(\nu\) on \(S\) is absolutely continuous with respect to \(\mu\), then there exists a finite valued measurable function \(f\) on \(X\) such that

\[
\nu(E) = \int_E f \, d\mu
\]

for every measurable set \(E\). The function \(f\) is unique in the sense that if also \(\nu(E) = \int_E g \, d\mu\), \(E \in \mathcal{S}\), then \(f = g[\mu]\).

### 6.32 Derivatives of signed measures

If \(\mu\) is a totally \(\sigma\)-finite measure and if \(\nu(E) = \int_E f \, d\mu\) for every measurable set \(E\), we shall write

\[
f = \frac{d\nu}{d\mu} \quad \text{or} \quad d\nu = f \, d\mu.
\]

All the properties of Radon-Nikodym integrands (which we may also call Radon-Nikodym derivatives), which are suggested by the well known differential formalism, correspond to true theorems.

**Theorem A.** If \(\lambda\) and \(\mu\) are totally \(\sigma\)-finite measures such that \(\mu \ll \lambda\) and if \(\nu\) is a totally \(\sigma\)-finite signed measure such that \(\nu \ll \mu\), then

\[
\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}[\lambda].
\]

**Theorem B.** If \(\lambda\) and \(\mu\) are totally \(\sigma\)-finite measures such that \(\mu \ll \lambda\), and if \(f\) is a finite valued measurable function for which \(\int f \, d\mu\) is defined, then

\[
\int f \, d\mu = \int \frac{d\mu}{d\lambda} f \, d\lambda.
\]

**Theorem C.** If \((X, S)\) is a measurable space and \(\mu\) and \(\nu\) are totally \(\sigma\)-finite signed measures on \(S\), then there exist two uniquely determined totally \(\sigma\)-finite signed measures \(\nu_0\) and \(\nu_1\) whose sum is \(\nu\), such that \(\nu_0 \perp \mu\) and \(\nu_1 \ll \mu\).
Lebesgue decomposition of a totally σ-finite signed measure into an absolutely continuous part and a singular part with respect to another totally σ-finite measure.

7 Product spaces

7.33 Cartesian products

Cartesian product, rectangle=set $E = A \times B$, sides $A$ and $B$

Theorem A. A rectangle is empty if and only if one of its sides is empty.

Theorem B. If $E_1 = A_1 \times B_1$ and $E_2 = A_2 \times B_2$ are non empty rectangles, then $E_1 \subset E_2$ if and only if $A_1 \subset A_2$ and $B_1 \subset B_2$.

Theorem C. If $A_1 \times B_1 = A_2 \times B_2$ is a non empty rectangle, then $A_1 = A_2$ and $B_1 = B_2$.

Theorem D. If $E = A \times B$, $E_1 = A_1 \times B_1$, and $E_2 = A_2 \times B_2$ are non empty rectangles, then a necessary and sufficient condition that $E$ be the disjoint union of $E_1$ and $E_2$, is that either $A$ is the disjoint union of $A_1$ and $A_2$ and $B = B_1 = B_2$, or else $B$ is the disjoint union of $B_1$ and $B_2$, and $A = A_1 = A_2$.

Theorem E. If $S$ and $T$ are rings of subsets of $X$ and $Y$ respectively, then the class $R$ of all finite, disjoint unions of rectangles of the form $A \times B$, where $A \in S$ and $B \in T$, is a ring.

Suppose now that in addition to the two sets $X$ and $Y$ we are also given two σ-rings $S$ and $T$ of subsets of $X$ and $Y$ respectively. We shall denote by $S \times T$ the σ-ring of subsets of $X \times Y$ generated by the class of all sets of the form $A \times B$, where $A \in S$ and $B \in T$.

Theorem F. If $(X,S)$ and $(Y,T)$ are measurable spaces, then $(X \times Y,S \times T)$ is a measurable space.

The measurable space $(X \times Y,S \times T)$ is the Cartesian product of the two given measurable spaces.

measurable rectangle: A rectangle in the Cartesian product of two measurable spaces $(X,S)$ and $(Y,T)$ is measurable if it belongs to $S \times T$. Equivalently: $A \in S$ and $B \in T$.

7.34 Sections

$E_x = \{ y : (x,y) \in E \}= a$ section of $E$ determined by $x$

$X$-section, $Y$-section

If $f$ is any function defined on a subset $E$ of the product space $X \times Y$ and $x$ is any point of $X$, we shall call the function $f_x$, defined on the section $E_x$ by

$$f_x(y) = f(x,y),$$

a section of $f$, or more precisely an $X$-section of $f$ determined by $x$.

Theorem A. Every section of a measurable set is a measurable set.

Theorem B. Every section of a measurable function is a measurable function.
7.35 Product measures

Theorem A. If \((X, S, \mu)\) and \((Y, T, \nu)\) are \(\sigma\)-finite measure spaces, and if \(E\) is any measurable subset of \(X \times Y\), then the functions \(f\) and \(g\) defined on \(X\) and \(Y\) respectively by \(f(x) = \nu(E_x)\) and \(g(y) = \mu(E^y)\), are non-negative measurable functions such that \(\int f \, d\mu = \int g \, d\nu\).

Theorem B. If \((X, S, \mu)\) and \((Y, T, \nu)\) are \(\sigma\)-finite measure spaces, then the set function \(\lambda\), defined for every set \(E\) in \(S \times T\) by

\[
\lambda(E) = \int \nu(E_x) \, d\mu(x) = \int \mu(E^y) \, d\nu(y),
\]

is a \(\sigma\)-finite measure with the property that, for every measurable rectangle \(A \times B\),

\[
\lambda(A \times B) = \mu(A) \nu(B).
\]

The latter condition determines \(\lambda\) uniquely.

The measure \(\lambda\) is called the product of the given measures \(\mu\) and \(\nu\), in symbols \(\lambda = \mu \times \nu\); the measure space is the Cartesian product of the given measure spaces.

7.36 Fubini’s theorem

Throughout this section we shall assume that \((X, S, \mu)\) and \((Y, T, \nu)\) are \(\sigma\)-finite measure spaces and \(\lambda\) is the product measure \(\mu \times \nu\) on \(S \times T\).

If a function \(h\) on \(X \times Y\) is such that its integral is defined, then the integral is denoted by

\[
\int h(x, y) \, d\lambda(x, y)
\]

and is called the double integral of \(h\). If \(h_x\) is such that

\[
\int h_x(y) \, d\nu(y) = f(x)
\]

is defined, and if it happens that \(\int f \, d\mu\) is also defined, it is customary to write

\[
\int f \, d\mu = \iint h(x, y) \, d\mu(x) \, d\nu(y) = \int d\mu(x) \int h(x, y) \, d\nu(y).
\]

The symbols \(\iint h(f, y) \, d\mu(x) \, d\nu(y)\) and \(\int d\nu(y) \int h(x, y) \, d\mu(x)\) are defined similarly, as the integral (if it exists) of the function \(g\) on \(Y\), defined by \(g(y) = \int h^y(x) \, d\mu(x)\). The integrals \(\iint h \, d\mu \, d\nu\) and \(\iint h \, d\nu \, d\mu\) are called the iterated integrals of \(h\).

Theorem A. A necessary and sufficient condition that a measurable subset of \(X \times Y\) have measure zero is that almost every \(X\)-section [or almost every \(Y\)-section] have measure zero.

Theorem B. If \(h\) is a non-negative, measurable function on \(X \times Y\), then

\[
\int h \, d(\mu \times \nu) = \iint h \, d\mu \, d\nu = \iint h \, d\nu \, d\mu.
\]
Theorem C (Fubini’s theorem). If $h$ is an integrable function on $X \times Y$, then almost every section of $h$ is integrable. If the functions $f$ and $g$ are defined by $f(x) = \int h(x,y)\,d\nu(y)$ and $g(y) = \int h(x,y)\,d\mu(x)$, then $f$ and $g$ are integrable and

\[ \int h \, d(\mu \times \nu) = \int f \, d\mu = \int g \, d\nu. \]

7.37 Finite dimensional product spaces

We may prove the analogs of the theorems of §33 by mathematical induction on $n$.

If $(X_i, S_i), i = 1, 2, \ldots, n$, are measurable spaces, we shall denote by

\[ S_1 \times \cdots \times S_n \quad \text{or} \quad \bigotimes_{i=1}^n S_i \quad \text{or} \quad \bigotimes \{ S_i : i = 1, 2, \ldots, n \} \]

the $\sigma$-ring generated by the class of all those rectangles $\times_{i=1}^n A_i$ for which $A_i \in S_i$, $i = 1, \ldots, n$, and we define the Cartesian product of the given measurable spaces as $(X_1 \times \cdots \times X_n, S_1 \times \cdots \times S_n)$. Proceeding by mathematical induction, it is now trivial to define the Cartesian product of $\sigma$-finite measure spaces $(X_i, S_i, \mu_i), i = 1, 2, \ldots, n$; there is one and only one measure $\mu$ (denoted $\mu_1 \times \cdots \times \mu_n$) on $S_1 \times \cdots \times S_n$ such that

\[ \mu(A_1 \times \cdots \times A_n) = \prod_{i=1}^n \mu_i(A_i) \]

for every measurable rectangle $A_1 \times \cdots \times A_n$.

It is customary to refer to a product space $X = \times_{i=1}^n X_i$ as $n$-dimensional.

7.38 Infinite dimensional product spaces

Suppose that, for each $i = 1, 2, \ldots, X_i$ is a set, $S_i$ is a $\sigma$-algebra of subsets of $X_i$, and $\mu_i$ is a measure on $S_i$ such that $\mu_i(X_i) = 1$. In this case we define a rectangle as a set of the form $\times_{i=1}^\infty A_i$, where $A_i \subseteq X_i$ for all $i$ and $A_i = X_i$ for all but finite number of values of $i$. We define a measurable rectangle as a rectangle $\times_{i=1}^\infty A_i$ for which each $A_i$ is a measurable subset of $X_i$. A subset of $\times_{i=1}^\infty X_i$ will be called measurable if it belongs to the $\sigma$-ring $S$ (which is in fact a $\sigma$-algebra generated by the class of all measurable rectangles; we shall write $S = \times_{i=1}^\infty S_i$).

