

Stone-Čech compactification of countable discrete space

Various notes on $\beta\omega$. (Basically the purpose of writing these notes was to get better acquainted with this material. I felt that it might help me in understanding and getting more insight if I go through some proofs again by myself. Writing them down is part of this.)

Literature: [HS], [T, Chapter II], [K], [B], [Z]

Hindman's theorem: [AT, Chapter B.III], [J, Theorem 29.1]

Van der Waerden Theorem: [J, Theorem 29.3]

1 Topology of $\beta\mathbb{N}$

Let $\beta\mathbb{N}$ be the set of all ultrafilters on \mathbb{N} . The set of free ultrafilters is denoted by $\beta\mathbb{N}^*$. The principal ultrafilter determined by n will be denoted by n^* .

For any $A \subseteq \mathbb{N}$ we denote ¹

$$\widehat{A} = \{\mathcal{F} \in \beta\mathbb{N}; A \in \mathcal{F}\}.$$

Notice that we have

$$\begin{aligned} \widehat{A \cap B} &= \widehat{A} \cap \widehat{B} \\ \widehat{A \cup B} &= \widehat{A} \cup \widehat{B} \\ \widehat{\mathbb{N} \setminus A} &= \beta\mathbb{N} \setminus \widehat{A} \end{aligned}$$

From the first property we see that the set $\{\widehat{A}; A \subseteq \mathbb{N}\}$ is a base for a topology on $\beta\mathbb{N}$.

1.1 Properties of $\beta\mathbb{N}$

The properties of this topology are summarized in the following results.

Theorem 1.1.1. *The topology on $\beta\mathbb{N}$ generated by $\{\widehat{A}; A \subseteq \mathbb{N}\}$ is compact and Hausdorff. This space is zero-dimensional.*

The map $e: \mathbb{N} \rightarrow \beta\mathbb{N}$ which maps n to n^ is a dense embedding if \mathbb{N} is endowed with the discrete topology. Moreover, for each sequence $x: \mathbb{N} \rightarrow K$ to a compact Hausdorff space K there is a unique continuous extension $\bar{x}: \beta\mathbb{N} \rightarrow K$ such that $\bar{x} \circ e = x$. (This extension can be expressed using \mathcal{F} -limit as $\bar{x}(\mathcal{F}) = \mathcal{F}\text{-lim } x$.)*

Proof can be found also e.g. in [HS, Chapter 3].

Proof. $\beta\mathbb{N}$ is Hausdorff. If $\mathcal{F} \neq \mathcal{G}$ are two ultrafilters on \mathbb{N} , then there is a set A such that $A \in \mathcal{F}$ and $A \notin \mathcal{G}$. Then \widehat{A} is a basic neighborhood of \mathcal{F} , and $\widehat{\mathbb{N} \setminus A}$ is a basic neighborhood of \mathcal{G} and these two neighborhoods are disjoint.

$\beta\mathbb{N}$ is compact. Suppose that there is an open cover $\{\widehat{A}_i; i \in I\}$ consisting of basic sets, which does not have a finite subcover. This means that for every finite set $F \subset I$ we have $\beta\mathbb{N} \setminus \bigcup_{i \in F} \widehat{A}_i \neq \emptyset$. Since $\beta\mathbb{N} \setminus \bigcup_{i \in F} \widehat{A}_i = \widehat{\mathbb{N} \setminus \bigcup_{i \in F} A_i}$, we see that $\mathbb{N} \setminus \bigcup_{i \in F} A_i = \bigcap_{i \in F} (\mathbb{N} \setminus A_i) \neq \emptyset$. Hence the system $\{\mathbb{N} \setminus A_i; i \in I\}$ has finite intersection property, and therefore there exists

¹Some authors use the notation A^* , e.g. [T]. I prefer to use different notation, since this might be confused with the notation for $\text{cl}_{\beta D}(A) \setminus A$, this subset of $\beta\mathbb{N}^*$ is commonly denoted by A^* , too.

an ultrafilter \mathcal{F} which contains this system. For this ultrafilter \mathcal{F} we have $\mathcal{F} \not\subseteq \bigcup_{i \in I} A_i$, which

contradicts the assumption that $\{\widehat{A}_i; i \in I\}$ is a cover of $\beta\mathbb{N}$.

$\beta\mathbb{N}$ is zero-dimensional. All basic set \widehat{A} are clopen, since $\widehat{\mathbb{N} \setminus A} = \beta\mathbb{N} \setminus \widehat{A}$.

