Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos and Vaclav Zizler: Banach Space Theory. The Basis for Linear and Nonlinear Analysis

Notes from [FHHMZ].

1 Basic concepts in Banach spaces

1.1 Basic definitions

Definition. (1.3) $[x,y]=\{\lambda x+(1-\lambda y);0\leq\lambda\leq 1\}$ = *closed segment*
$(x,y)=\{\lambda x+(1-\lambda y);0<\lambda<1\}$ = *open segment* (assuming $x\neq y$)

A set $M\subset X$ is called symmetric if $(-1)M\subset M$, and balanced if $\alpha M\subset M$ for all $\alpha\in\mathbb{K}$, $|\alpha|\leq 1$.

1.2 Hölder and Minkowski Inequalities, Classical Spaces $C[0,1]$, ℓ_p, c_0, $L_p[0,1]$

Theorem (Hölder inequality)**.** *(1.10) $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$*

$$\sum_{k=1}^{n}|a_kb_k|\leq\left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{q}}\tag{1.1}$$

Lemma. *(1.11) $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$,*

$$ab\leq\frac{a^p}{p}+\frac{a^q}{q}$$

for all $a,b\geq 0$.

Definition. (1.15) c_{00} = sequences with finite support

Theorem (Hölder inequality)**.** *(1.19) If $p>1$, $1/p+1/q=1$, $f\in L_p$ and $g\in L_q$, then $fg\in L_1$ and*

$$\int_0^1|f(t)g(t)|\,\mathrm{d}t\leq\left(\int_0^1|f(t)|^p\right)^{1/p}\left(\int_0^1|g(t)|^q\right)^{1/q}=\|f\|_p\|g\|_q.\tag{1.4}$$

Lemma. *(1.22) A normed space X is a Banach space if and only if every absolutely convergent series in X is convegent.*

1.3 Operators, Quotients, Finite-Dimensional Spaces

$\mathcal{B}(X,Y)$ = bounded operators form X to Y

$\mathcal{B}(X) = \mathcal{B}(X,X)$

isomorphism = bijection such that $T \in \mathcal{B}(X,Y)$ and $T^{-1} \in \mathcal{B}(X,Y)$

Definition. (1.30) Let X, Y be isomorphic normed spaces. The *Banach-Mazur distance* between X and Y is defined by

$$d(X,Y) = \inf\{\|T\|.\|T^{-1}\|; T \text{ an isomorphism of } X \text{ onto } Y\}$$

Note that $d(X,Y) \geq 1$ and we have $d(X,Z) \leq d(X,Y)d(Y,Z)$. [1]

Definition. (1.31) $T \in \mathcal{B}(X,Y)$ is called a *compact operator* if $\overline{T(B_X)}$ is compact in Y.

$\mathcal{K}(X,Y)$ = space of all compact operators from X into Y

Definition. (1.33) $X \oplus Y$; $\|(x,y)\| = \|x\|_X + \|y\|_Y$

Definition. (1.34) X/Y (for closed subspace Y of a normed space x); $\|\hat{x}\| = \inf\{\|y\|; y \in \hat{x}\}$

Proposition. *(1.35) Let Y be a closed subspace of a Banach space X. Then X/Y is a Banach space.*

$q\colon X \to X/Y$ is continous operator and $\|q\| = 1$ (This can be shown using Riesz's lemma.)

Lemma (Riesz)**.** *(1.37) Let X be a normed space. If Y is a proper closed subspace of X then for every $\varepsilon > 0$ there is $x \in S_X$ such that* $\operatorname{dist}(x,Y) \geq 1-\varepsilon$.

Theorem. *(1.38) Let X be a normed space. The space X is finite-dimensional if and only if the unit ball B_X of X is compact.*

Proposition. *(1.39) Every operator T from a finite-dimensional normed space X into a normed space Y is continuous.*

Proposition. *(1.40) $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$*

Proposition. *(1.42)*

(i) If $p \in [1,\infty)$, then the space ℓ_p is separable.

(ii) The spaces c and c_0 are separable.

(iii) The space ℓ_∞ is not separable.

Proposition. *(1.43)*

(i) The space $C[0,1]$ is separable.

(ii) If $p \in [1,\infty)$, then L_p is separable.

(iii) The space L_∞ is not separable.

Proposition. *(1.44) $\mathcal{B}(\ell_2)$ contains an isometric copy of ℓ_∞ and thus it is not separable*

[1]In both parts we can use $\|S \circ T\| \leq \|S\| \cdot \|T\|$.

