Stone-Čech compactification of countable discrete space

Various notes on $\beta\omega$. (Basically the purpose of writing these notes was to get better acquainted with this material. I felt that it might help me in understanding and getting more insight if I go through some proofs again by myself. Writing them down is part of this.)

Literature: [HS], [T, Chapter II], [K], [B], [Z]

Hindman's theorem: [AT, Chapter B.III], [J, Theorem 29.1]

Van der Waerden Theorem: [J, Theorem 29.3]

1 Topology of $\beta\mathbb{N}$

Let $\beta\mathbb{N}$ be the set of all ultrafilters on $\mathbb{N}$. The set of free ultrafilters is denoted by $\beta\mathbb{N}^*$. The principal ultrafilter determined by n will be denoted by n^*.

For any $A \subseteq \mathbb{N}$ we denote [1]

$$\widehat{A} = \{\mathcal{F} \in \beta\mathbb{N}; A \in \mathcal{F}\}.$$

Notice that we have

$$\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$$
$$\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$$
$$\widehat{\mathbb{N} \setminus A} = \beta\mathbb{N} \setminus \widehat{A}$$

(These conditions are just reformulations of:
- $A, B \in \mathcal{F} \Leftrightarrow A \cap B \in \mathcal{F}$;
- $A \in \mathcal{F} \vee B \in \mathcal{F} \Leftrightarrow A \cup B \in \mathcal{F}$;
- $\mathbb{N} \setminus A \in \mathcal{F} \Leftrightarrow A \notin \mathcal{F}$.

)

From the first property we see that the set $\{\widehat{A}; A \subseteq \mathbb{N}\}$ is a base for a topology on $\beta\mathbb{N}$.

1.1 Properties of $\beta\mathbb{N}$

The properties of this topology are summarized in the following results.

Theorem 1.1.1. *The topology on $\beta\mathbb{N}$ generated by $\{\widehat{A}; A \subseteq \mathbb{N}\}$ is compact and Hausdorff. This space is zero-dimensional.*

The map $e\colon \mathbb{N} \to \beta\mathbb{N}$ which maps n to n^* is a dense embedding if $\mathbb{N}$ is endowed with the discrete topology. Moreover, for each sequence $x\colon \mathbb{N} \to K$ to a compact Hausdorff space K there is a unique continuous extension $\overline{x}\colon \beta\mathbb{N} \to K$ such that $\overline{x} \circ e = x$. (This extension can be expressed using $\mathcal{F}$-limit as $\overline{x}(\mathcal{F}) = \mathcal{F}\text{-}\lim x$.)

Proof can be found also e.g. in [HS, Chapter 3].

Proof. *$\beta\mathbb{N}$ is Hausdorff.* If $\mathcal{F} \neq \mathcal{G}$ are two ultrafilters on $\mathbb{N}$, then there is a set A such that $A \in \mathcal{F}$ and $A \notin \mathcal{G}$. Then $\widehat{A}$ is a basic neighborhood of $\mathcal{F}$, and $\widehat{\mathbb{N} \setminus A}$ is a basic neighborhood of $\mathcal{G}$ and these two neighborhoods are disjoint.

[1] Some authors use the notation A^*, e.g. [T]. I prefer to use different notation, since this might be confused with the notation for $\mathrm{cl}_{\beta D}(A) \setminus A$, this subset of $\beta\mathbb{N}^*$ is commonly denoted by A^*, too.

$\beta\mathbb{N}$ is compact. Suppose that there is an open cover $\{\widehat{A_i}; i\in I\}$ consisting of basic sets, which does not have a finite subcover. This means that for every finite set $F\subset I$ we have $\beta\mathbb{N}\setminus \bigcup\limits_{i\in F} \widehat{A_i}\neq\emptyset$. Since $\beta\mathbb{N}\setminus \bigcup\limits_{i\in F} \widehat{A_i} = \widehat{\mathbb{N}\setminus \bigcup\limits_{i\in F} A_i}$, we see that $\mathbb{N}\setminus \bigcup\limits_{i\in F} A_i = \bigcap_{i\in F}(\mathbb{N}\setminus A_i)\neq\emptyset$. Hence the system $\{\mathbb{N}\setminus A_i; i\in I\}$ has finite intersection property, and therefore there exists an ultrafilter $\mathcal{F}$ which contains this system. For this ultrafilter $\mathcal{F}$ we have $\mathcal{F}\notin \bigcup\limits_{i\in I} A_i$, which contradicts the assumption that $\{\widehat{A_i}; i\in I\}$ is a cover of $\beta\mathbb{N}$.