The map e is an embedding. The map e is continuous, since \mathbb{N} has discrete topology. For each $n \in \mathbb{N}$ we have $e[\{n\}] = \{n^*\} = \widehat{\{n\}}$, therefore e is open.

$e[\mathbb{N}]$ is dense. If $A \neq \emptyset$, then there is $n \in A$ and we have $n^* \in \widehat{A}$.

Definition of \bar{x} . Let K be a compact Hausdorff space. Then for every ultrafilter and any sequence $x: \mathbb{N} \rightarrow K$ there exists a unique limit $\mathcal{F}\text{-lim } x_n$. A proof can be found e.g. in [D, Theorem 4.3.5], [HS, Theorem 3.48], [T, p.64, Claim 14.1], [F, 2A3Se(i)]².

Continuity of \mathcal{F} -lim. Let $x: \mathbb{N} \rightarrow K$, where K is compact. For any \mathcal{F} we denote $\bar{x}(\mathcal{F}) = \mathcal{F}\text{-lim } x$. Then the map \bar{x} is continuous. Let $\bar{x}(\mathcal{F}) = L$. Choose a neighborhood V of L such that $\overline{V} \subseteq U$. Then the set $A = x^{-1}(V)$ belongs to \mathcal{F} . We show that $\bar{x}(\mathcal{G}) \in U$ for any $\mathcal{G} \in \widehat{A}$. To see this, it suffices to notice that $\mathcal{G}\text{-lim } x \in \overline{V}$ for each such ultrafilter \mathcal{G} . Indeed, if $\mathcal{G}\text{-lim } x = L'$ then for any neighborhood U' of L' we have $x^{-1}(U') \in \mathcal{G}$. Since $x^{-1}(V) \in \mathcal{G}$, we get that $x^{-1}(U' \cap V) = x^{-1}(U') \cap x^{-1}(V) \in \mathcal{G}$. This implies that $U' \cap V \neq \emptyset$. Since every neighborhood of L' intersects V , we see that $L' \in \overline{V}$.

Uniqueness of \bar{x} follows from the fact that $e[\mathbb{N}]$ is dense in $\beta\mathbb{N}$. □

Note that the above construction can be considered a special case of Wallman extension, see [E, p.176–178], [N, Theorem IV.3], [W, 19K].

It is also interesting to notice that in the above proof we have in fact shown that if we fix a sequence $x: \mathbb{N} \rightarrow K$ then the map $\bar{x}: \beta\mathbb{N} \rightarrow K$ defined by $\bar{x}(\mathcal{F}) = \mathcal{F}\text{-lim } x$ is continuous. In the other words, \mathcal{F} -lim is continuous in \mathcal{F} . It is relatively easy to see that the topology of $\beta\mathbb{N}$ is precisely the coarsest topology which makes each $\bar{x}: \beta\mathbb{N} \rightarrow [0, 1]$ continuous; i.e., it is the initial topology w.r.t. these maps. It suffices to notice that $\overline{\chi_A}(\mathcal{F}) = \mathcal{F}\text{-lim } \chi_A = 1 \Leftrightarrow A \in \mathcal{F}$. (And, dually, $\overline{\chi_A}(\mathcal{F}) = \mathcal{F}\text{-lim } \chi_A = 0 \Leftrightarrow A \notin \mathcal{F}$; this second property is only true for ultrafilters, not for arbitrary filters.)

2 Algebra in $\beta\mathbb{N}$

Although bellow we will work with $(\mathbb{N}, +)$, it is worth noticing that the same can be done with arbitrary semigroup (endowed with discrete topology).

2.1 Definition of operations

For any $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we define

$$A - n = \{k \in \mathbb{N}; k + n \in A\}.$$

For ultrafilters \mathcal{U} and \mathcal{V} on \mathbb{N} we define

$$\mathcal{U} + \mathcal{V} = \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; A - n \in \mathcal{U}\} \in \mathcal{V}\}.$$

Lemma 2.1.1. *$\mathcal{U} + \mathcal{V}$ is an ultrafilter.*

Similar proof is given in [T, Section 15, Lemma 1].