Suppose that $J$ is any subset of the set $I$ of all positive integers; we shall say that two points

\[ x = (x_1, x_2, \ldots) \quad \text{and} \quad y = (y_1, y_2, \ldots) \]

agree on $J$, in symbols $x \equiv y(J)$, if $x_j = y_j$ for every $j \in J$. A set $E$ in $X$ is called a $J$-cylinder if $x \equiv y(J)$ implies that $x$ and $y$ belong or do not belong to $E$ simultaneously.

\[ X^{(n)} = \times_{i=n+1}^\infty X_i, \quad n = 0, 1, 2, \ldots \]
**Theorem A.** If \( J = \{1, \ldots, n\} \) and if a subset \( E \) of \( X \) is a measurable \( J \)-cylinder, then \( E = A \times X^{(n)} \), where \( A \) is a measurable subset of \( X_1 \times \ldots \times X_n \).

**Theorem B.** If \( \{(X_i, S_i, \mu_i)\} \) is a sequence of totally finite measure spaces with \( \mu_i(X_i) = 1 \), then there exists a unique measure \( \mu \) on the \( \sigma \)-algebra \( S = \bigotimes_{i=1}^\infty S_i \) with the property that, for every measurable set \( E \) of the form \( A \times X^{(n)} \),

\[
\mu(E) = (\mu_1 \times \ldots \mu_n)(A).
\]

The measure \( \mu \) is called the product of the given measures \( \mu_i \), \( \mu = \bigotimes_{i=1}^\infty \mu_i \); the measure space \( \bigotimes_{i=1}^\infty X_i \times \bigotimes_{i=1}^\infty S_i \times \bigotimes_{i=1}^\infty \mu_i \) is the Cartesian product of the given measure spaces.

8 Transformations and functions

8.39 Measurable transformations

A transformation is a function \( T \) defined for every point of a set \( X \) and taking values in a set \( Y \).

- domain, range, into, onto, image, inverse image, one to one

**Theorem A.** If \( T \) is a transformation from \( X \) into \( Y \), if \( g \) is a function on \( Y \), and if \( M \) is any subset of the space in which the values of \( g \) lie, then

\[
\{x : (gT)(x) \in M\} = T^{-1}\{\{y : g(y) \in M\}\}.
\]

If \( (X, S) \) and \( (Y, T) \) are measurable space, we shall say that \( T \) is a measurable transformation if the inverse image of every measurable set is measurable.

**Theorem B.** If \( T \) is a measurable transformation from \( (X, S) \) into \( (Y, T) \), and if \( g \) is an extended real valued measurable function on \( Y \), then \( gT \) is measurable with respect to the \( \sigma \)-ring \( T^{-1}(T) \).

**Theorem C.** If \( T \) is a measurable transformation from a measure space \( (X, S, \mu) \) into a measurable space \( (Y, T) \), and if \( g \) is an extended real valued measurable function on \( Y \), then

\[
\int g(d(\muT^{-1})) = \int (gT)d\mu,
\]

in the sense that if either integral exists, then so does the other and the two are equal.

**Theorem D.** If \( T \) is a measurable transformation from a measure space \( (X, S, \mu) \) into a totally \( \sigma \)-finite measure space \( (Y, T, \nu) \), such that \( \muT^{-1} \) is absolutely continuous with respect to \( \nu \), then there exists a non negative measurable function \( \phi \) on \( Y \) such that

\[
\int g(T(x))d\mu(x) = \int g(y)\phi(y)d\nu(y)
\]

for every measurable function \( g \), in the sense that if either integral exists, then so does the other and the two are equal.
The function $\phi$ plays the similar role as the Jacobian (or the absolute value of the Jacobian) in the theory of transformations of multiple integrals.

If $T$ is a one to one transformation from a measurable space $(X, S)$ onto a measurable space $(Y, T)$, and if both $T$ and $T^{-1}$ are measurable, we shall say that $T$ is *measurability preserving*. A measurability preserving transformation $T$ from a measure space $(X, S, \mu)$ onto a measure space $(Y, T, \nu)$ is *measure preserving* if $\mu T^{-1} = \nu$.

### 8.40 Measure rings

A *Boolean ring* is a ring in the usual algebraic sense, with the property that every element is idempotent. Equivalently, a Boolean ring is a set $\mathbb{R}$ and two algebraic operations (called addition and multiplication) defined for pairs of elements of $\mathbb{R}$, subject to the following restrictions. (a) Both addition and multiplication are commutative and associative, and multiplication is distributive with respect to addition. (b) There exists in $\mathbb{R}$ a unique element (denoted by 0) such the result of adding 0 to any element $E$ is $E$. (c) The result of adding any element to itself is 0. (d) The result of multiplying any element $E$ by itself is $E$.

A typical example of a Boolean ring is a ring of subsets of a set $X$ with $E \triangle F$ and $E \cap F$ playing the roles of the sum and the product of $E$ and $f$, respectively. We shall always denote addition and multiplication in Boolean rings by $\triangle$ and $\cap$.

A *Boolean $\sigma$-ring* is a Boolean ring $S$ with the property that every countable set of elements in $S$ has a union.

A *Boolean algebra* is a Boolean ring $\mathbb{R}$ in which there exists an element different from 0 (which, for obvious reasons, we shall denote by $X$), with the property that $E \subset X$ for every $E$ in $\mathbb{R}$. A Boolean $\sigma$-algebra is a Boolean $\sigma$-ring which is a Boolean algebra.

The definitions of the concepts of additivity, measure, $\sigma$-finiteness, etc. for functions defined on a Boolean ring are the same as the corresponding definitions for set functions on a ring of sets. A measure $\mu$ on a Boolean ring is *positive* if it vanishes for the zero element only.

If $(X, S, \mu)$ is a measure space, we shall use the symbol $S(\mu)$ to denote the $\sigma$-ring $S$ with equality interpreted modulo $\mu$.

A *measure ring* $(S, \mu)$ is a Boolean $\sigma$-ring $S$ and a positive measure $\mu$ on $S$. If $(X, S, \mu)$ is a measure space, then $(S(\mu), \mu)$ is a measure ring; we shall call it the measure ring *associated* with $X$ or simply the measure ring of $X$. A measure algebra is a Boolean algebra which is at the same time a measure ring. The phrases *totally* finite and $\sigma$-finite are used for measure rings and measure algebras in the same way as for measure spaces.

An *isomorphism* between two measure rings $(S, \mu)$ and $(T, \nu)$ is a one to one transformation $T$ from $S$ onto $T$ such that

$$T(E - F) = T(E) - T(F), \quad T \left( \bigcup_{n=1}^{\infty} E_n \right) = \bigcup_{n=1}^{\infty} T(E_n),$$

and

$$\mu(E) = \nu(T(E)).$$
whenever $E, F,$ and $E_n$ are elements of $S$, $n = 1, 2, \ldots$ Two measure rings are isomorphic if there exists an isomorphism between them. Two measure spaces are isomorphic if their associated measure rings are isomorphic.

An atom of a measure ring $(S, \mu)$ or of a measure $\mu$ is an element $E$ different from 0 such that if $F \subset E$, then either $F = 0$ or $F = E$; a measure ring with no atoms is non atomic.

If $(S, \mu)$ is a measure ring, we shall denote by $S$ [or $S(\mu)$] the set of elements of finite measure in $S$ and, for any two elements $E$ and $F$ in $S$, we shall write

$$g(E, F) = \mu(E \triangle F).$$

The function $g$ is a metric for $S$; we shall call $S$ the metric space associated with $(S, \mu)$ or, simply, the metric space of $(S, \mu)$. We shall also use the symbol $S(\mu)$ for the metric space associated with the measure ring $(S(\mu), \mu)$ of a measure space $(X, S, \mu)$. A measure ring or a measure space is called separable if the associated metric space is separable.

**Theorem A.** If $S$ is the metric space of a measure ring $(S, \mu)$, and if

$$f(E, F) = E \cup F \quad \text{and} \quad g(E, F) = E \cap F,$$

then $f$, $g$, and also $\mu$, are all uniformly continuous functions of their arguments.

**Theorem B.** If $(X, S, \mu)$ is a $\sigma$-finite measure space such that the $\sigma$-ring $S$ has a countable set of generators, then the metric space $S(\mu)$ of measurable sets of finite measure is separable.

### 8.41 The isomorphism theorem

In what follows we restrict our attention to totally finite measure algebras. If $(S, \mu)$ is a totally finite measure algebra, then, if not specified, the symbol $X$ will denote the maximal element of $S$. The algebra $S$ and the measure $\mu$ are called normalized if $\mu(X) = 1$. A partition of an element $E$ of $S$ is a finite set $P$ of disjoint elements of $S$ whose union is $E$. The norm of a partition $P = \{E_1, \ldots, E_k\}$, denoted by $|P|$, is the maximum of the numbers $\mu(E_1), \ldots, \mu(E_k)$.

If $P_1$ and $P_2$ are partitions, we shall write $P_1 \leq P_2$ if each element of $P_1$ is contained in some element of $P_2$; a sequence $\{P_n\}$ is decreasing if $P_{n+1} \leq P_n$ for $n = 1, 2, \ldots$. A sequence $\{P_n\}$ of partitions is dense if to every element $E$ of $S$ and to every positive number $\varepsilon$ there corresponds a positive integer $n$ and an element $E_0$ of $S$ which is equal to a union of elements of $P_n$ and is such that $g(E, E_0) = \mu(E \triangle E_0) < \varepsilon$.