$\beta\mathbb{N}$ is zero-dimensional. All basic set $\widehat{A}$ are clopen, since $\widehat{\mathbb{N}\setminus A}=\beta\mathbb{N}\setminus\widehat{A}$.

The map e is an embedding. The map e is continuous, since $\mathbb{N}$ has discrete topology. For each $n\in\mathbb{N}$ we have $e[\{n\}]=\{n^*\}=\widehat{\{n\}}$, therefore e is open.

$e[\mathbb{N}]$ is dense. If $A\neq\emptyset$, then there is $n\in A$ and we have $n^*\in\widehat{A}$.

Definition of $\overline{x}$. Let K be a compact Hausdorff space. Then for every ultrafilter and any sequence $x\colon\mathbb{N}\to K$ there exists a unique limit $\mathcal{F}\text{-}\lim x_n$. A proof can be found e.g. in [D, Theorem 4.3.5], [HS, Theorem 3.48], [T, p.64, Claim 14.1], [F, 2A3Se(i)][2].

Continuity of $\mathcal{F}$-lim. Let $x\colon\mathbb{N}\to K$, where K is compact. For any $\mathcal{F}$ we denote $\overline{x}(\mathcal{F})=\mathcal{F}\text{-}\lim x$. Then the map $\overline{x}$ is continuous. Let $\overline{x}(\mathcal{F})=\mathcal{F}\text{-}\lim x=L$. Choose a neighborhood V of L such that $\overline{V}\subseteq U$. Then the set $A=x^{-1}(V)$ belongs to $\mathcal{F}$. We show that $\overline{x}(\mathcal{G})\in U$ for any $\mathcal{G}\in\widehat{A}$. To see this, it suffices to notice that $\mathcal{G}\text{-}\lim x\in\overline{V}$ for each such ultrafilter $\mathcal{G}$. Indeed, if $\mathcal{G}\text{-}\lim x=L'$ then for any neighborhood U' of L' we have $x^{-1}(U')\in\mathcal{G}$. Since $x^{-1}(V)\in\mathcal{G}$, we get that $x^{-1}(U'\cap V)=x^{-1}(U')\cap x^{-1}(V)\in\mathcal{G}$. This implies that $U'\cap V\neq\emptyset$. Since every neighborhood of L' intersects V, we see that $L'\in\overline{V}$.

Uniqueness of $\overline{x}$ follows from the fact that $e[\mathbb{N}]$ is dense in $\beta\mathbb{N}$. □

Note that the above construction can be considered a special case of Wallman extension, see [E, p.176–178], [N, Theorem IV.3], [W, 19K].

It is also interesting to notice that in the above proof we have in fact shown that if we fix a sequence $x\colon\mathbb{N}\to K$ then the map $\overline{x}\colon\beta\mathbb{N}\to K$ defined by $\overline{x}(\mathcal{F})=\mathcal{F}\text{-}\lim x$ is continuous. In the other words, $\mathcal{F}$-lim is continuous in $\mathcal{F}$. It is relatively easy to see that the topology of $\beta\mathbb{N}$ is precisely the coarsest topology which makes each $\overline{x}\colon\beta\mathbb{N}\to[0,1]$ continuous; i.e., it is the initial topology w.r.t. these maps. It suffices to notice that $\overline{\chi_A}(\mathcal{F})=\mathcal{F}\text{-}\lim\chi_A=1\Leftrightarrow A\in\mathcal{F}$. (And, dually, $\overline{\chi_A}(\mathcal{F})=\mathcal{F}\text{-}\lim\chi_A=0\Leftrightarrow A\notin\mathcal{F}$; this second property is only true for ultrafilters, not for arbitrary filters.) (Hence for any neighborhood U of 1 which does not contain 0 we have $\overline{\chi_A}^{-1}(U)=\{\mathcal{F}\in\beta\mathbb{N};\mathcal{F}\text{-}\lim\chi_A=1\}=\{\mathcal{F}\in\beta\mathbb{N};A\in\mathcal{F}\}=\widehat{A}$.)