²or <http://thales.doa.fmph.uniba.sk/sleziak/texty/rozne/trf/iconv/notions.pdf>

Proof. $\emptyset \notin \mathcal{U} + \mathcal{V}$

If $A, B \in \mathcal{U} + \mathcal{V}$ and $C := A \cap B$, then $C - n = (A - n) \cap (B - n)$. This implies that $\{n \in \mathbb{N}; A - n \in \mathcal{U}\} \cap \{n \in \mathbb{N}; B - n \in \mathcal{U}\} \subseteq \{n \in \mathbb{N}; C - n \in \mathcal{U}\}$ and consequently $C \in \mathcal{U} + \mathcal{V}$.

If $A \in \mathcal{U} + \mathcal{V}$ and $A \subseteq B$, then $B \in \mathcal{U} + \mathcal{V}$. This follows from $A \subseteq B \Rightarrow A - n \subseteq B - n$.

Now let A be an arbitrary subset of \mathbb{N} , we want to show that either A or $\mathbb{N} \setminus A$ belongs to $\mathcal{U} + \mathcal{V}$.

Suppose that $A \notin \mathcal{U} + \mathcal{V}$. This means that

$$\begin{aligned} \{n \in \mathbb{N}; A - n \in \mathcal{U}\} &\notin \mathcal{V} \Leftrightarrow \\ \{n \in \mathbb{N}; A - n \notin \mathcal{U}\} &\in \mathcal{V} \Leftrightarrow \\ \{n \in \mathbb{N}; \mathbb{N} \setminus (A - n) \in \mathcal{U}\} &\in \mathcal{V} \Leftrightarrow \\ \{n \in \mathbb{N}; (\mathbb{N} \setminus A) - n \in \mathcal{U}\} &\in \mathcal{V}. \end{aligned}$$

The last line means that $\mathbb{N} \setminus A \in \mathcal{U} + \mathcal{V}$.

We have used $\mathbb{N} \setminus (A - n) = (\mathbb{N} \setminus A) - n$. □

Note that in the first part of the proof we only used the fact that \mathcal{U} and \mathcal{V} are filters. So we can get a filter $\mathcal{F} + \mathcal{G}$ for any two filters \mathcal{F}, \mathcal{G} .

It is also useful to notice that the above proof works for any semigroup (S, \cdot) instead of $(\mathbb{N}, +)$ without any changes.

2.1.1 Principal ultrafilters

Let us have a look at addition of ultrafilters when one of the ultrafilters is principal.

First let us try to express $a^* + \mathcal{V}$. Directly from the definition we get

$$\begin{aligned} a^* + \mathcal{V} &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; A - n \in a^*\} \in \mathcal{V}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; a \in A - n\} \in \mathcal{V}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; a + n \in A\} \in \mathcal{V}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; n + a \in A\} \in \mathcal{V}\} = \\ &= \{A \subseteq \mathbb{N}; A - a \in \mathcal{V}\}. \end{aligned}$$

After a little algebraic manipulation we can see from the above expression that $a^* + \mathcal{V}$ is precisely the filter generated by $a + \mathcal{V} = \{a + V; V \in \mathcal{V}\}$.

Similarly we get

$$\begin{aligned} \mathcal{U} + a^* &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; A - n \in \mathcal{U}\} \in a^*\} = \\ &= \{A \subseteq \mathbb{N}; a \in \{n \in \mathbb{N}; A - n \in \mathcal{U}\}\} = \\ &= \{A \subseteq \mathbb{N}; A - a \in \mathcal{U}\}. \end{aligned}$$

We have obtained precisely the same expression as above which shows that

$$a^* + \mathcal{U} = \mathcal{U} + a^*, \tag{1} \quad \{\text{EQPRINCOM}\}$$

i.e. principal ultrafilters commute with all ultrafilters.

If we add two principal filters, we get

$$\begin{aligned} a^* + b^* &= \{A \subseteq \mathbb{N}; A - b \in a^*\} = \\ &= \{A \subseteq \mathbb{N}; a \in A - b\} = \\ &= \{A \subseteq \mathbb{N}; a + b \in A\} = (a + b)^* \end{aligned}$$

i.e.

$$a^* + b^* = (a + b)^*.$$

This means that the addition of ultrafilters extends the usual addition on \mathbb{N} .

2.2 Associativity

Lemma 2.2.1. *For any ultrafilters $\mathcal{U}, \mathcal{V}, \mathcal{W}$ the equality*

$$(\mathcal{U} + \mathcal{V}) + \mathcal{W} = \mathcal{U} + (\mathcal{V} + \mathcal{W})$$

holds.

