Aliprantis, Border: Infinite-dimensional Analysis– A Hitchhiker's Guide

Notes from [AB2].

1 Odds and Ends

2 Topology

2.1 Topological spaces

Example. (2.2) A *semimetric* = triangle inequality and $d(x, x) = 0$.

2.6 Nets and filters

For each $\alpha \in D$ define the *section* or *tail* $F_\alpha = \{x_\beta; \beta \geq \alpha\}$ and consider the family of sets $\mathcal{B} = \{F_\alpha; \alpha \in D\}$. It is a routine matter to verify that $\mathcal{B}$ is a filter base. The filter $\mathcal{F}$ generated by $\mathcal{B}$ is called the *section filter* of $\{x_\alpha\}$ or the *filter generated by the net* $\{x_\alpha\}$.

2.13 Weak topologies

weak topology or *initial topology* on X generated by the family of functions $\{f_i\}_{i\in I}$

Let w denote this weak topology.

Lemma. *(2.52) A net satisfies* $x_\alpha \xrightarrow{\mathrm{w}} x$ *for the weak topology* w *if and only if* $f_i(x_\alpha) \xrightarrow{\tau_i} f_i(x)$ *for each* $i \in I$.

For a family $\mathcal{F}$ of real functions on X, the weak topology generated by $\mathcal{F}$ is denoted by $\sigma(\mathcal{F}, X)$.

3 Metrizable spaces

3.13 The Cantor set

Lemma. *(3.59) Any nonempty closed subset of* Δ *is a retract of* Δ.

3.14 The Baire space $\mathbb{N}^{\mathbb{N}}$

Lemma. *(3.64) Every nonempty closed subset of* $\mathcal{N}$ *is a retract of* $\mathcal{N}$.

3.15 Uniformities

3.16 The Hausdorff distance

Definition. (3.70) Let (X, d) be a semimetric space. For each pair of nonempty subsets A and B of X, define

$$h_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

The extended real number $h_d(A, B)$ is the *Hausdorff distance* between A and B relative to the semimetric d. The function h_d is the *Hausdorff semimetric* induced by d. By convention, $h_d(\emptyset, \emptyset) = 0$ and $h_d(A, \emptyset) = \infty$ for $A \neq \emptyset$.

Lemma. *(3.71) If A and B are nonempty subsets of a semimetric space (X, d), then*

$$h(A, B) = \inf\{\varepsilon > 0; A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}.$$

The function h has all the properties of semimetric except for the fact that it can take on the value ∞. [1]

Lemma. *(3.72)*

1. $h(A, B) \geq 0$ *and* $h(A, A) = 0$
2. $h(A, B) = h(B, A)$
3. $h(A, B) \leq h(A, C) + h(C, B)$
4. $h(A, B) = 0$ *if and only if* $\overline{A} = \overline{B}$

Lemma. *(3.74) Let (X, d) be a semimetric space. Then for any nonempty subsets A and B of X*

$$h(A, B) = \sup_{x \in X} |d(x, A) - d(x, B)|.$$

3.17 The Hausdorff metric topology

Given a metric space (X, d),

- $\mathcal{F}$ denotes the collection of *nonempty* closed subsets of X,
- $\mathcal{F}_d$ denotes the collection of *nonempty* d-bounded closed subsets of X,
- $\mathcal{K}$ denotes the collection of *nonempty* compact subsets of X.

Corollary. *(3.79) Let (X, d) be a metric space. Ten $F_n \xrightarrow{\tau_h} F$ in $\mathcal{F}$ if and only if the sequence $\{d(\cdot, F_n)\}$ of real functions converges uniformly to $d(\cdot, F)$ on X.*

[1] My question: Is also the function $\delta(A, B) = \sup_{a \in A} d(a, B)$ a semimetric "without symmetry"?

4 Measurability

4.1 Algebras of sets

Algebra of sets = complements and unions

4.2 Rings of sets

5 Topological vector spaces

5.7 Convex and concave functions

Definition. (5.38) A function $f: C \to \mathbb{R}$ on a convex set C in a vector space is

- *convex* if $f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in C$ and $0 \le \alpha \le 1$.
- *strictly convex*
- *concave*
- *strictly concave*

5.8 Sublinear functions and gauges

A real function f defined on a vector space is *subadditive* if

$$f(x+y) \le f(x) + f(y).$$

Recall that a nonempty subset C of a vector space is a *cone* if $x \in C$ implies $\alpha x \in C$ for every $\alpha \ge 0$. A real function f defined on a cone C is *positively homogeneous* if

$$f(\alpha x) = \alpha f(x)$$

for every $\alpha \ge 0$. Clearly, if f is positively homogeneous, then $f(0) = 0$ and f is completely determined by its values on any absorbing set.

