Atiyah, MacDonald: Introduction to commutative algebra

Notes from [AM]:

References

Bourbaki: Algebre commutative
Grothendieck, Dieudonne: Elements de Geometrie Algebraique
Northcott: Ideal theory
Zariski, Samuel: Commutative algebra I, II

1 Rings and ideals

Throughout the book the word "ring" shall mean a commutative ring with an identity element.

We do not exclude possibility $0 = 1$ – *zero ring.*

1.1 Rings and rings homomorphisms

1.2 Ideals, quotient rings

1.3 Zero-divisors, nilpotent elements, unis

1.4 Prime ideals and maximal ideals

prime $\mathfrak{p} \neq (1)$

A ring A with exactly one maximal ideal is called a *local ring.* (Every field is a local ring.)

1.5 Nilradical and Jacobson radical

Proposition. *(1.7) The set $\mathfrak{N}$ of all nilpotent elements in a ring A is an ideal, and $A/\mathfrak{N}$ has no nilpotent element $\neq 0$.*

The ideal $\mathfrak{N}$ is called *nilradical* of A.

Proposition. *(1.8) The nilradical of A is the intersection of all the prime ideals of A.*

The *Jacobson radical* $\mathfrak{R}$ of A is defined as the intersection of all the maximal ideals.

Proposition. *(1.9) $x \in \mathfrak{R} \Leftrightarrow 1 - xy$ is a unit in A for all $y \in A$.*

1.6 Operation on ideals

If $\mathfrak{a}$ is any ideal of A, the *radical* of $\mathfrak{a}$ is

$$r(\mathfrak{a}) = \{x \in A; x^n \in \mathfrak{a} \text{ for some } n > 0\}.$$

Proposition. *(1.14) The radical of an ideal $\mathfrak{a}$ is the intersection of prime ideals that contain $\mathfrak{a}$.*

Radical of $\mathfrak{a}$ is sometimes denoted by $\sqrt{\mathfrak{a}}$ (e.g., [IR]).

1.7 Extension and contraction

Let $f\colon A \to B$ be a ring homomorphism. If $\mathfrak{a}$ is an ideal in A, the set $f(\mathfrak{a})$ is not necessarily an ideal in B. We define the extension $\mathfrak{a}^e$ of $\mathfrak{a}$ to be the ideal $Bf(\mathfrak{a})$ generated by $f(\mathfrak{a})$ in B.

If $\mathfrak{b}$ is an ideal of B, then $f^{-1}(\mathfrak{b})$ is always an ideal of A, called the contraction $\mathfrak{b}^c$ of $\mathfrak{b}$.

If $\mathfrak{b}$ is prime then $\mathfrak{b}^c$ is prime. If $\mathfrak{a}$ is prime, $\mathfrak{a}^e$ need not be prime.

TODO Properties of extension and contraction

1.8 Exercises

8. $A \neq 0$ is a ring $\Rightarrow$ the set of prime ideals has a minimal element.

11. A ring A is *Boolean* if $x^2 = x$ for all $x \in A$. In a Boolean ring:
    $2x = 0$ for all $x \in A$
    Every prime ideal is maximal and $A/\mathfrak{p}$ is a field with two elements.
    Every finitely generated ideal in A is principal. (The basic idea of the proof of this fact was $(x, y) = (x + xy + y)$, which followed from the identities $x(x + xy + y) = x$ and $y(x + xy + y) = y$.)

The prime spectrum of a ring

15. For $E \subseteq A$ let $V(E)$ denote the prime ideals of A which contain E.
    If $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
    The sets $V(E)$ fulfills the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology.* The topological space X is called *prime spectrum* of A, and is written $\operatorname{Spec}(A)$.

16. Spectrum of $\mathbb{Z}[x]$ can be found as a worked example in [R].

17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$.
    The sets X_f form a basis for the Zariski topology.
    X is quasi-compact.
    More generally, each X_f is quasi-compact.

20. iv) The irreducible components of $\operatorname{Spec}(A)$ are the closed sets $V(\mathfrak{p})$ where $\mathfrak{p}$ is a minimal prime ideal.

2 Modules

2.1 Modules and module homomorphisms

Let G be a finite group and $A = k[G]$ be a group algebra over the field k. (Thus A is not commutative, unless G is.) Then A-module = k-representation of A. More on group algebras and their representations can be found in [S, II.3] or [T, Appendix III]

$\hom_A(M, N)$ = module of all A-module homomorphisms from M to N

2.2 Submodules and quotient modules

submodule

quotient

Let $(M_i)_{i \in I}$ be a family of submodules of M. Their *sum* $\sum M_i$ is the set of all finite sums $\sum x_i$ where $x_i \in M_i$ and almost all the x_i (that is, all but a finite number) are zero. $\sum M_i$ is the smallest submodule of M which contains all the M_i.

2.3 Operation on submodules

Proposition. *(2.1)*

(i) $(L/N)/(M/N) \cong L/M$

(ii) $(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2)$

TODO $(N : P)$ *annihilator*

If $M = \sum_{i \in I} Ax_i$, the x_i are said to be a *set of generators* of M. An A-module M is said to be *finitely generated* if it has a finite sets of generators.