Proof is given in [T, Section 15, Lemma 1].

Proof.

$$\begin{aligned} (\mathcal{U} + \mathcal{V}) + \mathcal{W} &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; A - n \in \mathcal{U} + \mathcal{V}\} \in \mathcal{W}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; \{k \in \mathbb{N}; (A - n) - k \in \mathcal{U}\} \in \mathcal{V}\} \in \mathcal{W}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; \{k \in \mathbb{N}; A - (k + n) \in \mathcal{U}\} \in \mathcal{V}\} \in \mathcal{W}\} \end{aligned}$$

We have used above the fact that $(A - n) - k = A - (k + n)$.

$$\begin{aligned} \mathcal{U} + (\mathcal{V} + \mathcal{W}) &= \{A \subseteq \mathbb{N}; \{k \in \mathbb{N}; A - k \in \mathcal{U}\} \in \mathcal{V} + \mathcal{W}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; \{k \in \mathbb{N}; A - k \in \mathcal{U}\} - n \in \mathcal{V}\} \in \mathcal{W}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; \{k \in \mathbb{N}; A - (k + n) \in \mathcal{U}\} \in \mathcal{V}\} \in \mathcal{W}\} \end{aligned}$$

We have used above the fact that $\{k \in \mathbb{N}; A - k \in \mathcal{U}\} - n = \{k \in \mathbb{N}; A - (k + n) \in \mathcal{U}\}$. \square

Note that the above proof did not use commutativity of $+$ anywhere. (So the same proof works for arbitrary semigroup instead of $(\mathbb{N}, +)$.)

2.3 Non-commutativity

It can be shown that the operation $+$ defined on $\beta\mathbb{N}$ is not commutative. See [HS, Theorem 4.27]. In fact, the center of $(\beta\mathbb{N}, +)$ is $Z(\beta\mathbb{N}) = \mathbb{N}$. See [HS, Theorem 6.10].³

2.4 Continuity

Lemma 2.4.1. *For every $\mathcal{U} \in \beta\mathbb{N}$ the map $L_{\mathcal{U}}: \beta\mathbb{N} \rightarrow \beta\mathbb{N}$ given by*

$$L_{\mathcal{U}}: \mathcal{V} \mapsto \mathcal{U} + \mathcal{V}$$

is continuous.

Proof. The topology of $\beta\mathbb{N}$ is generated by basic sets $\widehat{A} = \{\mathcal{V}; A \in \mathcal{V}\}$. It suffices to show that preimage of every basic set is open.

$$\begin{aligned} L_{\mathcal{U}}^{-1}(\widehat{A}) &= \{\mathcal{V}; A \in \mathcal{U} + \mathcal{V}\} \\ &= \{\mathcal{V}; \{n \in \mathbb{N}; A - n \in \mathcal{U}\} \in \mathcal{V}\} \\ &= \{n \in \mathbb{N}; \widehat{A - n} \in \mathcal{U}\} \end{aligned}$$

\square

[B, Theorem 3.2]

³See <http://math.stackexchange.com/questions/87109/addition-on-ultrafilters-is-non-commutative>

2.5 Relation of addition to tensor product

If \mathcal{U}, \mathcal{V} are ultrafilters on \mathbb{N} , then we define

$$\mathcal{U} \otimes \mathcal{V} = \{A \subseteq \mathbb{N} \times \mathbb{N}; \{n; \{k; (k, n) \in A\} \in \mathcal{V}\} \in \mathcal{U}\},$$

i.e. a set $A \subseteq \mathbb{N} \times \mathbb{N}$ belongs to $\mathcal{U} \otimes \mathcal{V}$ if and only if

$$\{n; \{k; (k, n) \in A\} \in \mathcal{V}\} \in \mathcal{U}.$$

It can be shown that $\mathcal{U} \otimes \mathcal{V}$ is an ultrafilter, and it is called *Fubini product* or *tensor product* of \mathcal{U} and \mathcal{V} .

If we think about \mathcal{F} -limits, then tensor product is a very natural operation, since $\mathcal{U} \otimes \mathcal{V}$ is the ultrafilter on $\mathbb{N} \times \mathbb{N}$ such that

$$\mathcal{U}\text{-}\lim_n \mathcal{V}\text{-}\lim_k x_{(k,n)} = \mathcal{U} \otimes \mathcal{V}\text{-}\lim x_{(k,n)}.$$

See [HS, Lemma 11.3], [J, Exercise 29.2] or [K, Remark 1.7].