Definition. A real function on a vector space is *sublinear* if it is both positively homogeneous and subadditive, or equivalently, if it is both positively homogeneous and convex.

5.9 The Hahn-Banach Extension Theorem

One of the most important and far-reaching results in analysis is the following seemingly mild theorem. It is usually stated for the case where p is sublinear, but this more general statement is as easy to prove. Recall that a real-valued function f dominates a real-valued function g on A if $f(x) \ge g(x)$ for all $x \in A$.

Theorem (Hahn-Banach Extension Theorem)**.** *(5.53) Let X be a vector space and let $p\colon X \to \mathbb{R}$ be any convex function. Let M be a vector subspace of X and let $f\colon M \to \mathbb{R}$ be a linear functional dominated by p on M. Then there is a* (*not generally unique*) *linear extension $\hat{f}$ of f to X that is dominated by p on X.*

In fact, the proofs shows that, for a given $v \in X$, there exists an extension such that the value $\hat{f}(v) = c$ can be chosen anywhere between

$$\sup_{x \in M, \lambda > 0} \frac{1}{\lambda}[f(x) - p(x - \lambda v)] \leq c \leq \inf_{y \in M, \mu > 0} \frac{1}{\mu}[p(y + \mu v) - f(y)].$$

5.10 Separating hyperplane theorems

5.11 Separation by continuous functionals

5.12 Locally convex spaces and seminorms

5.13 Separation in locally convex spaces

5.14 Dual pairs

Definition. (5.90) A *dual pair* (or a *dual system*) is a pair $\langle X, X' \rangle$ of vector spaces together with a bilinear functional $(x, x') \mapsto \langle x, x' \rangle$, from $X \times X'$ to $\mathbb{R}$, that separates the points of X and X'. That is:

1. The mapping $x' \mapsto \langle x, x' \rangle$ is linear for each $x \in X$.
2. The mapping $x \mapsto \langle x, x' \rangle$ is linear for each $x' \in X'$.
3. If $\langle x, x' \rangle = 0$ for each $x' \in X'$, then $x = 0$.
4. If $\langle x, x' \rangle = 0$ for each $x \in X$, then $x' = 0$.

Each space of a dual pair $\langle X, X' \rangle$ can be interpreted as a set of linear functionals on the other.

5.15 Topologies consistent with a given dual

Definition. (5.96) A locally convex topology τ on X is consistent (or compatible) with the dual pair $\langle X, X' \rangle$ if $(X, \tau)' = X'$. Consistent topologies on X' are defined similarly.

6 Normed spaces

6.1 Normed spaces and Banach spaces

6.2 Linear operators on normed spaces

6.3 The norm dual of a normed space

Definition. (6.7) The *norm dual* X' if a normed space $(X, \|\cdot\|)$ is Banach space $L(X, \mathbb{R})$. The operator norm on X' is also called the *dual norm,* also denoted $\|\cdot\|$. That is,

$$\|x'\| = \sup_{\|x\| \leq 1} |x'(x)| = \sup_{\|x\|=1} |x'(x)|$$

Theorem. *(6.8) The norm dual of a normed space is a Banach space.*

6.4 The uniform boundedness principle

6.5 Weak topologies on normed spaces

We have the following very important special case of Alaoglus Compactness Theorem 5.105.

Theorem (Alaoglu's theorem)**.** *(6.21) The closed unit ball of the norm dual of a normed space is weak* compact. Consequently, a subset of the norm dual of a normed space is weak* compact if and only if it is weak* closed and norm bounded.*

7 Convexity

7.1 Extended-valued convex functions

7.2 Lower semicontinuous convex functions

7.3 Support points

7.4 Subgradients

7.5 Supporting hyperplanes and cones

7.6 Convex functions on finite dimensional spaces

7.7 Separation and support in finite dimensional spaces

7.8 Supporting convex subsets of Hilbert spaces

7.9 The Bishop-Phelps Theorem

7.10 Support functionals

7.11 Extreme points of convex sets

Definition. (7.61) An *extreme subset* of a (not necessarily convex) subset C of a vector space, is a nonempty subset F of C with the property that if x belongs to F it cannot be written as a convex combination of points of C outside F. That is, if $x \in F$ and $x = \alpha y + (1-\alpha)z$, where $0 < \alpha < 1$ and $y, z \in C$, then $y, z \in F$. A point x is an *extreme point* of C if the singleton $\{x\}$ is an extreme set. The set of extreme points of C is denoted $\mathcal{E}(C)$.