**Theorem A.** If $(S, \mu)$ is a totally finite, non atomic measure algebra, and if $\{P_n\}$ is a dense, decreasing sequence of partitions of $X$, then $\lim_n |P_n| = 0$.

**Theorem B.** If $Y$ is the unit interval, $T$ is the class of all Borel subsets of $Y$, and $\nu$ is the Lebesgue measure on $T$, and if $\{Q_n\}$ is a sequence of partitions into intervals of the maximal element $Y$ of the measure algebra $(T, \nu)$, such that $\lim_n |Q_n|$ is a sequence of partitions into intervals of the maximal element $Y$ of the measure algebra $(T, \nu)$, such that $\lim_n |Q_n| = 0$, then $\{Q_n\}$ is dense.

**Theorem C.** Every separable, non atomic, normalized measure algebra $(S, \mu)$ is isomorphic to the measure algebra $(T, \nu)$ of the unit interval.
8.42 Function spaces

$L_1, L_p$ for real number $p > 1$.

$$\|f\|_p = \left( \int |f|^p d\mu \right)^{1/p}$$

**Theorem A (Hölder’s inequality).** If $p$ and $q$ are real numbers greater than 1 such that $\frac{1}{p} + \frac{1}{q} = 1$, and if $f \in L_p$ and $g \in L_q$, then $fg \in L_1$ and $\|fg\| \leq \|f\|_p \cdot \|g\|_q$.

**Theorem B (Minkowski’s inequality).** If $p$ is a real number greater than 1, and if $f$ and $g$ are in $L_p$, then $f + g \in L_p$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Another useful space is the set $\mathcal{M}$ of all essentially bounded measurable function. If we write, for any $f \in \mathcal{M}$

$$\|f\|_\infty = \text{esssup}\{ |f(x)| : x \in X \},$$

then $\mathcal{M}$ becomes a Banach space.

8.43 Set functions and point functions

Throughout this section we shall assume that $X$ is the real line, $\mathcal{S}$ is the class of all Borel sets, and $\mu$ is Lebesgue measure on $\mathcal{S}$. Monotone, non-decreasing functions will be referred for brevity just as monotone functions.

If $f$ is a bounded monotone function, then the following limits exists

$$\lim_{x \to -\infty} f(x), \lim_{x \to +\infty} f(x)$$

will be denoted as $f(-\infty)$ and $f(+\infty)$ respectively.

**Theorem A.** If $\mu$ is a finite measure on $\mathcal{S}$ and if, for every real number $x$,

$$f_\nu(x) = \nu(\{t : -\infty < t < x\}),$$

then $f_\nu$ is a bounded monotone function, continuous on the left and such that $f_\nu(-\infty) = 0$.

**Theorem B.** If $f$ is a bounded monotone function, continuous on the left and such that $f(-\infty) = 0$, then there exists a unique finite measure $\nu$ on $\mathcal{S}$ such that $f = f_\nu$.

**Theorem C.** If $\nu$ is a finite measure on $\mathcal{S}$, then a necessary and sufficient condition that $f_\nu$ be continuous is that $\nu(\{x\}) = 0$ for every point $x$.

A real valued function $f$ of a real variable is called absolutely continuous if for every positive number $\varepsilon$ there corresponds a positive number $\delta$ such that

$$\sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon$$

for every finite disjoint class $\{(a_i, b_i) : i = 1, \ldots, n\}$ of bounded open intervals for which $\sum_{i=1}^n (b_i - a_i) < \delta$. 25
Theorem D. If \( \nu \) is a finite measure on \( S \), then a necessary and sufficient condition that \( f_\nu \) be absolutely continuous is that \( \nu \) be absolutely continuous with respect to \( \mu \).

We shall say that a finite measure \( \nu \) on \( S \) is purely atomic if there exists a countable set \( C \) such that \( \nu(X - C) = 0 \).

Theorem E. If \( \nu \) is a finite measure measure on \( S \), then there exist three uniquely determined measures \( \nu_1, \nu_2 \) and \( \nu_3 \) on \( S \) whose sum is \( \nu \) and which are such that \( \nu_1 \) is absolutely continuous to \( \mu \), \( \nu_2 \) is purely atomic, and \( \nu_3 \) is singular with respect to \( \mu \) but \( \nu_3(\{x\}) = 0 \) for every point \( x \).

9 Probability

9.44 Heuristic introduction

9.45 Independence

A probability space is a totally finite measure space \((X, S, \mu)\) for which \( \mu(X) = 1 \); the measure on a probability space is called a probability measure.

If \( E \) is a finite or infinite class of measurable sets in a probability space \((X, S, \mu)\), the sets of the class \( E \) are (stochastically) independent if

\[
\mu \left( \bigcap_{i=1}^{n} E_i \right) = \prod_{i=1}^{n} \mu(E_i)
\]

for every finite class \( \{E_i : i = 1, 2, \ldots, n\} \) of distinct sets in \( E \).

Pairwise independence, i.e. the independence of all pairs of distinct elements, does not imply (stochastic) independent. \text{(reformulate)}

If \( E \) is a finite or infinite set of real valued measurable functions on a probability space \((X, S, \mu)\), the functions of the set \( E \) are (stochastically) independent if

\[
\mu \left( \bigcap_{i=1}^{n} \{x : f_i(x) \in M_i\} \right) = \prod_{i=1}^{n} \mu(\{x : f_i(x) \in M_i\})
\]

for every finite subset \( \{f_i : i = 1, 2, \ldots, n\} \) of distinct functions in \( E \) and every finite class \( \{M_i : i = 1, 2, \ldots, n\} \) of Borel sets on the real line.

Theorem A. If \( f_1 \) and \( f_2 \) are independent functions, neither of which vanishes a.e., then a necessary and sufficient condition that both \( f_1 \) and \( f_2 \) are integrable is that their product \( f_1 f_2 \) be integrable; if this condition is satisfied, then

\[
\int f_1 f_2 d\mu = \int f_1 d\mu \int f_2 d\mu.
\]

Theorem B. If \( \{f_{ij} : i = 1, 2, \ldots, k; j = 1, 2, \ldots, n_i\} \) is a set of independent functions, if \( \phi_i \) is a real valued, Borel measurable function of \( n_i \) real variables, \( i = 1, 2, \ldots, k \), and if

\[
f_i(x) = \phi_i(f_{i1}(x), \ldots, f_{in_i}(x)),
\]

then the functions \( f_1, \ldots, f_k \) are independent.
Let $f^2$ be integrable function. If $\int f \, d\mu = \alpha$, then the variance of $f$, denoted by $\sigma^2(f)$, is defined by $\int (f - \alpha)^2 \, d\mu$.

$$\sigma^2(f) = \left(\int f^2 \, d\mu\right) - \left(\int f \, d\mu\right)^2, \quad \sigma^2(cf) = c^2\sigma^2(f)$$

**Theorem C.** If $f$ and $g$ are independent functions with a finite variance, then

$$\sigma^2(f + g) = \sigma^2(f) + \sigma^2(g).$$

If $F$ is a measurable set of positive measure in a probability space $(X, \mathcal{S}, \mu)$, and $F$ is a probability measure of $\mathcal{S}$ such that $\mu(F) = 1$. The sets $E$ and $F$ are independent if and only if $\mu(F(E)) = \mu(E)$. The number $\mu(F(E))$ is called the conditional probability of $E$ given $F$ (podmienená pravdepodobnosť $E$ vzťahom na $F$).

**Multiplicative theorem** for conditional probabilities:

If $\{E_i : i = 1, \ldots, n\}$ is a finite class of measurable sets of positive measure, then $\mu(E_1 \cap E_2 \cap \cdots \cap E_n) = \mu(E_1)\mu(E_2)\cdots\mu(E_1 \cap \cdots \cap E_{n-1}(E_n))$.

The Bayes’s theorem (1. Bayesova veta):

If $\{E_i : i = 1, \ldots, n\}$ is a finite, disjoint class of measurable sets of positive measure whose union is $X$ (i.e. $\{E_i\}$ is a partition of $X$), then, for every measurable set $F$, $\mu(F) = \sum_{i=1}^n \mu(E_i)\mu(E,F)$, and, if $F$ has positive measure,

$$\mu_F(E_i) = \frac{\mu(E_i)\mu(E,F)}{\sum_{i=1}^n \mu(E_i)\mu(E,F)}.$$

Let $X = \{x : 0 \leq x \leq 1\}$ be the unit interval with Lebesgue measure. For every positive integer $n$ define a function $f_n$ on $X$ by setting $f_n(x) = +1$ or $-1$ according as the integer $i$ for which $\frac{i-1}{2^{n-1}} \leq x < \frac{i}{2^n}$ is odd or even. The functions $f_n$ are called *Rademacher functions*. Any two of the functions $f_1$, $f_2$ and $f_1f_2$ are independent, but the three together are not.

**Coefficient of correlation:**

$$r(f, g) = \frac{\int fg \, d\mu - \int f \, d\mu \cdot \int g \, d\mu}{\sigma(f)\sigma(g)},$$

where $\sigma(f) = \sqrt{\sigma^2(f)}$ is standard deviation of $f$. The functions $f$ and $g$ are **uncorrelated** if $r(f, g) = 0$. If $f$, $g$ are independent functions, then $f$, $g$ are uncorrelated. The converse is not true in general. A necessary and sufficient condition that $\sigma^2(f + g) = \sigma^2(f) + \sigma^2(g)$ is that $f$, $g$ are uncorrelated.

### 9.46 Series of independent functions

Throughout this section we shall work with a fixed probability space $(X, \mathcal{S}, \mu)$.