If we have a map $f: X \rightarrow Y$ and an ultrafilter on Y , we can transfer this ultrafilter in a natural way to X :

Lemma 2.5.1. *Let $f: X \rightarrow Y$ be a map, \mathcal{F} be a filter on Y . Then*

$$\mathcal{G} = \{B \subseteq Y; f^{-1}(B) \in \mathcal{F}\}$$

is a filter on Y ,

$$\mathcal{B} = \{f[A]; A \in \mathcal{F}\}$$

is a filterbase on Y and it generates \mathcal{G} .

Moreover, if \mathcal{F} is an ultrafilter then \mathcal{G} is an ultrafilter.

In this way, if we use the filter $\mathcal{V} \otimes \mathcal{U}$ on $\mathbb{N} \times \mathbb{N}$ and the map $+: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, we get precisely the filter $\mathcal{U} + \mathcal{V}$ since

$$\begin{aligned} A \in \mathcal{U} + \mathcal{V} &\Leftrightarrow \{n \in \mathbb{N}; A - n \in \mathcal{U}\} \in \mathcal{V} \\ &\Leftrightarrow \{n \in \mathbb{N}; \{k \in \mathbb{N}; k + n \in A\} \in \mathcal{U}\} \in \mathcal{V}. \end{aligned}$$

3 Idempotent ultrafilters and their applications

3.1 Existence of idempotent ultrafilters

Theorem 3.1.1 (Glazer). *There exists $\mathcal{U} \in \beta\mathbb{N}$ such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$.*

Glazer's theorem follows from the following more general result. By saying that $(X, +)$ is left-topological semigroup we mean that left multiplication is continuous for every $a \in X$; i.e. for each $a \in X$ the map $x \mapsto a + x$ is continuous.

The following theorem can be found in [T, Section II.15], [B, Theorem 3.3], [Z, Theorem 6.12].

Theorem 3.1.2 (Auslander-Ellis-Numakura). *Every left-topological semigroup $(X, +)$ on a compact Hausdorff space X has an idempotent, i.e., an element a such that $a + a = a$.*

The following proof is taken from [T, Section II.15, Lemma 3].

Proof. Denote $\mathcal{Z} = \{Z \subseteq X; Z \neq \emptyset; Z \text{ is closed and } Z + Z \subseteq Z\}$. We first check that (\mathcal{Z}, \supseteq) fulfills the assumptions of Zorn Lemma.

Since $X \in \mathcal{Z}$, we see that \mathcal{Z} is non-empty.

Let \mathcal{C} be a non-empty chain in \mathcal{Z} and $B := \bigcap \mathcal{C}$. Then B is closed and non-empty (by compactness). We also show that $B + B \subseteq B$. Indeed, we have $B \subseteq Z$ for every $Z \in \mathcal{C}$, which implies $B + B \subseteq Z + Z \subseteq Z$. Hence $B + B \subseteq \bigcap \mathcal{C} = B$. Thus we have shown that $B \in \mathcal{Z}$.

Now by Zorn lemma there is a minimal element of \mathcal{Z} w.r.t. inclusion. Let Y be such set. Let $a \in Y$. Then $a + Y$ is closed subset of X . We also have $(a + Y) + (a + Y) \subseteq a + (Y + Y + Y) \subseteq a + Y$. From this we get $a + Y \in \mathcal{Z}$.

Since $a + Y \subseteq Y$, the minimality of Y implies $a + Y = Y$. Thus $Z := \{y \in Y; a + y = a\}$ is non-empty closed set. For $y, z \in Z$ we have $a + (y + z) = (a + y) + z = a + z = a$, i.e. $y + z \in Z$. This shows that $Z + Z \subseteq Z$, i.e. $Z \in \mathcal{Z}$. Again from $Z \subseteq Y$ and the minimality of Y we get that $Y = Z$.

Now $a \in Y = Z$ implies $a + a = a$. □

Remark 3.1.3. In case we assume $0 \notin \mathbb{N}$, then every idempotent ultrafilter is free.

3.2 Hindman's theorem

Definition 3.2.1. For $D \subseteq \mathbb{N}$ we define

$$\text{FS}(D) = \left\{ \sum_{i \in F} i; F \subseteq D; F \text{ is finite and non-empty} \right\}.$$

(I.e., $\text{FS}(D)$ is the set of all finite *non-repetitive* sums of elements from D .)