Theorem (The Krein-Milman Theorem)**.** *(7.68) In a locally convex Hausdorff space X each nonempty convex compact subset is the closed convex hull of its extreme points.*

If X is finite dimensional, then every nonempty convex compact subset is the convex hull of its extreme points.

8 Riesz spaces

8.1 Orders, lattices and cones

8.2 Riesz spaces

An ordered vector space that is also a lattice is called a *Riesz space* or a *vector space.*

For a vector x in a Riesz space, the positive part x^+, the negative part x^-, and the absolute value $|x|$ are defined by

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0 \qquad \text{and} \qquad |x| = x \vee (-x).$$

Note that

$$x = x^+ - x^- \text{ and } |x| = x^+ + x^-.$$

Also note that $|x| = 0$ if and only if $x = 0$.

Examples: $\mathbb{R}^n$, $C(X)$, $C_b(X)$, $L_p(\mu)$ $(0 \le p \le \infty)$ with almost everywhere pointwise ordering, $ba(\mathcal{A})$ signed charges of bounded variations on a given algebra $\mathcal{A}$ of subsets of a set X, ℓ_p $(0 < p \le \infty)$ under pointwise ordering, c_0 under pointwise ordering

8.3 Order bounded sets

8.4 Order and lattice properties

8.5 The Riesz decomposition property

8.6 Disjointness

8.7 Riesz subspaces and ideals

8.8 Order converges and order continuity

8.9 Bands

If S is a nonempty subset of a Riesz space E, then its *disjoint complement* S^d, defined by

$$S^d = \{x \in E; |x| \wedge |y| = 0 \text{ for all } y \in S\}$$

is necessarily a band.

8.10 Positive functionals

8.11 Extending positive functionals

We can now state a more general form of the HahnBanach Extension Theorem. Its proof is a Riesz space analogue of the proof of Theorem 5.53; see [AB1, Theorem 2.1, p. 21].

TODO 8.30,

Theorem. *(8.31) Let F be a Riesz subspace of a Riesz space E and let $f\colon F \to \mathbb{R}$ be a positive linear functional. Then f extends to a positive linear functional on all of E if and only if there is a monotone sublinear function $p\colon E \to \mathbb{R}$ satisfying $f(x) \le p(x)$ for all $x \in F$.*

8.12 Positive operators

8.13 Topological Riesz spaces

Definition. (8.44) TODO

Definition. (8.45) A seminorm p on a Riesz space is a *lattice seminorm* (or a *Riesz seminorm*) if $|x| \le |y|$ implies $p(x) \le p(y)$ or, equivalently, if

1. p is absolute, $p(x) = p(|x|)$ for all x; and
2. p is monotone on the positive cone, $0 \leq x \leq y$ implies $p(x) \leq p(y)$.

Theorem. *(8.46) A linear topology on a Riesz space is locally convex-solid if and only if it is generated by a family of lattice seminorms.*

Example (Locally convex-solid Riesz spaces)**.** (8.47)
4. $ba(\mathcal{A})$=signed measures of bounded variation with the topology generated by the lattice norm $\|\mu\| = |\mu|(X)$. For details see Theorem 10.53.

9 Banach lattices

9.1 Fréchet and Banach lattices

Recall that a lattice norm $\|\cdot\|$ has the property that $|x| \leq |y|$ in E implies $\|x\| \leq \|y\|$. A Riesz space equipped with a lattice norm is called a *normed Riesz space.* A complete normed Riesz space is called a *Banach lattice.*

9.2 The StoneWeierstrass Theorem

9.3 Lattice homomorphisms and isometries

9.4 Order continuous norms

9.5 AM- and AL-spaces

10 Charges and measures

10.1 Set functions

$\mu\colon \mathcal{S} \to [-\infty, \infty]$ where $\mathcal{S}$ is a semiring.
signed charge = additive, assumes at most one of values $\pm\infty$ and $\mu(\emptyset) = 0$
charge = only nonnegative values
signed measure, measure

10.10 The AL-space of charges

If $\mu\colon X \to \langle -\infty, \infty \rangle$ is a signed charge, then *total variation* (or simply the *variation*) of μ is defined by

$$V_\mu = \sup\{\sum_{i=1}^{n} |\mu(A_i)|; \{A_1, \ldots, A_n\} \text{ is a partition of } X\}$$

A signed charge is of bounded variation if $V_\mu < \infty$.