**Theorem A (Kolmogoroff’s inequality).** If $f_i, i = 1, 2, \ldots, n$ are independent functions such that $\int f_i \, d\mu = 0$, and $\int f_i^2 \, d\mu < \infty$, $i = 1, 2, \ldots, n$, and if $f(x) = |\sum_{i=1}^k f_i(x)|$ (i.e. $f$ is the maximum of the absolute values of the partial sums of the $f_i$’s), then, for every positive number $\varepsilon$,

$$\mu(\{x : |f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon^2} \sum_{k=1}^n \sigma^2(f_k).$$
Theorem B. If \( (f_n) \) is a sequence of independent functions such that \( \int f_n \, d\mu = 0 \) and \( \sum_{n=1}^{\infty} \sigma^2(f_n) < +\infty \), then the series \( \sum_{n=1}^{\infty} f_n(x) \) converges a.e.

Theorem C. If \( (f_n) \) is a sequence of independent functions and \( c \) is a positive constant such that \( \int f_n \, d\mu = 0 \) and \( \sigma(f_n(x)) \leq c \) a.e., \( n = 1, 2, \ldots \), and if \( \sum_{n=1}^{\infty} f_n(x) \) converges on a set of positive measure, then \( \sum_{n=1}^{\infty} \sigma^2(f_n) < \infty \).

Theorem D. If \( (f_n) \) is a sequence of independent functions and \( c \) is a positive constant such that \( \sigma(f_n(x)) \leq c \) a.e., \( n = 1, 2, \ldots \), then \( \sum_{n=1}^{\infty} f_n(x) \) converges a.e. if and only if both the series \( \sum_{n=1}^{\infty} \int f_n \, d\mu \) and \( \sum_{n=1}^{\infty} \sigma^2(f_n) \) are convergent.

Theorem E (Three series theorem). If \( (f_n) \) is a sequence of independent functions and \( c \) is a positive constant, and if \( E_n = \{ x : |f_n(x)| \leq c \} \), \( n = 1, 2, \ldots \), then a necessary and sufficient condition for the convergence a.e. of \( \sum_{n=1}^{\infty} f_n(x) \) is the convergence of all the series

\[
\sum_{n=1}^{\infty} \mu(E_n'), \\
\sum_{n=1}^{\infty} \int_{E_n} f_n \, d\mu, \\
\sum_{n=1}^{\infty} \left( \int_{E_n} f_n^2 \, d\mu - \left( \int_{E_n} f_n \, d\mu \right)^2 \right).
\]

Tchebycheff’s inequality: If \( f \) is a measurable function with finite variance, then, for every positive number \( \varepsilon \),

\[
\mu(\{ x : |f(x) - \int f \, d\mu| \geq \varepsilon \}) \leq \frac{1}{\varepsilon^2} \sigma^2(f).
\]

Zero-one law: Suppose that the probability space \( X \) is the Cartesian product of a sequence \( (X_n) \) of probability spaces. If, for each positive integer \( n \), \( J_n = \{ n+1, n+2, \ldots \} \), and if a measurable set \( E \) in \( X \) is a \( J_n \)-cylinder for every \( n \), then \( \mu(E) = 0 \) or 1.

Borel-Cantelli lemma:

If \( (E_n) \) is a sequence of independent sets, then \( \mu(\limsup_n E_n) = 0 \) if and only if \( \sum_{n=1}^{\infty} \mu(E_n) < +\infty \).

9.47 The law of large numbers

Theorem A (Bernoulli’s theorem, Weak law of large numbers). If \( (f_n) \) is a sequence of independent functions with finite variances, such that \( \int f_n \, d\mu = 0 \), \( n = 1, 2, \ldots \), and \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \sigma^2(f_i) = 0 \), then the sequence \( \left( \frac{1}{n} \sum_{i=1}^{n} f_i \right) \) of averages converges to 0 in measure.

Two real valued measurable functions \( f \) and \( g \) on a probability space \( (X, S, \mu) \) have the same distribution if \( \mu(f^{-1}(M)) = \mu(g^{-1}(M)) \) for all real Borel sets \( M \).

Theorem B. If \( (y_n) \) is a sequence of real numbers which converges to a finite limit \( y \), then \( \lim_n \frac{1}{n} \sum_{i=1}^{n} y_i = y \).
Theorem C. If \((y_n)\) is a sequence of real numbers such that the series \(\sum_{n=1}^{\infty} \frac{1}{n} y_n\) is convergent, then \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} y_i = 0\).

Theorem D (Strong law of large numbers). If \((f_n)\) is a sequence of independent functions with finite variances, such that \(\int f_n \, d\mu = 0\), \(n = 1, 2, \ldots\), and \(\sum_{n=1}^{\infty} \sigma^2(f_n) < \infty\), then the sequence \(\left\{ \frac{1}{n} \sum_{i=1}^{n} f_i \right\}\) converges to 0 almost everywhere.

9.48 Conditional probabilities and expectations

Let \(T\) be a measurable transformation from the probability space \((X, S, \mu)\) into a measurable space \((Y, \mathcal{T})\). It follows from the Radon-Nikodym theorem that there exists an integrable function \(\rho_E\) on \(Y\) such that

\[
\mu(E \cap T^{-1}(F)) = \int_p \rho_E(y) d\mu_{T^{-1}}(y)
\]

for every \(F\) in \(\mathcal{T}\); the function \(\rho_E\) is uniquely determined modulo \(\mu_{T^{-1}}\). We shall call \(\rho_E(y)\) the conditional probability of \(E\) given \(y\) or the conditional probability of \(E\) given that \(T(x) = y\).

We shall generally write \(p(E, y)\) for \(\rho_E(y)\).

Theorem A. For each fixed measurable set \(E\) in \(X\),

\[
0 \leq p(E, y) \leq 1 [\mu T^{-1}] ;
\]

for each fixed disjoint sequence \((E_n)\) of measurable sets in \(X\),

\[
p \left( \bigcup_{n=1}^{\infty} E_n, y \right) = \sum_{n=1}^{\infty} p(E_n, y) [\mu T^{-1}] .
\]

If \(f\) is any integrable function \(X\), then we may consider its indefinite integral \(\nu\), defined by

\[
\nu(F) = \int_{T^{-1}(F)} f(x) d\mu(x),
\]

as a signed measure on \(\mathcal{T}\). Since clearly \(\nu \ll \mu T^{-1}\), it follows from the Radon-Nikodym theorem that there exists an integrable function \(\rho_f\) on \(Y\) such that

\[
\int_{T^{-1}(F)} f(x) d\mu(x) = \int_F \rho_f(y) d\mu_{T^{-1}}(y)
\]

for every \(F\) in \(\mathcal{T}\). The function \(\rho_f\) is uniquely determined modulo \(\mu T^{-1}\). We shall call \(\rho_f(y)\) the conditional expectation of \(f\) given \(y\); we shall also write \(e(f, y)\) instead of \(\rho_f(y)\).

Theorem B. If \(f\) is an integrable function on \(Y\), then \(fT\) is an integrable function on \(X\) and \(e(fT, y) = f(y) [\mu T^{-1}]\).
9.49 Measures on product spaces

Suppose that, for each positive integer \( n \), \( X_n \) is the closed interval and \( S_n \) is the class of all Borel sets in \( X_n \), and write \((X, S) = \times_{n=1}^\infty (X_n, S_n)\). Let \( F_n \) be the \( \sigma \)-ring of all measurable \( \{1, \ldots, n\} \)-cylinders in \( X \) and let \( F = \bigcup_{n=1}^\infty F_n \) be the ring of all measurable, finite dimensional subsets of \( X \).

**Theorem A.** If \( \mu \) is a set function on \( F \) such that, for each positive integer \( n \), \( \mu \) is a probability measure on \( F_n \), then \( \mu \) has a unique extension to a probability measure on \( S \).

**Theorem B.** For every measurable set \( E \) in \( X \),

\[
\lim_{n \to \infty} p(E, T_n(X)) = \chi_E(x)[\mu];
\]

in other words, the condition probabilities of \( E \), for given values of the first \( n \) coordinates of a point \( x \), converge (except perhaps on a set of \( x \)'s of measure zero) to 0 or 1 according as \( x \in E \) or \( x \notin E \).

10 Locally compact spaces

10.50 Topological lemmas

Throughout this chapter, unless we explicitly say otherwise, we shall assume that \( X \) is a locally compact Hausdorff spaces. We shall use the symbol \( F \) for the class of all real valued, continuous functions \( f \) on \( X \) such that \( 0 \leq f(x) \leq 1 \) for all \( x \) in \( X \).

**Theorem A.** If \( C \) is a compact set and \( U \) and \( V \) are open sets such that \( C \subset U \cup V \), then there exists compact sets \( D \) and \( E \) such that \( D \subset U \), \( E \subset V \), and \( C = D \cup E \).

**Theorem B.** If \( C \) is a compact set, \( F \) is a closed set, and \( C \cap F = 0 \), then there exist a function \( f \) in \( F \) such that \( f(x) = 0 \) for \( x \) in \( C \) and \( f(x) = 1 \) for \( x \) in \( F \).

**Theorem C.** If \( f \) is a real valued continuous function on \( X \) and \( c \) is a real number, then each of the three sets

\[
\{x : f(x) \geq c\}, \quad \{x : f(x) \leq c\}, \quad \text{and} \quad \{x : f(x) = c\}
\]

is a closed \( G_\delta \). If, conversely, \( C \) is a compact \( G_\delta \), then there exists a function \( f \) in \( F \) such that \( C = \{x : f(x) = 0\} \).

**Theorem D.** If \( C \) is compact, \( U \) is open, and \( C \subset U \), then there exist sets \( C_0 \) and \( U_0 \) such that \( C_0 \) is a compact \( G_\delta \), \( U_0 \) is a \( \sigma \)-compact open set and

\[
C \subset U_0 \subset C_0 \subset U.
\]

**Theorem E.** If \( X \) is separable, then every compact subset \( C \) of \( X \) is a \( G_\delta \).
10.51 Borel sets and Baire sets

We continue with our study of a fixed locally compact Hausdorff space \( X \).