2.4 Direct sum and product

2.5 Finitely generated modules

free A-module

Proposition. *(2.4) Let M be a finitely generated A-module, let $\mathfrak{a}$ be an ideal of A and let ϕ be an A-module endomorphism of M with $\phi(M) \subseteq \mathfrak{a}M$. Then ϕ satisfies an equation of the form*

$$\phi^n + a_1\phi^{n-1} + \ldots + a_n = 0$$

where a_i are in $\mathfrak{a}$.

The proof uses Cramer's rule $A.\,\mathrm{adjA} = |A|.I$ in the ring of all A-morphisms of M. (This theorem should be valid in any commutative ring with unity.)

Corollary. *(2.5) Let M be a finitely generated module and let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M = M$. Then there exists $x \equiv 1 \pmod{\mathfrak{a}}$ such that $xM = 0$.*

Proposition (Nakayama's lemma). *(2.6) Let M be a finitely generated A-module and $\mathfrak{a}$ an ideal of A contained in the Jacobson radical $\mathfrak{R}$ of A. Then $\mathfrak{a}M = M$ implies $M = 0$.*

Corollary. *(2.7) Let M be a finitely generated module, N a submodule of M, $\mathfrak{a} \subset \mathfrak{R}$ an ideal. Then $M = \mathfrak{a}M + N$ implies $M = N$.*

Proposition. *(2.8) Let $\mathfrak{m}$ be a maximal ideal of M. Let x_i, $i = 1, 2, \ldots, n$, be elements of M whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then x_i generate M.*

2.6 Exact sequences

2.7 Tensor product of modules

2.8 Restriction and extension of scalars

Let $f \colon A \to B$ be a homomorphism of rings and let N be a B-module. Then N has an A-module structure defined as follows: if $a \in A$ and $x \in X$ then ax is defined to be $f(a)x$. This A-module is said to be obtained from N by *restriction of scalars.* In particular, f defines in this way an A-module structure on B.

Proposition. *(2.16) Suppose N is finitely generated as a B-module and that B is finitely generated as an A-module. Then N is finitely generated as an B-module.*

Let M be an A-module. We can define $M_B = B \otimes_A M$ (since B can be regarded as an A-module.) In fact M_B carries a B-module structure such that $b(b' \otimes x) = bb' \otimes x$ for all $b, b' \in B$ and $x \in M$. The B-module M_B is said to be obtained from M by *extension of scalars.*

Proposition. *(2.17) If M is finitely generated as an A-module, then M_B is finitely generated as a B-module.*

2.9 Exactness properties of the tensor product

Proposition. *(2.9)*

(i) *Let*

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0 \tag{1}$$

be a sequence of a modules and homomorphisms. The sequence (1) *is exact $\Leftrightarrow$ for all A-modules N, the sequence*

$$0 \longrightarrow \hom(M'', N) \xrightarrow{\overline{v}} \hom(M, N) \xrightarrow{\overline{u}} \hom(M', N) \tag{2}$$

is exact.

(ii) *Let*

$$0 \longrightarrow N' \xrightarrow{u} N \xrightarrow{v} N'' \qquad (3)$$

be a sequence of a modules and homomorphisms. The sequence (3) *is exact* $\Leftrightarrow$ *for all* A*-modules* M*, the sequence*

$$0 \longrightarrow \hom(M, N') \xrightarrow{\overline{u}} \hom(M, N) \xrightarrow{\overline{v}} \hom(M, N'') \qquad (4)$$

is exact.

2.10 Algebras

2.11 Tensor product of algebras

2.12 Exercises

1. $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.
   $am + bn = 1 \Rightarrow am = 1$ v $\mathbb{Z}_n$ and $bn = 1$ v $\mathbb{Z}_m$
   $1 \otimes 1 = n \times b \otimes m \times a = m \times b \otimes n \times a = 0 \otimes 0 = 0$
   $x \otimes y = xy(1 \otimes 1) = 0 \otimes 0$

3 Rings and modules of fractions

$S =$ *multiplicative closed subset* = closed under multiplication and $1 \in S$

$(a, s) \equiv (b, t) \Leftrightarrow (at - bs)u = 0$ for some $u \in S$[1]

$S^{-1}A$ = *ring of fractions* of A with respect to S

Proposition. *(3.1) Let* $g\colon A \to B$ *be a ring homomorphism such that* $g(s)$ *is a unit in* B *for all* $s \in S$*. then there exists a unique ring homomorphism* $h\colon S^{-1}A \to B$ *such that* $g = h \circ f$*.*

Examples: Localization $A_{\mathfrak{p}}$, $\mathfrak{p}$ is a prime ideal $S = A \setminus \mathfrak{p}$.
A_f for $S = \{f^n; n \in \mathbb{N}_0\}$
If M is a module we define $S^{-1}M$
$(m, s) \equiv (m', s') \Leftrightarrow (\exists t \in S)\ t(sm' - s'm) = 0$.
Examples: $M_{\mathfrak{p}}$, M_f.

Contents

[1] My question: Do we have to assume that all units are in S?

References

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[R] Miles Reid. *Undergraduate Commutative Algebra*. Cambridge University Press, Cambridge, 1995. London Mathematical Society Stuent Texts 29.

[S] Barry Simon. *Representation of Finite and Compact Groups*. AMS, 1996. Graduate Studies in Mathematics 10.

[T] Wu-Ki Thung. *Group Theory in Physics*. World Scientific, 1985.