Theorem. *(10.53) If $\mathcal{A}$ is an algebra of subsets of some set X, then its space of charges $ba(\mathcal{A})$ is an AL-space. Specifically:*

1. *The lattice operations on* $ba(\mathcal{A})$ *are given by*

$$(\mu \vee \nu)(A) = \sup\{\mu(B) + \nu(A \setminus B); B \in \mathcal{A} \text{ and } B \subset A\}; \text{ and}$$
$$(\mu \wedge \nu)(A) = \inf\{\mu(B) + \nu(A \setminus B); B \in \mathcal{A} \text{ and } B \subset A\}$$

2. *The Riesz space* $ba(\mathcal{A})$ *is order complete and* $\mu_\alpha \uparrow \mu$ *in the lattice sense if and only if* $\mu_\alpha(A) \uparrow \mu(A)$ *for each* $A \in \mathcal{A}$.

3. *The total variation* $|\mu| = V_\mu = |\mu|(X)$ *is the L-norm on* $ba(\mathcal{A})$.

11 Integrals

11.1 The integral of a step function

representation for a μ-step function

standard representation

The integral is defined using the standard representation, but Lemma 11.4 shows that the same value is obtained for any representation.

11.2 Finitely additive integration of bounded functions

finite charge μ on an algebra $\mathcal{A}$ of subsets of a set X

TODO *lower integral*, *upper integral*

Theorem. *(11.6) For a bounded function* $f\colon X \to \mathbb{R}$ *and a finite charge* μ *on an algebra of subsets of* X*, the following statements are equivalent.*

1. *The function* f *is integrable.*

2. *For each* $\varepsilon > 0$ *there exist two step functions* φ *and* ψ *satisfying* $\varphi \leq f \leq \psi$ *and* $\int(\psi - \varphi)d\mu < \varepsilon$.

3. *There exist sequences* $(\varphi_n)_{n=1}^{\infty}$ *and* $(\psi_n)_{n=1}^{\infty}$ *of step functions satisfying* $\varphi_n \uparrow\leq f$, $\psi_n \downarrow\geq f$, *and* $\int(\varphi_n - \psi_n)d\mu \downarrow 0$.

Let $\mathcal{A}_\mathbb{R}$ denote the algebra generated in $\mathbb{R}$ by the collection of all half open intervals $\{[a,b); a < b\}$.

Theorem. *(11.7) The collection of all bounded integrable functions with respect to a finite charge is a Riesz space, and in fact, a function space. Moreover, the integral is a* $\|\cdot\|_\infty$*-continuous positive linear functional on the vector space of bounded integrable functions.*

Theorem. *(11.8) Every bounded* $(\mathcal{A}, \mathcal{A}_\mathbb{R})$*-measurable function is integrable with respect to any finite charge.*

12 Measures and topology

13 L_p-spaces

14 Riesz Representation Theorems

14.1 The AM-space $B_b(\Sigma)$ and its dual

14.2 The dual of $C_b(X)$ for normal spaces

Theorem (Positive functionals on $C_b(X)$)**.** *Let X be a normal Hausdorff topological space and let $\Lambda\colon C_b(X) \to \mathbb{R}$ be a positive linear functional. Then there exists a unique finite normal charge μ on the algebra $\mathcal{A}_X$ generated by the open sets satisfying $\mu(X) = \|\Lambda\| = \Lambda(\mathbf{1})$ and*

$$\lambda(f) = \int f d\mu$$

for each $f \in C_b(X)$.

Theorem (Dual of $C_b(X)$, with X normal)**.** *(14.10) LetX be a Hausdorff normal topological space and let $\mathcal{A}_X$ be the algebra generated by the open subsets of X. Then the mapping $\Lambda\colon ba_n(\mathcal{A}_X) \to C_b'(X)$, defined by*

$$\lambda_\mu(f) = \int f d\mu = \int f d\mu^+ - \inf f d\mu^-,$$

is a surjective lattice isometry. In other words, the norm dual of the AM-space $C_b(X)$ can be identified with the AL-space $ba_n(\mathcal{A}_X)$.