2 Algebra in $\beta\mathbb{N}$

Although bellow we will work with $(\mathbb{N},+)$, it is worth noticing that the same can be done with arbitrary semigroup (endowed with discrete topology).

2.1 Definition of operations

For any $A\subseteq\mathbb{N}$ and $n\in\mathbb{N}$ we define

$$A-n=\{k\in\mathbb{N}; k+n\in A\}.$$

For ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ we define

$$\mathcal{U}+\mathcal{V}=\{A\subseteq\mathbb{N};\{n\in\mathbb{N}; A-n\in\mathcal{U}\}\in\mathcal{V}\}.$$

[2] or http://thales.doa.fmph.uniba.sk/sleziak/texty/rozne/trf/iconv/notions.pdf

Lemma 2.1.1. *$\mathcal{U}+\mathcal{V}$ is an ultrafilter.*

Similar proof is given in [T, Section 15, Lemma 1].

Proof. $\emptyset \notin \mathcal{U}+\mathcal{V}$ (We have $\emptyset - n = \emptyset$ and thus $\{n\in\mathbb{N}; \emptyset - n \in \mathcal{U}\} = \emptyset \notin \mathcal{V}$.)

If $A, B \in \mathcal{U}+\mathcal{V}$ and $C := A\cap B$, then $C-n = (A-n)\cap(B-n)$. ($k+n\in A\cap B \Leftrightarrow k+n\in A \wedge k+n \in B$) This implies that $\{n\in\mathbb{N}; A-n\in\mathcal{U}\}\cap\{n\in\mathbb{N}; B-n\in\mathcal{U}\}\subseteq\{n\in\mathbb{N}; C-n\in\mathcal{U}\}$ and consequently $C\in\mathcal{U}+\mathcal{V}$.

If $A\in\mathcal{U}+\mathcal{V}$ and $A\subseteq B$, then $B\in\mathcal{U}+\mathcal{V}$. This follows from $A\subseteq B \Rightarrow A-n\subseteq B-n$.

Now let A be an arbitrary subset of $\mathbb{N}$, we want to show that either A or $\mathbb{N}\setminus A$ belongs to $\mathcal{U}+\mathcal{V}$.

Suppose that $A\notin\mathcal{U}+\mathcal{V}$. This means that

$$\begin{aligned}
&\{n\in\mathbb{N}; A-n\in\mathcal{U}\}\notin\mathcal{V} \Leftrightarrow\\
&\{n\in\mathbb{N}; A-n\notin\mathcal{U}\}\in\mathcal{V} \Leftrightarrow\\
&\{n\in\mathbb{N}; \mathbb{N}\setminus(A-n)\in\mathcal{U}\}\in\mathcal{V} \Leftrightarrow\\
&\{n\in\mathbb{N}; (\mathbb{N}\setminus A)-n\in\mathcal{U}\}\in\mathcal{V}.
\end{aligned}$$

The last line means that $\mathbb{N}\setminus A\in\mathcal{U}+\mathcal{V}$.

We have used $\mathbb{N}\setminus(A-n) = (\mathbb{N}\setminus A)-n$. ($k\in\mathbb{N}\setminus(A-n) \Leftrightarrow k\notin A-n \Leftrightarrow k+n\notin A \Leftrightarrow k+n\in(\mathbb{N}\setminus A)$) □

Note that in the first part of the proof we only used the fact that $\mathcal{U}$ and $\mathcal{V}$ are filters. So we can get a filter $\mathcal{F}+\mathcal{G}$ for any two filters $\mathcal{F}$, $\mathcal{G}$.

It is also useful to notice that the above proof works for any semigroup $(S,\cdot)$ instead of $(\mathbb{N},+)$ without any changes.

2.1.1 Principal ultrafilters

Let us have a look at addition of ultrafilters when one of the ultrafilters is principal.

