Bressoud: A Radical Approach to Real Analysis

Notes from [B].

1 Crisis in Mathematics: Fourier's Series

2 Infinite Summations

2.1 The Archimedean Understanding

2.2 Geometric Series

2.3 Calculating π

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \tag{1}$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \arctan x \tag{2}$$

2.4 Logarithms and Harmonic Series

Taylor series for logarithms at the point 1.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} \tag{2.27}$$

Radius of convergence: 1. Converges for $x \in (-1,1]$.

Error: $\ln(1+x) = x - \frac{x^2}{2} + \dots + \frac{(-1)^{n-1}x^n}{n} + R_n$.

For $x > 0$ we have an alternating series and $|R_n| \le \frac{|x|^{n+1}}{n+1}$.

For $x < 0$ we get

$$\begin{aligned}
|R_n| &\le \frac{|x|^{n+1}}{n+1} + \frac{|x|^{n+2}}{n+2} + \frac{|x|^{n+3}}{n+3} + \dots \\
&\le \frac{|x|^{n+1} + |x|^{n+2} + |x|^{n+3} + \dots}{n+1} \\
&= \frac{|x|^{n+1}}{n+1}(1 + |x| + |x|^2 + \dots) \\
&= \frac{1}{1-|x|} \cdot \frac{|x|^{n+1}}{n+1}
\end{aligned}$$

So we have

$$|R_n| \le \frac{1}{1-|x|} \cdot \frac{|x|^{n+1}}{n+1}.$$

Problem 2.4.4. Let $S_n(x) = \sum\limits_{k=1}^{n} \frac{(-1)^{n-1}x^n}{n}$

	$\ln(1+x)$	$S_{10}(x)$	$S_{100}(x)$
$x = 1.1$	0·741,937	0·612,543	−71·098,341
$x = 1.01$	0·698,135	0·645,378	0·684,611
$x = 1.001$	0·693,647	0·645,632	0·692,289
$x = 1$	0·693,147	0·645,635	0·688,172
$x = 0.999$	0·692,647	0·645,632	0·688,148
$x = 0.99$	0·688,135	0·645,391	0·686,323
$x = 0.9$	0·641,854	0·626,198	0·641,854
$x = 0.5$	0·405,465	0·405,434	0·405,465

	$\ln(1+x)$	$S_{10}(x)$	$S_{100}(x)$	$S_{1000}(x)$
$x = -0.9$	−2·302,585	−2·118,748	−2·302,583	−2·302,585
$x = -0.99$	−4·605,170	−2·831,179	−4·389,453	−4·605,166
$x = -0.999$	−6·907,755	−2·918,997	−5·089,800	−6·688,739

	$\ln(1+x) - S_{10}(x)$	$\ln(1+x) - S_{100}(x)$	$\ln(1+x) - S_{1000}(x)$
$x = 1$	$5 \cdot 10^{-2}$	$5 \cdot 10^{-3}$	$5 \cdot 10^{-4}$
$x = 0.999$	$5 \cdot 10^{-2}$	$5 \cdot 10^{-3}$	$5 \cdot 10^{-4}$
$x = 0.5$	$3 \cdot 10^{-3}$	0	0
$x = -0.999$	-4	-2	-0.2

Problem 2.4.5.

$$\ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \dots\right) = 2\sum_{k=1}^{\infty} \frac{z^{2k-1}}{2k-1} \tag{2.37}$$

$$1 + x = \frac{1+z}{1-z} = -1 - \frac{2}{z-1} \Leftrightarrow z = 1 - \frac{2}{x+2} = \frac{x}{x+2}$$

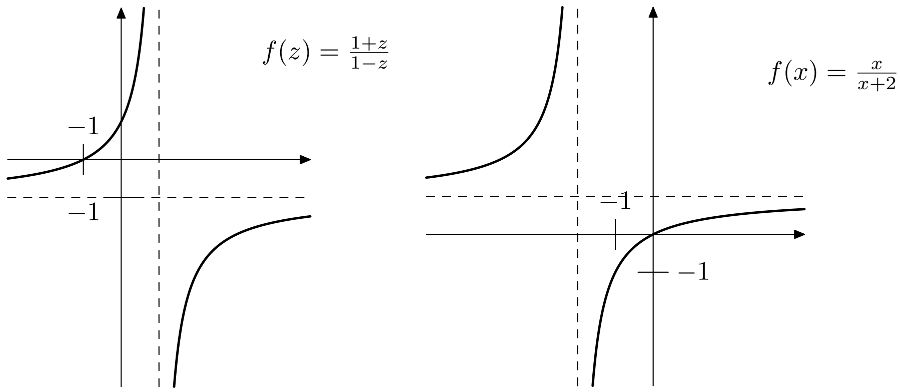
$$\ln(1+x) = 2\left(\frac{x}{x+2} + \frac{1}{3}\left(\frac{x}{x+2}\right)^3 + \frac{1}{5}\left(\frac{x}{x+2}\right)^5 + \dots\right) \tag{2.38}$$