We shall denote by \( C \) the class of all compact subsets of \( X \), by \( S \) the \( \sigma \)-ring generated by \( C \), and by \( U \) the class of all open sets belonging to \( S \). We shall call the sets of \( S \) the **Borel sets** of \( X \). A real valued function is **Borel measurable** (or simply a **Borel function**) if it is measurable with respect to the \( \sigma \)-ring \( S \).

**Theorem A.** Every Borel set is \( \sigma \)-bounded; every \( \sigma \)-bounded open set is a Borel set.

We shall denote by \( C_0 \) the class of all compact subsets of \( X \), by \( S_0 \) the \( \sigma \)-ring generated by \( C_0 \), and by \( U_0 \) the class of all open sets belonging to \( S_0 \). We shall call the sets of \( S_0 \) the **Baire sets** of \( X \). A real valued function is **Baire measurable** (or simply a **Baire function**) if it is measurable with respect to the \( \sigma \)-ring \( S_0 \).

**Theorem B.** If a real valued continuous function \( f \) on \( X \) is such that the set \( N(f) = \{ x : f(x) \neq 0 \} \) is \( \sigma \)-bounded, then \( f \) is Baire measurable.

**Theorem C.** If \( B \) is a subbase and if \( \hat{S} \) is a \( \sigma \)-ring containing \( B \), then \( \hat{S} \supset S_0 \).

**Theorem D.** Every compact Baire set is a **G\( \delta \)**.

**Theorem E.** If \( X \) and \( Y \) are locally compact Hausdorff spaces, and if \( A_0, B_0 \) and \( S_0 \) are the \( \sigma \)-rings of Baire sets in \( X \), \( Y \) and \( X \times Y \) respectively, then \( S_0 = A_0 \times B_0 \).

**Theorem F.** The class of all finite, disjoint unions of proper differences of sets of \( C \) [or of \( C_0 \)] is a ring; the \( \sigma \)-ring it generates coincides with \( S \) [or, respectively, with \( S_0 \)].

10.52 Regular measures

A **Borel measure** is a measure \( \mu \) defined on the class \( S \) of all Borel sets and such that \( \mu(C) < \infty \) for every \( C \in C \); a **Baire measure** is a measure \( \mu \) defined on the class \( S_0 \) of all Baire sets and such that \( \mu(C_0) < \infty \) for every \( C_0 \in C_0 \).

Throughout this section we shall use \( \hat{C}, \hat{U} \) and \( \hat{S} \) to stand either for \( C, U \) and \( S \) or else for \( C_0, U_0 \) and \( S_0 \), respectively, and we shall study a measure \( \hat{\mu} \) which is a Borel measure if \( \hat{S} = S \) and a Baire measure if \( \hat{S} = S_0 \).

A set \( E \) in \( \hat{S} \) is **outer regular** (with respect to the measure \( \hat{\mu} \)) if

\[
\hat{\mu}(E) = \inf \{ \hat{\mu}(U) : E \subset U \in \hat{U} \};
\]

a set \( E \) in \( \hat{S} \) is **inner regular** (with respect to the measure \( \hat{\mu} \)) if

\[
\hat{\mu}(E) = \sup \{ \hat{\mu}(C) : E \supset C \in \hat{C} \}.
\]

A set \( E \) in \( \hat{S} \) is **regular** if it is both inner regular and outer regular; a measure \( \hat{\mu} \) is regular if every set \( E \) in \( \hat{S} \) is regular.

**Theorem A.** If every set in \( \hat{C} \) is outer regular, then so is every proper difference of two sets of \( \hat{C} \); if every bounded set in \( \hat{U} \) is inner regular, then so is every proper difference of two sets of \( \hat{C} \).
**Theorem B.** A finite, disjoint union of inner regular sets of finite measure is inner regular.

**Theorem C.** The union of a sequence of outer regular sets is outer regular; the union of an increasing sequence of inner regular sets is inner regular.

**Theorem D.** The intersection of a sequence of inner regular sets of finite measure is inner regular; the intersection of a decreasing sequence of outer regular sets of finite measure is outer regular.

**Theorem E.** A necessary and sufficient condition that every set in \( \hat{C} \) be outer regular is that every bounded set in \( \hat{U} \) be inner regular.

**Theorem F.** Either the outer regularity of every set in \( \hat{C} \) or the inner regularity of every bounded set in \( \hat{U} \) is a necessary and sufficient condition for the regularity of the measure \( \hat{\mu} \).

**Theorem G.** Every Baire measure \( \nu \) is regular; if \( C \in C \), then
\[
\nu^*(C) = \inf \{ \nu(U_0) : C \subset U_0 \in U_0 \},
\]
and, if \( U \in U \), then
\[
\nu_* (U) = \sup \{ \nu(C_0) : U \supset C_0 \in C_0 \}.
\]

**Theorem H.** Let \( \mu \) be a Borel measure and let \( \nu \) be the Baire contraction of \( \mu \) (defined for every Baire set \( E \) by \( \nu(E) = \mu(E) \)). Either of the two conditions,
\[
\mu(C) = \nu^*(C) \quad \text{for all } C \ \text{in } C,
\]
\[
\mu(U) = \nu_*(U) \quad \text{for all } U \ \text{in } U,
\]
is necessary and sufficient for the regularity of \( \mu \). If two regular Borel measures agree on all Baire sets, then they agree on all Borel sets.

If \( \mu \) is any Borel measure, its Baire contraction \( \mu_0 \), defined for all \( E \) in \( S_0 \) by \( \mu_0(E) = \mu(E) \), is a Baire measure associated with \( \mu \) in a natural way. If it happens that every set in \( C \) or every bounded set in \( U \), and therefore in either case, every set in \( S \) is \( \mu_0^* \)-measurable (i.e. if all compact sets, and therefore all Borel sets, belong to the domain of definition of the completion of \( \mu_0 \)), then we shall say that the Borel measure \( \mu \) is completion regular.

**10.53 Generation of Borel measures**

We define a content as a non negative, finite, monotone, additive, and subadditive set function on the class \( C \) of all compact sets. In other words, a content is a set function \( \lambda \) on \( C \) which is such that

(a) \( 0 \leq \lambda(C) < \infty \) for all \( C \in C \),

(b) if \( C \) and \( D \) are compact sets for which \( C \subset D \), then \( \lambda(C) \leq \lambda(D) \)

(c) if \( C \) and \( D \) are disjoint compact sets, then \( \lambda(C \cup D) = \lambda(C) + \lambda(D) \), and

(d) if \( C \) and \( D \) are any two compact sets, then \( \lambda(C \cup D) \leq \lambda(C) + \lambda(D) \).
The inner content $\lambda_*$ induced by a content $\lambda$, is the set function defined for every $U \in \mathbf{U}$ by

$$\lambda_*(U) = \sup\{\lambda(C) : U \supset C \in \mathbf{C}\}.$$  

**Theorem A.** The inner content $\lambda_*$ induced by a content $\lambda$ vanishes at 0, and is monotone, countably subadditive, and countably additive.

If $\lambda$ is a content and $\lambda_*$ is the inner content induced by $\lambda$, we define a set function $\mu^*$ on the hereditary ring of all $\sigma$-bounded sets by

$$\mu^*(E) = \inf\{\lambda_*(U) : E \subset U \in \mathbf{U}\}.$$  

The set function $\mu^*$ is called the outer measure induced by $\lambda$.

**Theorem B.** The outer measure $\mu^*$ induced by a content $\lambda$ is an outer measure.

**Theorem C.** If $\lambda_*$ is the inner content and $\mu^*$ is the outer measure induced by a content $\lambda$, then $\mu^*(U) = \lambda_*(U)$ for every $U$ in $\mathbf{U}$ and $\mu^*(C^0) \leq \lambda(C) \leq \mu^*(C)$ for every $C$ in $\mathbf{C}$.

(We recall that $C^0$ denotes the interior of the set $C$.)

**Theorem D.** If $\mu^*$ is the outer measure induced by a content $\lambda$, then a $\sigma$-bounded set $E$ is $\mu^*$-measurable if and only if

$$\mu^*(U) \geq \mu^*(U \cap E) + \mu^*(U \cap E')$$

for every $U$ in $\mathbf{U}$.

**Theorem E.** If $\mu^*$ is the outer measure induced by a content $\lambda$, then the set function $\mu$, defined for every Borel set $E$ by $\mu(E) = \mu^*(E)$, is a regular Borel measure.

We shall call $\mu$ the Borel measure induced by the content $\lambda$.

**Theorem F.** Suppose that $T$ is a homeomorphism of $X$ onto itself and that $\lambda$ is a content. If, for every $C$ in $\mathbf{C}$, $\lambda(T(C)) = \lambda(C)$, and if $\mu$ and $\tilde{\lambda}$ are the Borel measures induced by $\lambda$ and $\lambda$ respectively, then $\tilde{\mu}(E) = \mu(T(E))$ for every Borel set $E$. If, in particular, $\lambda$ is invariant under $T$, then the same is true of $\mu$.

## 10.54 Regular contents

A content $\lambda$ is regular if, for every $C$ in $\mathbf{C}$,

$$\lambda(C) = \inf\{\lambda(D) : C \subset D^0 \subset D \in \mathbf{C}\}.$$  

**Theorem A.** If $\mu$ is the Borel measure induced by a regular content $\lambda$, then $\mu(C) = \lambda(C)$ for every $C$ in $\mathbf{C}$.

**Theorem B.** If $\mu$ is a regular Borel measure and if, for every $C$ in $\mathbf{C}$, $\lambda(C) = \mu(C)$, then $\lambda$ is a regular content and the Borel measure induced by $\lambda$ coincides with $\mu$.

**Theorem C.** If $\mu_0$ is a Baire measure and if, for every $C$ in $\mathbf{C}$,

$$\lambda(C) = \inf\{\mu_0(U_0) : C \subset U_0 \in \mathbf{U}_0\},$$

then $\lambda$ is a regular content.