First let us try to express $a^*+\mathcal{V}$. Directly from the definition we get

$$\begin{aligned}
a^*+\mathcal{V} = &\{A\subseteq\mathbb{N}; \{n\in\mathbb{N}; A-n\in a^*\}\in\mathcal{V}\} =\\
&\{A\subseteq\mathbb{N}; \{n\in\mathbb{N}; a\in A-n\}\in\mathcal{V}\} =\\
&\{A\subseteq\mathbb{N}; \{n\in\mathbb{N}; a+n\in A\}\in\mathcal{V}\} =\\
&\{A\subseteq\mathbb{N}; \{n\in\mathbb{N}; n+a\in A\}\in\mathcal{V}\} =\\
&\{A\subseteq\mathbb{N}; A-a\in\mathcal{V}\}.
\end{aligned}$$

After a little algebraic manipulation we can see from the above expression that $a^*+\mathcal{V}$ is precisely the filter generated by $a+\mathcal{V} = \{a+V; V\in\mathcal{V}\}$. (It is easy to see that every set $A\supseteq a+V$ belongs to $\{A\subseteq\mathbb{N}; A-a\in\mathcal{V}\}$. On the other hand if $A-a\in\mathcal{V}$, then $A\supseteq a+(A-a)\in a+\mathcal{V}$.)

Similarly we get

$$\begin{aligned}
\mathcal{U}+a^* = &\{A\subseteq\mathbb{N}; \{n\in\mathbb{N}; A-n\in\mathcal{U}\}\in a^*\} =\\
&\{A\subseteq\mathbb{N}; a\in\{n\in\mathbb{N}; A-n\in\mathcal{U}\}\} =\\
&\{A\subseteq\mathbb{N}; A-a\in\mathcal{U}\}.
\end{aligned}$$

We have obtained precisely the same expression as above which shows that

$$a^*+\mathcal{U} = \mathcal{U}+a^*, \qquad (1) \quad \{\texttt{EQPRINCOM}\}$$

i.e. principal ultrafilters commute with all ultrafilters.

If we add two principal filters, we get

$$\begin{aligned} a^* + b^* = &\{A \subseteq \mathbb{N}; A - b \in a^*\} = \\ &\{A \subseteq \mathbb{N}; a \in A - b\} = \\ &\{A \subseteq \mathbb{N}; a + b \in A\} = (a+b)^* \end{aligned}$$

i.e.

$$a^* + b^* = (a+b)^*.$$

This means that the addition of ultrafilters extends the usual addition on $\mathbb{N}$.

2.2 Associativity

Lemma 2.2.1. *For any ultrafilters $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ the equality*

$$(\mathcal{U} + \mathcal{V}) + \mathcal{W} = \mathcal{U} + (\mathcal{V} + \mathcal{W})$$

holds.

Proof is given in [T, Section 15, Lemma 1].

Proof.

$$\begin{aligned} (\mathcal{U} + \mathcal{V}) + \mathcal{W} &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; A - n \in \mathcal{U} + \mathcal{V}\} \in \mathcal{W}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; \{k \in \mathbb{N}; (A - n) - k \in \mathcal{U}\} \in \mathcal{V}\} \in \mathcal{W}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; \{k \in \mathbb{N}; A - (k+n) \in \mathcal{U}\} \in \mathcal{V}\} \in \mathcal{W}\} \end{aligned}$$

We have used above the fact that $(A - n) - k = A - (k + n)$. ($s \in (A-n) - k \Leftrightarrow s + k \in A - n \Leftrightarrow s + k + n \in A \Leftrightarrow s \in A - (k+n)$)

$$\begin{aligned} \mathcal{U} + (\mathcal{V} + \mathcal{W}) &= \{A \subseteq \mathbb{N}; \{k \in \mathbb{N}; A - k \in \mathcal{U}\} \in \mathcal{V} + \mathcal{W}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; \{k \in \mathbb{N}; A - k \in \mathcal{U}\} - n \in \mathcal{V}\} \in \mathcal{W}\} = \\ &= \{A \subseteq \mathbb{N}; \{n \in \mathbb{N}; \{k \in \mathbb{N}; A - (k+n) \in \mathcal{U}\} \in \mathcal{V}\} \in \mathcal{W}\} \end{aligned}$$

We have used above the fact that $\{k \in \mathbb{N}; A - k \in \mathcal{U}\} - n = \{k \in \mathbb{N}; A - (k+n); \in \mathcal{U}\}$. ($s \in \{k \in \mathbb{N}; A - k \in \mathcal{U}\} - n \Leftrightarrow s + n \in \{k \in \mathbb{N}; A - k \in \mathcal{U}\} \Leftrightarrow A - (s+n) \in \mathcal{U} \Leftrightarrow s \in \{k \in \mathbb{N}; A - (k+n) \in \mathcal{U}\}$.) □

Note that the above proof did not use commutativity of + anywhere. (So the same proof works for arbitrary semigroup instead of $(\mathbb{N}, +)$.)