Estimate for the error is the sum of errors of $(2n+1)$-th order or the positive and negative part of (2.27).

$$E_n \le \left(1 + \frac{1}{1-|z|}\right)\frac{|z|^{2n+1}}{2n+1} = \frac{2-|z|}{1-|z|} \cdot \frac{|z|^{2n+1}}{2n+1}$$

For $x = 4$ we have $z = \frac{2}{3}$ and $\frac{2-|z|}{1-|z|} = 4$, so

$$E_n \le \frac{4}{2n+1}\left(\frac{2}{3}\right)^n < \varepsilon \Leftrightarrow \frac{2n+1}{4}\left(\frac{3}{2}\right)^{2n+1} > \frac{1}{\varepsilon}$$



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For $x = -\frac{4}{5}$ we have $z = -\frac{2}{3}$ and, again, $|z| = \frac{2}{3}$. So we get the same estimates on the error term.

If we want to estimate the error term by hand, we might use inequalities $\left(\frac{3}{2}\right)^2 = \frac{9}{2} > 2$ and $\left(\frac{3}{2}\right)^4 = \frac{81}{16} > 5$. (Which together give $\left(\frac{3}{2}\right)^6 > 10$.)

n	$\frac{2n+1}{4}\left(\frac{3}{2}\right)^{2n+1}$	
3	$\frac{7}{4}\left(\frac{3}{2}\right)^7 \approx 30$	$\frac{7}{4}\cdot\frac{3}{2}\left(\frac{3}{2}\right)^6 > \frac{7}{4}\cdot\frac{3}{2}\cdot 10 > 10$
4	$\frac{9}{4}\left(\frac{3}{2}\right)^9 \approx 86$	
5	$\frac{11}{4}\left(\frac{3}{2}\right)^{11} \approx 238$	$\frac{11}{4}\left(\frac{3}{2}\right)^{11} > \left(\frac{3}{2}\right)^{12} > 10^2$
7	$\frac{15}{4}\left(\frac{3}{2}\right)^{15} \approx 1600$	$\frac{15}{4}\left(\frac{3}{2}\right)^{15} > \frac{15}{4}\cdot\frac{3}{2}\cdot 2\cdot 10^2 > 10^3$
8	$\frac{17}{4}\left(\frac{3}{2}\right)^{17} \approx 4000$	
9	$\frac{19}{4}\left(\frac{3}{2}\right)^{19} \approx 10000$	
12	$\frac{25}{4}\left(\frac{3}{2}\right)^{25} \approx 160000$	$\frac{25}{4}\left(\frac{3}{2}\right)^{25} > \left(\frac{3}{2}\right)^{24} > 10^6$

n	$G_n(4)$	$\log(5) - G_n(4)$
$\log 5$	1·609,437,912	
3	1·583,539,095	$\approx 3\cdot 10^{-2}$
5	1·606,041,794	$\approx 3\cdot 10^{-3}$
7	1·608,934,295	$\approx 5\cdot 10^{-4}$
11	1·609,424,767	$\approx 1.3\cdot 10^{-5}$
12	1·609,432,515	$\approx 5\cdot 10^{-6}$

For $x = -\frac{4}{5}$ and $z = -\frac{2}{3}$ we get exactly opposite values.

Problem 2.4.6.

	$\ln(1+x) - G_{10}(x)$	$\ln(1+x) - G_{100}(x)$	$\ln(1+x) - G_{1000}(x)$
$x = 1$	10^{-11}	0	0
$x = 0.999$	10^{-11}	0	0
$x = 0.5$	0	0	0
$x = -0.999$	-2.5	-0.7	$-4 \cdot 10^{-3}$

Problem 2.4.7.

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right) = \frac{1}{2}\log 2$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$$

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots = \frac{\pi}{4} + \frac{\ln 2}{2}$$

Estimate with the first 1000 terms:

$$\frac{\pi}{4} + \frac{\ln 2}{2} = 1{\cdot}131{,}971{,}753{,}677{,}421$$
$$S_{1000} = 1{\cdot}130{,}972{,}254{,}176{,}419$$

Problem 2.4.10. Kempner series[1]

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} \leq 8$$
$$\frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{88} \leq \frac{8 \cdot 9}{10}$$
$$\frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{888} \leq \frac{8 \cdot 9^2}{10^2}$$
$$\vdots$$

$$S \leq 8\left(1 + \frac{9}{10} + \frac{9^2}{10^2} + \ldots\right) = \frac{8}{1 - \frac{9}{10}} = 80$$

Problem 2.4.11. Neither 8 nor 9:
$7 + \frac{7\cdot 8}{10} + \frac{7\cdot 8^2}{10^2} + \cdots = \frac{7}{1-\frac{8}{10}} = 35$

Problem 2.4.12. No digit 1:
$\frac{8}{2} + \frac{8\cdot 9}{2\cdot 10} + \frac{8\cdot 9^2}{2\cdot 10^2} + \cdots = 40$

[1] `https://en.wikipedia.org/wiki/Kempner_series`

Problem 2.4.13.

