

Bressoud: A Radical Approach to Real Analysis

Notes from [B].

1 Crisis in Mathematics: Fourier's Series

2 Infinite Summations

2.1 The Archimedean Understanding

2.2 Geometric Series

2.3 Calculating π

2.4 Logarithms and Harmonic Series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots \quad (2.27)$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \quad (2.28)$$

Euler's Constant

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} - \ln n \right) \quad (1)$$

$$\gamma \approx 0.577,215,665$$

Estimating Euler's Gamma

$$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n-1} - \ln n \quad (2)$$

$$u_n = 1 + \frac{1}{2} + \dots + \frac{1}{n-1} + \frac{1}{n} - \ln n = x_n + \frac{1}{n} \quad (3)$$

$x_n < \gamma < y_n$, x_n is increasing and y_n is decreasing

$$\begin{aligned} \ln x &\leq x - 1 \\ \ln \frac{1}{x} &\leq \frac{1}{x} - 1 \\ -\ln x &\leq \frac{1}{x} - 1 \\ \ln x &\geq 1 - \frac{1}{x} \end{aligned}$$

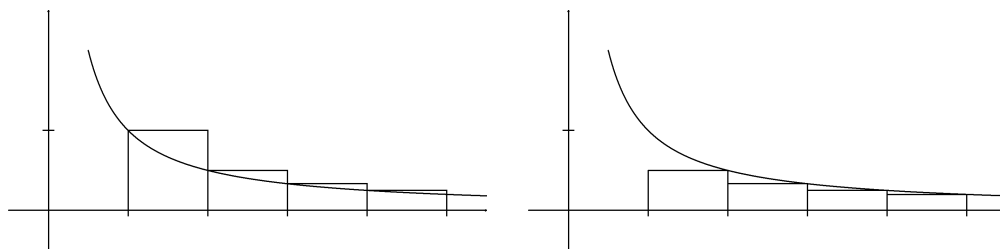


Figure 1: Harmonic series nad the function $\frac{1}{x}$

$$1 - \frac{1}{x} \leq \ln x \leq x - 1$$

$$1 - \frac{1}{x+1} \leq \ln(1+x) \leq x$$

So we get $\ln\left(1 + \frac{1}{n}\right) \leq \frac{1}{n}$ and

$$\ln\left(1 + \frac{1}{n}\right) \geq 1 - \frac{1}{1 + \frac{1}{n}} = 1 - \frac{n}{n+1} = \frac{1}{n+1},$$

i.e.,

$$\frac{1}{n+1} \leq \ln\left(1 + \frac{1}{n}\right) \leq \frac{1}{n}$$

From this we can get that $x_{n+1} > x_n$ and $y_n < y_{n+1}$.

$$x_{n+1} - x_n = \frac{1}{n} - \ln(n+1) + \ln n = \frac{1}{n} - \ln\left(1 + \frac{1}{n}\right) > 0$$

$$y_n - y_{n+1} = \ln(n+1) - \ln n - \frac{1}{n+1} = \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} > 0$$

The Nested Interval Principle

Approximating Partial Sums of the Harmonic Series

References

- [B] David M. Bressoud. *A Radical Approach to Real Analysis*. The Mathematical Association of America, 2nd edition, 2007.

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