2.3 Non-commutativity

It can be shown that the operation + defined on $\beta\mathbb{N}$ is not commutative. See [HS, Theorem 4.27]. In fact, the center of $(\beta\mathbb{N}, +)$ is $Z(\beta\mathbb{N}) = \mathbb{N}$. See [HS, Theorem 6.10]. [3]

[3] See `http://math.stackexchange.com/questions/87109/addition-on-ultrafilters-is-non-commutative`

2.4 Continuity

Lemma 2.4.1. *For every $\mathcal{U} \in \beta\mathbb{N}$ the map $L_{\mathcal{U}} \colon \beta N \to \beta N$ given by*

$$L_{\mathcal{U}} \colon \mathcal{V} \mapsto \mathcal{U} + \mathcal{V}$$

is continuous.

Proof. The topology of $\beta\mathbb{N}$ is generated by basic sets $\widehat{A} = \{\mathcal{V}; A \in \mathcal{V}\}$. It suffices to show that preimage of every basic set is open.

$$\begin{aligned} L_{\mathcal{U}}^{-1}(\widehat{A}) &= \{\mathcal{V}; A \in \mathcal{U} + \mathcal{V}\} \\ &= \{\mathcal{V}; \{n \in \mathbb{N}; A - n \in \mathcal{U}\} \in \mathcal{V}\} \\ &= \{n \in \widehat{\mathbb{N}; A - n} \in \mathcal{U}\} \end{aligned}$$

□

[B, Theorem 3.2]

2.5 Relation of addition to tensor product

If $\mathcal{U}$, $\mathcal{V}$ are ultrafilters on $\mathbb{N}$, then we define

$$\mathcal{U} \otimes \mathcal{V} = \{A \subseteq \mathbb{N} \times \mathbb{N}; \{n; \{k; (k,n) \in A\} \in \mathcal{V}\} \in \mathcal{U}\},$$

i.e. a set $A \subseteq \mathbb{N} \times \mathbb{N}$ belongs to $\mathcal{U} \otimes \mathcal{V}$ if and only if

$$\{n; \{k; (k,n) \in A\} \in \mathcal{V}\} \in \mathcal{U}.$$

It can be shown that $\mathcal{U} \otimes \mathcal{V}$ is an ultrafilter, and it is called *Fubini product* or *tensor product* of $\mathcal{U}$ and $\mathcal{V}$.

If we think about $\mathcal{F}$-limits, then tensor product is a very natural operation, since $\mathcal{U} \otimes \mathcal{V}$ is the ultrafilter on $\mathbb{N} \times \mathbb{N}$ such that

$$\mathcal{U}\text{-}\lim_n \mathcal{V}\text{-}\lim_k x_{(k,n)} = \mathcal{U} \otimes \mathcal{V}\text{-}\lim x_{(k,n)}.$$

See [HS, Lemma 11.3], [J, Exercise 29.2] or [K, Remark 1.7].

If we have a map $f \colon X \to Y$ and an ultrafilter on Y, we can transfer this ultrafilter in a natural way to X:

Lemma 2.5.1. *Let $f \colon X \to Y$ be a map, $\mathcal{F}$ be a filter on Y. Then*

$$\mathcal{G} = \{B \subseteq Y; f^{-1}(B) \in \mathcal{F}\}$$

is a filter on Y,

$$\mathcal{B} = \{f[A]; A \in \mathcal{F}\}$$

is a filterbase on Y and it generates $\mathcal{G}$.

Moreover, if $\mathcal{F}$ is an ultrafilter then $\mathcal{G}$ is an ultrafilter.

(

Proof. *$\mathcal{G}$ is a filter.* $f^{-1}(\emptyset)=\emptyset\notin\mathcal{F}$, hence $\emptyset\notin\mathcal{G}$. Also $f^{-1}(Y)=X$, hence $Y\in\mathcal{G}$ and $\mathcal{G}\neq\emptyset$.