**Theorem D.** If $\mu_0$ is a Baire measure, then there exists a unique, regular Borel measure $\mu$ such that $\mu(E) = \mu_0(E)$ for every Baire set $E$. 

33
10.55 Classes of continuous functions

If $X$ is, as usual, a locally compact Hausdorff space, we shall denote by $\mathcal{L}(X)$ or simply by $\mathcal{L}$ the class of all those real valued, continuous functions on $X$ which vanish outside a compact set.

If $X$ is not compact and if $X^*$ is the one-point compactification of $X$ by $x^*$, then the point $x^*$ is frequently called the point at infinity.

We shall denote by $\mathcal{L}_+(X)$ or simply by $\mathcal{L}_+$ the subclass of all nonnegative functions in $\mathcal{L}$.

**Theorem A.** If $C$ is any compact Baire set, then there exists a decreasing sequence $(f_n)$ of functions in $\mathcal{L}_+$ such that
\[
\lim_{n \to \infty} f_n(x) = \chi_C(x)
\]
for every $x$ in $X$.

**Theorem B.** If a Baire measure $\mu$ is such that the measure of every nonempty Baire open set is positive, and if $f \in \mathcal{L}_+$, then a necessary and sufficient condition that $\int f \, d\mu = 0$ is that $f(x) = 0$ for every $x$ in $X$.

**Theorem C.** If $\mu_0$ is a Baire measure and $\varepsilon > 0$, then, corresponding to every integrable simple Baire function $f$, there exists an integrable simple function $g$,
\[
g = \sum_{i=1}^{\infty} \alpha_i \chi_{C_i},
\]
such that $C_i$ is a compact Baire set, $i = 1, \ldots, n$, and
\[
\int |f - g| \, d\mu_0 \leq \varepsilon.
\]

**Theorem D.** If $\mu_0$ is a Baire measure, if $\varepsilon > 0$ and if $g = \sum_{i=1}^{\infty} \alpha_i \chi_{C_i}$ is a simple function such that $C_i$ is a compact Baire set, $i = 1, \ldots, n$, then there exists a function $h$ in $\mathcal{L}$ such that
\[
\int |g - h| \, d\mu_0 \leq \varepsilon.
\]

Lusin’s theorem: If $\mu$ is a regular Borel measure, $E$ is a Borel set of finite measure, and $f$ is a Borel measurable function on $E$, then, for every $\varepsilon > 0$, there exists a compact set $C$ in $E$ such that $\mu(E - C) \leq \varepsilon$ and such that $f$ is continuous on $C$.

10.56 Linear functionals

A linear functional on $\mathcal{L}$ is a real valued function $\Lambda$ of the functions in $\mathcal{L}$ such that
\[
\Lambda(\alpha f + \beta g) = \alpha \Lambda(f) + \beta \Lambda(g)
\]
for every $f, g \in \mathcal{L}$ and $\alpha, \beta \in \mathbb{R}$. A linear functional $\Lambda$ on $\mathcal{L}$ is positive if $\Lambda(f) \geq 0$ for every $f$ in $\mathcal{L}_+$.

If $E$ is any subset of $X$ and $f$ is any real valued function on $X$, then we shall write $E \subset f$ [or $E \supset f$] if $\chi_E(x) \leq f(x)$ [$\chi_E(x) \geq f(x)$] for every $x$ in $X$.

34
Theorem A. If \( \Lambda \) is a positive linear functional on \( \mathcal{L} \) and if, for every \( C \) in \( \mathcal{C} \),
\[
\lambda(C) = \inf \{ \lambda(f) : C \subset f \in \mathcal{L}_+ \},
\]
then \( \lambda \) is a regular content. If \( \mu \) is the Borel measure induced by \( \lambda \), then
\[
\mu(U) \leq \Lambda(f)
\]
for every bounded open set \( U \) and for every \( f \) in \( \mathcal{L}_+ \) for which \( U \subset f \).

Theorem B. If \( \Lambda \) is a positive linear functional on \( \mathcal{L} \), if, for every \( C \) in \( \mathcal{C} \),
\[
\lambda(C) = \inf \{ \Lambda(f) : C \subset f \in \mathcal{L}_+ \},
\]
and if \( \mu \) is the Borel measure induced by the content \( \lambda \), then
\[
\int f \, d\mu \leq \Lambda(f)
\]
for every \( f \) in \( \mathcal{L}_+ \).

Theorem C. If \( \Lambda \) is a positive linear functional on \( \mathcal{L} \), if, for every \( C \) in \( \mathcal{C} \),
\[
\lambda(C) = \inf \{ \Lambda(f) : C \subset f \in \mathcal{L}_+ \},
\]
and if \( \mu \) is the Borel measure induced by the content \( \lambda \), then, corresponding to every compact set \( C \) and every positive number \( \varepsilon \), there exists a function \( f_0 \) in \( \mathcal{L}_+ \) such that \( C \subset f_0 \), \( f_0 \leq 1 \), and
\[
\Lambda(f_0) \leq \int f_0 \, d\mu + \varepsilon.
\]

Theorem D. If \( \Lambda \) is a positive linear functional on \( \mathcal{L} \), then there exists a Borel measure \( \mu \) such that, for every \( f \) in \( \mathcal{L} \),
\[
\Lambda(f) = \int f \, d\mu.
\]

Theorem E. If \( \mu \) is a regular Borel measure, if, for every \( f \) in \( \mathcal{L} \), \( \Lambda(f) = \int f \, d\mu \), and if, for every \( C \) in \( \mathcal{C} \)
\[
\lambda(C) = \inf \{ \Lambda(f) : C \subset f \in \mathcal{L}_+ \},
\]
then \( \mu(C) = \lambda(C) \) for every \( C \) in \( \mathcal{C} \). Hence, in particular, the representation of a positive linear functional as an integral with respect to a regular Borel measure is unique.

11 Haar measure

11.57 Full subgroups

A subgroup \( Z \) of a topological group \( X \) is full if it has a non empty interior.

Theorem A. If \( Z \) is a full subgroup of a topological group \( X \), then every union of left cosets of \( Z \) is both open and closed in \( X \).

Theorem B. If \( E \) is any Borel set in a locally compact topological group \( X \), then there exists a \( \sigma \)-compact full subgroup \( Z \) of \( X \) such that \( E \subset Z \).
11.58 Existence

A Haar measure is a Borel measure $\mu$ in a locally compact topological group $X$, such that $\mu(U) > 0$ for every non empty Borel open set $U$, and $\mu(xE) = \mu(E)$ for every Borel set $E$.

If $E$ is any bounded set and $F$ is any set with a non empty interior, we define the “ratio” $E : F$ as the least non negative integer $n$ with the property that $E$ may be covered by $n$ left translations of $F$. It is easy to verify that (since $E$ is bounded and $F^0$ is non empty) $E : F$ is always finite, and that, if $A$ is a bounded set with a non empty interior, then

$$E : F \leq (E : A)(A : F).$$

**Theorem A.** For each fixed, non empty open set $U$ and compact set $A$ with a non empty interior, the set function $\lambda_U$, defined for all compact sets $C$ by

$$\lambda_U(C) = \frac{C : U}{A : U}$$

is non negative, finite, monotone, subadditive, and left invariant; it is additive in the restricted sense that if $C$ and $D$ are compact sets for which $CU^{-1} \cap DU^{-1} = 0$, then

$$\lambda_U(C \cup D) = \lambda_U(C) + \lambda_U(D).$$

**Theorem B.** In every locally compact topological group $X$ there exists at least one regular Haar measure.

11.59 Measurable groups

A measurable group is a $\sigma$-finite measure space $(X, S, \mu)$ such that

(a) $\mu$ is not identically zero,

(b) $X$ is a group,

(c) the $\sigma$-ring $S$ and the measure $\mu$ are invariant under left translations, and

(d) the transformation $S$ of $X \times X$ onto itself, defined by $S(x, y) = (x, xy)$, is measurability preserving.

(To say that $S$ is invariant under left translations means that $xE \in S$ for every $x$ in $X$ and $E$ in $S$; by a measurable subset of $X \times X$ we mean, as always, a set in the $\sigma$-ring $S \times S$.)

If $X$ is any measurable space (and hence, in particular, if $X$ is any measurable group), then the one to one transformation $R$ of $X \times X$ onto itself, defined by $R(x, y) = (y, x)$, is measurability preserving. We shall frequently use the transformation $T = R^{-1}SR$; we observe that $T(x, y) = (yx, y)$.

**Theorem A.** If $E$ is any subset of $X \times X$, then

$$(S(E))_x = xE_x \quad \text{and} \quad (T(E))^y = yE^y$$

for every $x, y \in X$.

**Theorem B.** The transformations $S$ and $T$ are measure preserving transformation of the measure space $(X \times X, S \times S, \mu \times \nu)$ onto itself.
Theorem C. If \( Q = S^{-1}RS \) then
\[
(Q(A \times B))_{x^{-1}} = xA \cap B^{-1},
\]
and
\[
(Q(A \times B))_{y^{-1}} = \begin{cases} 
y & \text{if } y \in B, \\
0 & \text{if } y \notin B.
\end{cases}
\]

Theorem D. If \( A \) is a measurable subset of \( X \) [of positive measure], and \( y \in X \), then \( Ay \) is a measurable set [of positive measure] and \( A^{-1} \) is a measurable set [of positive measure]. If \( f \) is a measurable function, \( A \) is a measurable set of positive measure, and, for every \( x \) in \( X \), \( g(x) = \frac{f(x^{-1})}{\mu(A^{-1})} \), then \( g \) is measurable.