A set $A \subseteq \mathbb{N}$ is called an IP set if there exists an infinite subset $D \subseteq A$ such that $\text{FS}(D) \subseteq A$.

The following lemma is [T, Section II.15, Lemma 4].

Lemma 3.2.2. *If A belongs to an idempotent free ultrafilter (i.e. a free ultrafilter \mathcal{U} such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$), then A is an IP set.*

Proof. We construct by induction a system of sets $A_i \subseteq A$ and integers $n_i \in A_i$ such that:

- $A_k \in \mathcal{U}$;
- $A_k - n_k \in \mathcal{U}$;
- $A_{k+1} \subseteq A_k$, i.e. this system is decreasing;
- $A_{k+1} \subseteq A_k - n_k$;
- $n_k < n_{k+1}$, i.e. the sequence $(n_k)_{k=1}^{\infty}$ is strictly increasing.

In the first step we can choose $A_0 = A$. Since $A \in \mathcal{U} = \mathcal{U} + \mathcal{U}$, we get that the set $\{n \in \mathbb{N}; A - n \in \mathcal{U}\} \cap A$ is infinite and thus there is at least one n_0 in this set.

Inductive step: Suppose we already have A_0, A_1, \dots, A_k and n_0, n_1, \dots, n_k . Now since both A_k and $A_k - n_k$ belong to \mathcal{U} , we get that $A_{k+1} = A_k \cap (A_k - n_k) \in \mathcal{U}$. Clearly, $A_{k+1} \subseteq A_k$ and $A_{k+1} \subseteq A_k - n_k$.

Again, since $A_{k+1} \in \mathcal{U} = \mathcal{U} + \mathcal{U}$, the set $\{n \in \mathbb{N}; A_{k+1} - n \in \mathcal{U}\} \cap A_{k+1}$ is infinite and we can choose n_{k+1} from this set which is greater than n_k .

Now we claim that if we put $D = \{n_0 < n_1 < \dots < n_k < \dots\}$, then $\text{FS}(D) \subseteq A$. We will show by induction on s that for any choice $n_{i_1} < n_{i_2} < \dots < n_{i_s}$ we have $n_{i_1} + n_{i_2} + \dots + n_{i_s} \in A_{i_1}$.

1° For $s = 1$ this is clearly true.

2° Suppose the claim is true for any choice of s elements from D . Let $n_{i_1} < n_{i_2} < \dots < n_{i_s} < n_{i_{s+1}}$. By the inductive hypothesis $N := n_{i_2} + \dots + n_{i_s} + n_{i_{s+1}} \in A_{i_2}$. From $N \in A_{i_2} \subseteq A_{i_1+1} \subseteq A_{i_1} - n_{i_1}$, we get

$$n_{i_1} + N = n_{i_1} + n_{i_2} + \dots + n_{i_s} + n_{i_{s+1}} \in A_{i_1}.$$

□

Note that we have used the commutativity of $+$ in the last step; the definition of $A_{i_1} - n_{i_1}$ would give us $N + n_{i_1} \in A_{i_1}$.

Theorem 3.2.3 (Hindman). *If $\mathbb{N} = A_1 \cup A_2 \cup \dots \cup A_k$, then one of the sets is an IP-set.*

Different interpretation: If we color integers by finitely many colors then there is an infinite set of integers such that all finite (non-repeating) sums of integers from this set have the same color.

Proof. If we fix an idempotent free ultrafilter \mathcal{U} on \mathbb{N} , then one of the sets A_1, \dots, A_k must belong to \mathcal{U} . □

3.3 Corollaries of Hindman's theorem

From Hindman theorem we can obtain some well-known results from Ramsey theory.

Theorem 3.3.1 (Folkman). *If \mathbb{N} is colored by finitely many colors then there is an arbitrary large set $A \subseteq \mathbb{N}$ such that $\text{FS}(A)$ is monochromatic.*

Theorem 3.3.2 (Schur). *If \mathbb{N} is colored by finitely many colors then there exist $x, y, z \in \mathbb{N}$ such that $x + y = z$ and they have the same color.*

4 Minimal ideals

4.1 Van der Waerden Theorem

5 Other topological semigroups

Example 5.0.1. $x * y = x$ left/right zero semigroup

5.1 One-point compactification

Example 5.1.1.

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