If $B\in\mathcal{G}$ and $B\subseteq C$, then we have $f^{-1}(B)\in\mathcal{F}$ and $f^{-1}(B)\subseteq f^{-1}(C)$. This implies $f^{-1}(C)\in\mathcal{F}$ and $C\in\mathcal{G}$.

If $B,C\in\mathcal{G}$ then $f^{-1}(B\cap C)=f^{-1}(B)\cap f^{-1}(C)\in\mathcal{F}$, and thus $B\cap C\in\mathcal{G}$.

$\mathcal{B}$ is a filterbase. For any $A,B\in\mathcal{F}$ we have $f[A]\cap f[B]\supseteq f[A\cap B]$. (And each $f[A]$ is non-empty.)

$\mathcal{B}$ generates $\mathcal{G}$. We have $f[f^{-1}(B)]\subseteq B$, and thus $\mathcal{G}\subseteq\mathcal{F}_{\mathcal{B}}$.

We also have $f^{-1}(f[A])\supseteq A$, which means $\mathcal{B}\subseteq\mathcal{G}$.

If $\mathcal{F}$ is an ultrafilter, then $\mathcal{G}$ is an ultrafilter. $B\notin\mathcal{G}\Rightarrow f^{-1}(B)\notin\mathcal{F}\Rightarrow X\setminus f^{-1}(B)=f^{-1}(Y\setminus B)\in\mathcal{F}\Rightarrow Y\setminus B\in\mathcal{G}$. □

)

In this way, if we use the filter $\mathcal{V}\otimes\mathcal{U}$ on $\mathbb{N}\times\mathbb{N}$ and the map $+\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, we get precisely the filter $\mathcal{U}+\mathcal{V}$ since

$$\begin{aligned}A\in\mathcal{U}+\mathcal{V}&\Leftrightarrow\{n\in\mathbb{N}; A-n\in\mathcal{U}\}\in\mathcal{V}\\&\Leftrightarrow\{n\in\mathbb{N};\{k\in\mathbb{N};k+n\in A\}\in\mathcal{U}\}\in\mathcal{V}.\end{aligned}$$

($+^{-1}(A)\in\mathcal{V}\otimes\mathcal{U}\Leftrightarrow\{(k,n);k+n\in A\}\in\mathcal{V}\otimes\mathcal{U}\Leftrightarrow\{n;\{k;k+n\in A\}\in\mathcal{U}\}\in\mathcal{V}$)

3 Idempotent ultrafilters and their applications

3.1 Existence of idempotent ultrafilters

Theorem 3.1.1 (Glazer)**.** *There exists $\mathcal{U}\in\beta\mathbb{N}$ such that $\mathcal{U}+\mathcal{U}=\mathcal{U}$.*

Glazer's theorem follows from the following more general result. By saying that $(X,+)$ is left-topological semigroup we mean that left multiplication is continuous for every $a\in X$; i.e, for each $a\in X$ the map $x\mapsto a+x$ is continuous.

The following theorem can be found in [T, Section II.15], [B, Theorem 3.3], [Z, Theorem 6.12].

Theorem 3.1.2 (Auslander-Ellis-Numakura)**.** *Every left-topological semigroup $(X,+)$ on a compact Hausdorff space X has an idempotent, i.e., an element a such that $a+a=a$.*

The following proof is taken from [T, Section II.15, Lemma 3].

Proof. Denote $\mathcal{Z}=\{Z\subseteq X; Z\neq\emptyset; Z\text{ is closed and }Z+Z\subseteq Z\}$. We first check that $(\mathcal{Z},\supseteq)$ fulfills the assumptions of Zorn Lemma.

Since $X\in\mathcal{Z}$, we see that $\mathcal{Z}$ is non-empty.

Let $\mathcal{C}$ be a non-empty chain in $\mathcal{Z}$ and $B:=\bigcap\mathcal{C}$. Then B is closed and non-empty (by compactness). We also show that $B+B\subseteq B$. Indeed, we have $B\subseteq Z$ for every $Z\in\mathcal{C}$, which implies $B+B\subseteq Z+Z\subseteq Z$. Hence $B+B\subseteq\bigcap\mathcal{C}=B$. Thus we have shown that $B\in\mathcal{Z}$.