Theorem E. If \( A \) and \( B \) are measurable sets of positive measure, then there exist measurable sets \( C_1 \) and \( C_2 \) of positive measure and elements \( x_1, y_1, x_2, \) and \( y_2 \), such that
\[
x_1C_1 \subset A, \ y_1C_1 \subset B, \ C_2x_2 \subset A, \ C_2y_2 \subset B.
\]

Theorem F (average theorem). If \( A \) and \( B \) are measurable sets and if \( f(x) = \mu(x^{-1}A \cap B) \), then \( f \) is a measurable function and
\[
\int f \, d\mu = \mu(A)\mu(B^{-1}).
\]
If \( g(x) = \mu(xA \triangle B) \), and if \( \varepsilon < \mu(A) + \mu(B) \), then the set \( \{ x : g(x) < \varepsilon \} \) is measurable.

11.60 Uniqueness

Theorem A. If \( \mu \) and \( \nu \) are two measures such that \((X, S, \mu)\) and \((X, S, \nu)\) are measurable groups, and if \( E \) in \( S \) is such that \( 0 < \nu(E) < \infty \), then, for every non negative measurable function \( f \) on \( X \),
\[
\int f(x) \, d\mu(x) = \mu(E) \int \frac{f(y^{-1})}{\nu(Ey)} \, d\nu(y).
\]

Theorem B. If \( \mu \) and \( \nu \) are two measures such that \((X, S, \mu)\) and \((X, S, \nu)\) are measurable groups, and if \( E \) in \( S \) is such that \( 0 < \nu(E) < \infty \), then, for every \( F \) in \( S \), \( \mu(E)\nu(F) = \nu(E)\mu(F) \).

Theorem C. If \( \mu \) and \( \nu \) are regular Haar measures on a locally compact topological group \( X \), then there exists a positive finite constant \( c \) such that \( \mu(E) = cv(E) \) for every Borel set \( E \).

12 Measure and topology in group

12.61 Topology in terms of measure

Throughout this section we shall assume that \( X \) is a locally compact topological group, \( \mu \) is a regular haar measure on \( X \), and \( g(E, F) = \mu(E \triangle F) \) for any two Borel sets \( E \) and \( F \).
Theorem A. If $E$ is a Borel set of finite measure and if $f(x) = \varrho(xE, E)$, for every $x$ in $X$, then $f$ is continuous.

Theorem B. If $U$ is any neighborhood of $e$, then there exist a Baire set $E$ of positive, finite measure and a positive number $\varepsilon$ such that $\{x : \varrho(xE, E) < \varepsilon\} \subset U$.

Theorem C. A necessary and sufficient condition that a set $A$ be bounded is that there exist a Baire set $E$ of positive, finite measure and a number $\varepsilon$, $0 \leq \varepsilon < 2\mu(E)$, such that $A \subset \{x : \varrho(xE, E) \leq \varepsilon\}$.

Density theorem for topological groups: If $E$ is any bounded Borel set, and if, for every $x \in X$ and every bounded neighborhood $U$ of $e$,

$$f_U(x) = \frac{\mu(E \cap Ux)}{\mu(Ux)},$$

then $f_U$ converges in the mean (and therefore in measure) to $\chi_E$ as $U \to e$. In other words, for every positive number $\varepsilon$ there exists a bounded neighborhood $V$ of $e$ such that if $U \subset V$, then $\int |f_U - \chi_E|d\mu < \varepsilon$.

12.62 Weil topology

Throughout this section we shall work with a fixed measurable group $(X, S, \mu)$; we shall write $\varrho(E, F) = \mu(E \triangle F)$ for any two measurable sets $E$ and $F$. We shall denote by $A$ the class of all sets of the form $EE^{-1}$, where $E$ is a measurable set of positive, finite measure, and by $N$ the class of all sets of the form $\{x : \varrho(xE, E) < \varepsilon\}$, where $E$ is a measurable set of positive, finite measure and $\varepsilon$ is a real number such that $0 < \varepsilon < 2\mu(E)$.

Theorem A. If $N = \{x : \varrho(xE, E) < \varepsilon\} \in N$, then every measurable set $F$ of positive measure contains a measurable subset $G$ of positive, finite measure such that $GG^{-1} \subset N$.

Theorem B. If $A = EE^{-1} \in A$ and $0 < \varepsilon < 2\mu(E)$, and if

$$N = \{x : \varrho(xE, E) < \varepsilon\},$$

then $N \in N$ and $N \subset A$.

Theorem C. If $N = \{x : \varrho(xE, E) < \varepsilon\} \in N$, then $N$ is a measurable set of positive measure. If $\mu(E^{-1}) < \infty$, then $\mu(N) < \infty$.

Theorem D. If $A$ and $B$ are any two sets of $A$, then there exists a set $C$ in $A$ such that $C \subset A \cap B$.

We shall say that a measurable group $X$ is separated if whenever an element $x$ of the group is different from $e$, then there exists a measurable set $E$ of positive, finite measure such that $\varrho(xE, E) > 0$.

Theorem E. If $X$ is separated, and if the class $N$ is taken for a base at $e$, then, with respect to the induced topology, $X$ is a topological group.
We shall refer to this topology of the measurable group as the Weil topology.

**Theorem F.** If $X$ is a separated, measurable group, then $X$ is locally bounded with respect to its Weil topology. If a measurable set $E$ has a non empty interior, then $\mu(E) > 0$; if a measurable set $E$ is bounded, then $\mu(E) < \infty$.

**Theorem G.** If $\mu$ is any Baire measure in a locally compact topological group $X$, and if $Y$ is the set of all those elements $y$ for which $\mu(\pi E) = \mu(\pi E)$ for all Baire sets $E$, then $Y$ is a closed subgroup of $X$.

By a thick subgroup of a measurable group we mean a subgroup which is a thick set.

**Theorem H.** If $(X, S, \mu)$ is a separated, measurable group, then there exists a locally compact topological group $\hat{X}$ with a Haar measure $\hat{\mu}$ on the class $\hat{S}$ of all Baire sets, such that $X$ is a thick subgroup of $\hat{X}$, $S \supset \hat{S} \cap X$, and $\mu(E) = \hat{\mu}(\pi E)$ whenever $\hat{E} \in \hat{S}$ and $E = \pi \hat{E} \cap X$.

### 12.63 Quotient groups

Throughout this section we shall assume that $X$ is a locally compact topological group and $\mu$ is a Haar measure in $X$; $Y$ is a compact invariant subgroup of $X$, $\nu$ is a Haar measure in $Y$ such that $\nu(Y) = 1$, and $\pi$ is the projection from $X$ onto the quotient group $\hat{X} = X/Y$.

**Theorem A.** If a compact set $C$ is a union of cosets of $Y$ and if $U$ is an open set containing $C$, then there exists an open set $\hat{V}$ in $\hat{X}$ such that $C \subset \pi^{-1}(\hat{V}) \subset U$.

**Theorem B.** If $\hat{C}$ is a compact subset of $\hat{X}$, then $\pi^{-1}(\hat{C})$ is a compact subset of $X$; if $\hat{E}$ is a Baire set [or a Borel set] in $\hat{X}$, then $\pi^{-1}(\hat{E})$ is a Baire set [or a Borel set] in $X$.

**Theorem C.** If $\hat{\mu} = \mu \pi^{-1}$, then $\hat{\mu}$ is a Haar measure in $\hat{X}$.

**Theorem D.** If $f \in L_+^+(X)$ and if

$$g(x) = \int_Y f(xy)d\nu(y),$$

then $g \in L_+^+(X)$ and there exists a (uniquely determined) function $\hat{g}$ in $L^+(\hat{X})$ such that $g = \hat{g} \pi$.

**Theorem E.** If $C$ is a compact Baire set in $X$ and if $g(x) = \nu(x^{-1}C \cap Y)$, then there exists a (uniquely determined) Baire measurable and integrable function $\hat{g}$ on $\hat{X}$ such that $g = \hat{g} \pi$. If $C$ is a union of cosets of $Y$, then $\int \hat{g}d\hat{\mu} = \mu(C)$.

**Theorem F.** If, for each Baire set $E$ in $X$,

$$g_E(x) = \nu(x^{-1}E \cap Y),$$

then there exists a (uniquely determined) Baire measurable function $\hat{g}_E$ on $\hat{X}$ such that $g_E = \hat{g}_E \pi$. 39
Theorem G. If, for each Baire set $E$ in $X$, $\hat{g}_E$ is the unique Baire measurable function on $\hat{X}$ for which
$$\hat{g}_E(\pi(x)) = \nu(x^{-1}E \cap Y) = g_E(x)$$
for every $x \in X$, then
$$\int \hat{g}_E d\hat{\mu} = \mu(E)$$
for every Baire set $E$.

12.64 The regularity of Haar measure
Throughout this section, up to the statement of the final, general result, we shall assume that $X$ is a locally compact and $\sigma$-compact topological group, and $\mu$ is a left invariant Baire measure in $X$ which is not identically zero (and which, therefore, is positive on all non empty open Baire sets).

By an invariant $\sigma$-ring we shall mean a $\sigma$-ring $T$ of Baire sets such that if $E \in T$ and $x \in X$, then $xE \in T$. Since the class of all Baire sets is an invariant $\sigma$-ring, and since the intersection of any collection of invariant $\sigma$-rings is itself an invariant $\sigma$-ring, we may define the invariant $\sigma$-ring generated by any class $E$ of Baire sets as the intersection of all invariant $\sigma$-rings containing $E$.

Theorem A. If $E$ is a class of Baire sets and $T$ is the invariant $\sigma$-ring generated by $E$, then $T$ coincides with the $\sigma$-ring $T_0$ generated by the class \{ $xE : x \in X, E \in E$ \}.

Theorem B. If $E$ is a countable class of Baire sets of finite measure and $T$ is the invariant $\sigma$-ring generated by $E$, then the metric space $J$ of all sets of finite measure in $T$ (with the metric defined by $\rho(E, F) = \mu(E \triangle F)$) is separable.