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

n	S_n	$2\sqrt{n+1}-2$	$2\sqrt{n}$
10	5·020,998	4·633	6·325
100	18·589,604	18·100	20
1,000	61·801,009	61·277	63·246
10,000	198·544,645	198·010	200
100,000	630·996,759	630·459	632·455
1,000,000	1,998·540,145	1,998·001	2,000
10,000,000	6,323·095,123,940	6,322·556	6,324·555

Problem 2.4.14.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \ln(2) + O\left(\frac{1}{n}\right)$$
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} = \ln(2n) + \gamma$$

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} = \ln 2 + \frac{\gamma}{2} + \frac{1}{2}\ln n + O\left(\frac{1}{n}\right)$$
$$\lim_{n\to\infty}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2}\ln n\right) = \ln 2 + \frac{\gamma}{2}$$

Problem 2.4.15.

$$\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2nr-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2ns}\right)$$

$$\begin{aligned} S &\approx \left(\ln 2 + \frac{\gamma}{2} + \frac{\ln(nr)}{2}\right) - \frac{1}{2}(\gamma + \ln(ns)) \\ &= \ln 2 + \frac{1}{2}\ln(nr) - \ln(ns) = \ln 2 + \frac{1}{2}\ln\frac{r}{s} \end{aligned}$$

Problem 2.4.16.

$$\sum_{m=n}^{\infty} \frac{1}{m^2} = \frac{1}{n^2} + \sum_{m=n+1}^{\infty} \frac{1}{m^2} < \frac{1}{n^2} + \int_n^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{n^2} + \frac{1}{n} = \frac{1}{n}\left(1 + \frac{1}{n}\right)$$

$$H_{n-1} > \ln n + \gamma - \frac{1}{2}\sum_{m=n+1}^{\infty} \frac{1}{m^2}$$
$$H_{n-1} > \ln n + \gamma - \frac{1}{2n}\left(1 + \frac{1}{n}\right)$$

$$
\begin{aligned}
\ln n - \ln(n-1) > \frac{1}{n} > \frac{1}{n} \cdot \frac{1+\frac{1}{n}}{2} \\
\ln n - \frac{1}{2n}\left(1+\frac{1}{n}\right) > \ln(n-1) \\
H_{n-1} > \ln(n-1) + \gamma
\end{aligned}
$$

Problem 2.4.18. Ant on a rubber rope `https://en.wikipedia.org/wiki/Ant_on_a_rubber_rope`

$$\frac{1}{2000}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

$$
\begin{aligned}
\ln n + \gamma &\approx 2000 \\
n &\approx e^{2000-\gamma}
\end{aligned}
$$

More generally:
$\alpha =$ ant's speed
$c =$ initial rope length
$v =$ how much length grows
$c + vn =$ length before the n'th step
$\theta(n) =$ ratio covered after the n'th set

Discrete case.

$$
\begin{aligned}
\theta(n-1) &= \frac{\alpha}{c+v} + \frac{\alpha}{c+2v} + \cdots + \frac{\alpha}{c+(n-1)v} \\
\theta(n-1) &\geq \int_1^n \frac{\alpha}{c+vt}\,\mathrm{d}t = \left[\frac{\alpha}{v}\ln(c+vt)\right]_1^n \\
\theta(n-1) &\geq \frac{\alpha}{v}\ln\left(\frac{c+vn}{c+v}\right)
\end{aligned}
$$

Continuous case.

$c + vt =$ length at time t
$y(t) =$ position
$\phi(t) =$ ratio

$$y'(t) = \alpha + \frac{vy(t)}{c+vt}$$

$$\phi(t) = \frac{y(t)}{c+vt}$$

$$
\begin{aligned}
y(t) &= \phi(t)(c+vt)\\
y'(t) &= \phi'(t)(c+vt) + v\phi(t)\\
y'(t) &= \phi'(t)(c+vt) + \frac{vy(t)}{c+vt}\\
y'(t) - \frac{vy(t)}{c+vt} &= \phi'(t)(c+vt)\\
\alpha &= \phi'(t)(c+vt)\\
\frac{\alpha}{c+vt} &= \phi'(t)
\end{aligned}
$$

$$
\begin{aligned}
\phi'(t) &= \frac{\alpha}{c+vt}\\
\phi(t) &= \int_0^t \frac{\alpha}{c+vx}\,\mathrm{d}x\\
\phi(t) &= \frac{\alpha}{v}\ln\left(\frac{c+vt}{c}\right)
\end{aligned}
$$

$$
\frac{\alpha}{v}\ln\left(\frac{c+vT}{c}\right) = 1 \qquad \Leftrightarrow \qquad T = \frac{c}{v}\left(e^{v/\alpha}-1\right)
$$

References

[B] David M. Bressoud. *A Radical Approach to Real Analysis*. The Mathematical Association of America, 2nd edition, 2007.

Contents