Corollary (Dual of $\ell_\infty(\mathbb{N})$)**.** *(14.11) Let X be a set and let $\ell_\infty(X)$ denote the AM-space of all bounded real real functions on X. Then the norm dual of $\ell_\infty(X)$ coincides with $ba(X)$, the AL-space of all signed measures of bounded variation defined on the power set of X.*

14.3 The dual of $C_c(X)$ for locally compact spaces

14.4 Baire vs. Borel measures

15 Probability measures

15.1 The weak* topology on $\mathcal{P}(X)$

15.2 Embedding X in $\mathcal{P}(X)$

Theorem (Point masses are extreme)**.** *(15.9) If X is a separable metrizable topological space, then the set of extreme points of $\mathcal{P}(X)$ is identified with X under the embedding $x \mapsto \delta_x$.*

15.3 Properties of $\mathcal{P}(X)$

16 Spaces of sequences

16.1 The basic sequences spaces

Given a sequence x we define the *n-tail of x* by

$$x^{(n)} = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)$$

and the *n-head* by

$$^{(n)} = (x_1, \ldots, x_n, 0, 0, \ldots).$$

$\langle x, y \rangle = \sum\limits_{n=1}^{\infty} x_n y_n$

16.2 The sequence spaces $\mathbb{R}^{\mathbb{N}}$ and φ

16.6 ℓ_1 and the symmetric Riesz pair $\langle \ell_\infty, \ell_1 \rangle$

16.7 The sequence space ℓ_∞

Theorem. *(16.28) The AM-space ℓ_∞ is not separable.*

$\ell'_\infty = ba(\mathbb{N}) = ca(\mathbb{N}) \oplus pa(\mathbb{N})$

Lemma. *(16.29) A signed charge in $ba(\mathbb{N})$ is purely finitely additive if and only if it vanishes on the finite subsets of $\mathbb{N}$.*

Theorem. *(16.31) The norm dual of the AM-space ℓ_∞ is given by*

$$\ell'_\infty = \ell_1 \oplus \ell_1^d = ca \oplus pa,$$

with the following identifications

1. *The AL-spaces ℓ_1 and ca are identified via the lattice isometry $x \mapsto \mu_x$ defined by $\mu_x(A) = \sum_{n \in A} x_n$; and*

2. *The AL-spaces ℓ_1^d and pa are identified via the lattice isometry $\theta \mapsto \mu_\theta$ defined by $\mu_\theta(A) = \theta(\chi_A)$.*

Moreover, we have $(\ell_\infty)\tilde{}_n = \ell_1 = ca$ and $(\ell_\infty)\tilde{}_s = \ell_1^d = pa$.

To put it another way: *Every countably additive finite signed measure on $\mathbb{N}$ corresponds to exactly one sequence belonging to ℓ_1, and every purely additive finite signed charge corresponds to exactly one extension of a scalar multiple of the limit functional on c.*

16.8 More on $\ell'_\infty = ba(\mathbb{N})$

By mimicking the proof of Theorem 15.9, we see that the zero-one charges are the extreme points of the set of probability charges. They are also the charges generated by ultrafilters.

Lemma. *(16.35) A charge $\mu \in ba(\mathbb{N})$ is a zero-one-charge if and only if $\mu = \pi_{\mathcal{U}}$ for a unique ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Moreover for an ultrafilter $\mathcal{U}$:*

1. *If $\mathcal{U}$ is free, then $\pi_{\mathcal{U}}$ is purely finitely additive.*
2. *If $\mathcal{U}$ is fixed, then $\pi_{\mathcal{U}}$ is countably additive.*

16.9 Embedding sequence spaces

16.10 Banach-Mazur limits and invariant measures

Theorem. *(16.48) Every continuous function on a compact metrizable topological space has an invariant measure.*

17 Correspondences

18 Measurable correspondences

19 Markov transitions

20 Ergodicity

Contents

References

[AB1] Charalambos D. Aliprantis and Owen Burkinshaw. *Positive Operators.* Academic Press, New York, 1985. Pure and Applied Mathematics.

[AB2] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimentional Analysis, A Hitchhiker's Guide.* Springer, Berlin, 3rd edition, 2006.