Now by Zorn lemma there is a minimal element of $\mathcal{Z}$ w.r.t. inclusion. Let Y be such set. Let $a\in Y$. Then $a+Y$ is closed subset of X. (The set $a+Y$ is an image of the set Y in the continuous map $x\mapsto a+x$, hence it is compact. Since X is compact Hausdorff, any compact subset of X is closed.) We also have $(a+Y)+(a+Y)\subseteq a+(Y+Y+Y)\subseteq a+Y$. From this we get $a+Y\in\mathcal{Z}$.

Since $a+Y\subseteq Y$, the minimality of Y implies $a+Y=Y$. Thus $Z:=\{y\in Y; a+y=a\}$ is non-empty closed set. (It is closed since it is the intersection of Y with the preimage of $\{a\}$ in the continuous map $x\mapsto a+x$; i.e. $Y\cap L_a^{-1}(\{a\})$.) For $y,z\in Z$ we have $a+(y+z)=(a+y)+z=a+z=a$, i.e. $y+z\in Z$. This shows that $Z+Z\subseteq Z$, i.e. $Z\in\mathcal{Z}$. Again from $Z\subseteq Y$ and the minimality of Y we get that $Y=Z$.

Now $a\in Y=Z$ implies $a+a=a$. □

Remark 3.1.3. In case we assume $0\notin\mathbb{N}$, then every idempotent ultrafilter is free.

3.2 Hindman's theorem

Definition 3.2.1. For $D \subseteq \mathbb{N}$ we define

$$\mathrm{FS}(D) = \{\sum_{i\in F} i; F \subseteq D; F \text{ is finite and non-empty}\}.$$

(I.e., $\mathrm{FS}(D)$ is the set of all finite *non-repetitive* sums of elements from D.)

A set $A \subseteq \mathbb{N}$ is called an IP set if there exists an infinite subset $D \subseteq A$ such that $\mathrm{FS}(D) \subseteq A$.

(Note that this formulation of the definition makes sense only for commutative semigroups, since we have not specified the order in which element of F are added.)

The following lemma is [T, Section II.15, Lemma 4].

Lemma 3.2.2. *If A belongs to an idempotent free ultrafilter (i.e. a free ultrafilter $\mathcal{U}$ such that $\mathcal{U}+\mathcal{U}=\mathcal{U}$), then A is an IP set.*

Proof. We construct by induction a system of sets $A_i \subseteq A$ and integers $n_i \in A_i$ such that:

- $A_k \in \mathcal{U}$;
- $A_k - n_k \in \mathcal{U}$;
- $A_{k+1} \subseteq A_k$, i.e. this system is decreasing;
- $A_{k+1} \subseteq A_k - n_k$;
- $n_k < n_{k+1}$, i.e. the sequence $(n_k)_{k=1}^{\infty}$ is strictly increasing.

In the first step we can choose $A_0 = A$. Since $A \in \mathcal{U} = \mathcal{U}+\mathcal{U}$, we get that the set $\{n \in \mathbb{N}; A - n \in \mathcal{U}\} \cap A$ is infinite (since it belongs to the free ultrafilter $\mathcal{U}$) and thus there is at least one n_0 in this set.

Inductive step: Suppose we already have $A_0, A_1, \dots, A_k$ and $n_0, n_1, \dots, n_k$. Now since both A_k and $A_k - n_k$ belong to $\mathcal{U}$, we get that $A_{k+1} = A_k \cap (A_k - n_k) \in \mathcal{U}$. Clearly, $A_{k+1} \subseteq A_k$ and $A_{k+1} \subseteq A_k - n_k$.

Again, since $A_{k+1} \in \mathcal{U} = \mathcal{U}+\mathcal{U}$, the set $\{n \in \mathbb{N}; A_{k+1} - n \in \mathcal{U}\} \cap A_{k+1}$ is infinite and we can choose n_{k+1} from this set which is greater than n_k.

Now we claim that if we put $D = \{n_0 < n_1 < \dots < n_k < \dots\}$, then $\mathrm{FS}(D) \subseteq A$. We will show by induction on s that for any choice $n_{i_1} < n_{i_2} < \dots < n_{i_s}$ we have $n_{i_1}+n_{i_2}+\dots+n_{i_s} \in A_{i_1}$. (Since $A_{i_1} \subseteq A$, this implies that the sum belongs to A and A is an IP set.)