Theorem C. If $T$ is an invariant $\sigma$-ring, if $f$ is a function in $L$ which is measurable ($T$), and if $y$ in $X$ is such that $\rho(yE, E) = 0$ for every $E$ in $T$, then $f(y^{-1}x) = f(x)$ for every $x$ in $X$.

Theorem D. If $T$ is an invariant $\sigma$-ring generated by its sets of finite measure and containing at least one bounded set of positive measure, if $E$ is a class of sets dense in the metric space of sets of finite measure in $T$, and if
$$Y = \{ y : \rho(yE, E) = 0, E \in E \},$$
then $Y$ is a compact, invariant subgroup of $X$.

Theorem E. If $E$ is any Baire set in $X$, then there exists a compact, invariant Baire subgroup $Y$ of $X$ such that $E$ is a union of cosets of $Y$.

Theorem F. If $\{ e \}$ is a Baire set, then $X$ is separable.

Theorem G. If $E$ is any Baire set in $X$, then there exists a compact, invariant subgroup $Y$ of $X$ such that $E$ is a union of cosets of $Y$, and such that the quotient group $X/Y$ is separable.

Theorem H. Every Haar measure in $X$ is completion regular.

Theorem I. If $X$ is an arbitrary (not necessarily $\sigma$-compact) locally compact topological group, and if $\mu$ is a left invariant Borel measure in $X$, then $\mu$ is completion regular.
Index

σ-finite measure, 3
absolutely continuous, 17, 25
additive function, 3
algebra, 2
almost uniform convergence, 11
atom, 24
average theorem, 37
Baire function, 31
Baire measurable function, 31
Baire set, 31
Boolean σ-algebra, 23
Boolean σ-ring, 23
Boolean algebra, 2, 23
Boolean ring, 2, 23
Borel function, 31
Borel measurable function, 31
Borel measure, 31
Borel measure induced by a content, 33
Borel set, 31
Borel sets, 8
Borel-Cantelli lemma, 28
bounded set, 1
Cantor set, 9
Cartesian product, 19, 20, 22
complete measure, 3
completion, 1
completion of a measure, 7
completion regular measure, 32
conditional expectation, 29
conditional probability, 29
content, 32
continuous from above, 5
continuous from below, 5
corvergence in measure, 12
coset, 1
countably additive function, 3
countably subadditive set function, 5
dense sequence of partitions, 24
density theorem, 38
difference set, 9
distance, 13
double integral, 20
equicontinuous, 15
equivalent signed measures, 17
essential supremum, 11
essentially bounded, 11
extension, 7
Fatou’s lemma, 16
finite measure, 3
finitely additive function, 3
finitely subadditive set function, 5
Fubini’s theorem, 21
full subgroup, 35
fundamental in measure, 12
generated σ-ring, 2
generated hereditary σ-ring, 5
generated invariant σ-ring, 40
generated ring, 2
Haar measure, 36
Hahn decomposition, 17
Hausdorff measure, 7
hereditary, 5
indefinite integral, 13
independent functions, 26
independent sets, 26
induced measure, 6
inner content, 33
inner measure, 7
inner regular set, 31
integrable function, 14
integral, 14
integral of simple function, 12
invariant σ-ring, 40
isomorphism between measure rings, 23
iterated integrals, 20
J-cylinder, 21
Jordan decomposition, 17
Lebesgue decomposition, 19
Lebesgue measurable set, 8
Lebesgue measure, 8
Lebesgue’s bounded convergence theorem, 15
left translation, 1
linear functional, 34
locally bounded, 1
locally compact space, 1
lower variation, 17
Lusin’s theorem, 34

$\mu^*$-measurable set, 5
mean converges, 14
mean fundamental, 13
measurable cover, 6
measurable function, 10
measurable group, 36
measurable kernel, 8
measurable rectangle, 19, 21
measurable space, 9
measurable transformation, 22
measure, 3
purely atomic, 26
measure algebra, 23
normalized, 24
measure ring, 23
non atomic, 24
separable, 24
measure space, 10
separable, 24
metric outer measure, 6
Minkowski’s inequality, 25
monotone class, 3
monotone set function, 4

negative set, 17
norm of a partition, 24
outer measure, 5
outer measure induced by a content, 33
outer regular set, 31
partition, 24
point at infinity, 34
positive measure, 23
positive set, 17
probability measure, 26
probability space, 26
product measure, 20, 22
proper difference, 2
Radon-Nikodym derivatives, 18
Radon-Nikodym theorem, 18
rectangle, 21
regular content, 33
regular measure, 7, 31
regular set, 31
right translation, 1
ring, 2

$\sigma$-compact set, 1
$\sigma$-ring, 2
same distribution, 28
section, 19
semiring, 2
separated measurable group, 38
set function, 3
signed measure, 16
signed measures
mutually singular, 18
singular, 18
simple function, 11
subadditive set function, 5
subtractive set function, 4
superior limit, 1

Tchebycheff’s inequality, 28
thick, 10
thick subgroup, 39
three series theorem, 28
topological group, 1
total variation, 17
totally $\sigma$-finite measure, 3
totally finite, 3
transformation, 22
measurability preserving, 23
measure preserving, 23

uniform convergence a.e., 11
uniformly absolutely continuous, 14
upper variation, 17

variance, 27

Weil topology, 39

Zero-one law, 28
## Contents

1 Sets and classes  
1.3 Limits, complements, and differences ........................................ 1  
1.4 Ring and algebras ........................................................................ 2  
1.5 Generated rings and $\sigma$-rings .................................................. 2  
1.6 Monotone classes ......................................................................... 3  

2 Measures and outer measures  
2.7 Measure on rings ......................................................................... 3  
2.8 Measure on intervals ...................................................................... 4  
2.9 Properties of measures .................................................................. 4  
2.10 Outer measures ............................................................................ 5  
2.11 Measurable sets .......................................................................... 5  

3 Extension of measures  
3.12 Properties of induced measures ..................................................... 6  
3.13 Extension, completion and approximation ..................................... 7  
3.14 Inner measures ........................................................................... 7  
3.15 Lebesgue measure ......................................................................... 8  
3.16 Non measurable sets ..................................................................... 9  

4 Measurable functions  
4.17 Measure spaces ........................................................................... 9  
4.18 Measurable functions .................................................................... 10  
4.19 Combination of measurable functions .......................................... 10  
4.20 Sequences of measurable functions .............................................. 11  
4.21 Pointwise convergence ................................................................ 11  
4.22 Convergence in measure ............................................................... 12  

5 Integration  
5.23 Integrable simple functions .......................................................... 12  
5.24 Sequences of integrable simple functions ....................................... 13  
5.25 Integrable functions ...................................................................... 14  
5.26 Sequences of integrable functions ................................................ 15  
5.27 Properties of integrals .................................................................. 15  

6 General set functions  
6.28 Signed measures ......................................................................... 16  
6.29 Hahn and Jordan decomposition ................................................... 17  
6.30 Absolute continuity ...................................................................... 17  
6.31 The Radon-Nikodym theorem ....................................................... 18  
6.32 Derivatives of signed measures .................................................... 18  

7 Product spaces  
7.33 Cartesian products ....................................................................... 19  
7.34 Sections ....................................................................................... 19  
7.35 Product measures ......................................................................... 20  
7.36 Fubini’s theorem .......................................................................... 20  
7.37 Finite dimensional product spaces ................................................. 21  
7.38 Infinite dimensional product spaces ............................................... 21
8 Transformations and functions
  8.39 Measurable transformations . . . . . . . . . . . . . . . . . . . . 22
  8.40 Measure rings . . . . . . . . . . . . . . . . . . . . . . . . . . . . 23
  8.41 The isomorphism theorem . . . . . . . . . . . . . . . . . . . . . 24
  8.42 Function spaces . . . . . . . . . . . . . . . . . . . . . . . . . . 25
  8.43 Set functions and point functions . . . . . . . . . . . . . . . . 25

9 Probability
  9.44 Heuristic introduction . . . . . . . . . . . . . . . . . . . . . . . 26
  9.45 Independence . . . . . . . . . . . . . . . . . . . . . . . . . . . . 26
  9.46 Series of independent functions . . . . . . . . . . . . . . . . . . 27
  9.47 The law of large numbers . . . . . . . . . . . . . . . . . . . . . . 28
  9.48 Conditional probabilities and expectations . . . . . . . . . . . . 29
  9.49 Measures on product spaces . . . . . . . . . . . . . . . . . . . . 30

10 Locally compact spaces
  10.50 Topological lemmas . . . . . . . . . . . . . . . . . . . . . . . . 30
  10.51 Borel sets and Baire sets . . . . . . . . . . . . . . . . . . . . . 31
  10.52 Regular measures . . . . . . . . . . . . . . . . . . . . . . . . . 31
  10.53 Generation of Borel measures . . . . . . . . . . . . . . . . . . . 32
  10.54 Regular contents . . . . . . . . . . . . . . . . . . . . . . . . . . 33
  10.55 Classes of continuous functions . . . . . . . . . . . . . . . . . . 34
  10.56 Linear functionals . . . . . . . . . . . . . . . . . . . . . . . . . 34

11 Haar measure
  11.57 Full subgroups . . . . . . . . . . . . . . . . . . . . . . . . . . . . 35
  11.58 Existence . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 36
  11.59 Measurable groups . . . . . . . . . . . . . . . . . . . . . . . . . 36
  11.60 Uniqueness . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 37

12 Measure and topology in group
  12.61 Topology in terms of measure . . . . . . . . . . . . . . . . . . . 37
  12.62 Weil topology . . . . . . . . . . . . . . . . . . . . . . . . . . . . 38
  12.63 Quotient groups . . . . . . . . . . . . . . . . . . . . . . . . . . . 39
  12.64 The regularity of Haar measure . . . . . . . . . . . . . . . . . . 40