1° For $s = 1$ this is clearly true.

2° Suppose the claim is true for any choice of s elements from D. Let $n_{i_1} < n_{i_2} < \dots < n_{i_s} < n_{i_{s+1}}$. By the inductive hypothesis $N := n_{i_2} + \dots + n_{i_s} + n_{i_{s+1}} \in A_{i_2}$. From $N \in A_{i_2} \subseteq A_{i_1+1} \subseteq A_{i_1} - n_{i_1}$, we get

$$n_{i_1} + N = n_{i_1} + n_{i_2} + \dots + n_{i_s} + n_{i_{s+1}} \in A_{i_1}.$$

□

Note that we have used the commutativity of + in the last step; the definition of $A_{i_1} - n_{i_1}$ would give us $N + n_{i_1} \in A_{i_1}$.

Theorem 3.2.3 (Hindman)**.** *If $\mathbb{N} = A_1 \cup A_2 \cup \dots \cup A_k$, then one of the sets is an IP-set.*

Different interpretation: If we color integers by finitely many colors then there is an infinite set of integers such that all finite (non-repeating) sums of integers from this set have the same color.

Proof. If we fix an idempotent free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, then one of the sets $A_1, \dots, A_k$ must belong to $\mathcal{U}$. □

3.3 Corollaries of Hindman's theorem

From Hindman theorem we can obtain some well-known results from Ramsey theory.

Theorem 3.3.1 (Folkman)**.** *If* $\mathbb{N}$ *is colored by finitely many colors then there is an arbitrary large set* $A \subseteq \mathbb{N}$ *such that* $\operatorname{FS}(A)$ *is monochromatic.*

Theorem 3.3.2 (Schur)**.** *If* $\mathbb{N}$ *is colored by finitely many colors then there exist* $x, y, z \in \mathbb{N}$ *such that* $x + y = z$ *and they have the same color.*

4 Minimal ideals

4.1 Van der Waerden Theorem

5 Other topological semigroups

Example 5.0.1. $x * y = x$ left/right zero semigroup

5.1 One-point compactification

Example 5.1.1.

References

[AT] Spiros A. Argyros and Stevo Todorcevic. *Ramsey Methods in Analysis.* Birkhäuser, Basel, 2005.

[B] Vitaly Bergelson. Combinatorial and Diophantine applications of ergodic theory. In *Handbook of Dynamical systems, Vol. 1B*, pages 745–869. Elsevier, Amsterdam, 2006. available at `http://people.brandeis.edu/~kleinboc/211a/berg_review.pdf`.

[D] Jacques Dixmier. *General Topology.* Springer-Verlag, New York, 1984. Undergraduate Texts in Mathematics.

[E] Ryszard Engelking. *General Topology.* Heldermann Verlag, Berlin, 1989. Revised and completed edition, Sigma Series in Pure Mathematics, Vol. 6.

[F] D. H. Fremlin. *Measure theory, Volume 2: Broad Foundations.* Torres Fremlin, Essex, 2001.

[HS] Neil Hindman and Dona Strauss. *Algebra in the Stone-Čech Compactification: Theory and Applications.* Walter de Gruyter, Berlin–New York, 1998.

[J] T. Jech. *Set theory.* Springer, Berlin, 2002.

[K] Peter Krautzberger. *Idempotent Filters and Ultrafilters.* PhD thesis, Freie Universität Berlin, 2009. available at `http://www.diss.fu-berlin.de/diss/servlets/MCRFileNodeServlet/FUDISS_derivate_000000006649/Krautzberger_Idempotent_Filters_and_Ultrafilters.pdf`.

[N] Jun-Iti Nagata. *Modern General Topology.* North-Holland, Amsterdam, 2nd edition, 1985.

[T] Stevo Todorcevic. *Topics in Topology.* Springer-Verlag, Berlin–Heidelberg, 1997. Lecture Notes in Mathematics 1652.

[W] S. Willard. *General topology.* Addison-Wesley, Massachussets, 1970.

[WIK] Wikipedia. `http://en.wikipedia.org`.

[Z] Yevhen G. Zelenyuk. *Ultrafilters and Topologies on Groups.* De Gruyter, Berlin–New York, 2011. De Gruyter Expositions in Mathematics 50.