1.4 Hilbert spaces

Theorem (Riesz)**.** *(1.49) Let F be a subspace of a Hilbert space H. If F is closed, then $F + F^{\perp} - H$. Thus $T\colon F \oplus F^{\perp} \to H$ defined by $T(x, y) = x + y$ is an isomorphism of $F \oplus F^{\perp}$ onto H, and so H is the topological direct sum of F and $F^{\perp}$.*

Corollary. *(1.50) If F is a closed subspace of a Hilbert space H, then F is one-complemented in H, i.e., there is a liner projection of norm* 1 *from H onto F.*

Proposition. *(1.51) Let H be a Hilbert space and F be a subspace of H. Then F is linearly isometric to $H/F^{\perp}$.*

Definition. (1.52) A maximal orthonormal set in H is called an *orthonormal basis* of H.

Theorem. *(1.53) Every Hilbert space has an orthonormal basis.*

Theorem. *(1.54) Let H be a Hilbert space, and let H_0 be a closed subspace of H_0 can be extended to an orthonormal basis of H.*

Theorem. *(1.55) Every separable infinite-dimensional Hilbert space H has an orthonormal basis $(e_i)_{i=1}^{\infty}$.*

Moreover, if $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of H, then for every $x \in H$,

$$x = \sum_{i=1}^{\infty} (x, e_i) e_i.$$

Proposition. *(1.56) Let $(e_i)_{i=1}^{\infty}$ be an orthonormal set in a Hilbert space H and $x \in H$. then*

(i) (The Bessel inequality)

$$\sum_{i=1}^{\infty} |(x, e_i)|^2 \leq \|x\|^2 \tag{1}$$

(ii) (The Parseval equiality) If $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of H, then

$$\|x\|^2 = \sum_{i=1}^{\infty} |(x, e_i)|^2 \tag{2}$$

(iii) If the Parseval equality holds for every $x \in H$, then $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of H.

(iv) If $\overline{\operatorname{span}}((e_i)_{i=1}^{\infty}) = H$, then $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of H.

Theorem (Riesz, Fischer)**.** *(1.57) Every separable infinite-dimensional Hilbert space H is linearly isometric to ℓ_2.*

1.5 Remarks and Open Problems

Exercises for Chapter 1

(1.15) Let $1 \leq p \leq q \leq \infty$. Then $\|x\|_{\ell_q} \leq \|x\|_{\ell_p}$ for $x \in \ell_p$ and $\|f\|_{L_p} \leq \|f\|_{L_q}$ for $f \in L_p[0,1]$. [2]

In particular, $\ell_p \subset \ell_q$ and, if $1 \leq p < \infty$, then $\ell_p \subset c_0$. Moreover, $L_q[0,1] \subset L_p[0,1]$. All the corresponding operators have norm one.

(1.15) Hilbert cube $Q = \{x = (x_i) \in \ell_2; (\forall i)|x_i| \leq 2^{-i}\}$ is a compact set in ℓ_2.[3]

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[2] If we use (1.4) for functions $|f|^p$ and $g(x) = 1$ and for $r = \frac{q}{p} > 1$, we get

$$\int_0^1 |f(x)|^p \, \mathrm{d}x \leq \int_0^1 (|f(x)|^q)^{p/q} \, \mathrm{d}x,$$

which is equivalent to $\|f\|_{L_p}^p \leq \|f\|_{L_q}^p$.
Proof using Jensen's inequality:

$$\int_0^1 |f(x)|^q \, \mathrm{d}x = \int_0^1 (|f(x)|^p)^{\frac{q}{p}} \, \mathrm{d}x \geq \int_0^1 (|f(x)|^p)^{\frac{q}{p}}$$

(using the fact that $q/p \geq 1$ and the function $x^{q/p}$ is convex.) The above inequality is equivalent to $\|f\|_{L_q}^q \geq \|f\|_{L_p}^p$.
Similarly one can show $\|f\|_p \leq \mu(S)^{(1/p)-(1/q)}\|f\|_q$ for spaces $L_p(S,\mu)$, $L_q(S,\mu)$ such that $\mu(S) < \infty$.

[3] As a topological space, Q is homeomorphic to $[0,1]^\omega$.

2 Hahn-Banach and Banach Open Mapping Theorems

3 Weak Topologies and Banach Spaces

3.1 Dual Pairs,Weak Topologies

3.2 Topological Vector Spaces

3.3 Locally Convex Spaces

3.4 Polarity

3.5 Topologies Compatible with a Dual Pair

3.6 Topologies of Subspaces and Quotients

3.7 Weak Compactness

3.8 Extreme Points, KreinMilman Theorem

3.9 Representation and Compactness

3.10 The Space of Distributions

3.11 Banach Spaces

3.11.1 Banach-Steinhaus Theorem

3.11.2 BanachDieudonné Theorem

3.11.3 The Bidual Space

3.11.4 The Completion of a Normed Space

3.11.5 Separability and Metrizability

3.11.6 Weak Compactness

3.11.7 Reflexivity

3.11.8 Boundaries

Dirac Deltas and Extreme Points of $B_{C(K)^*}$

James Boundaries

Strong James Boundaries

James Boundaries and James Theorem

The RainwaterSimons Theorem

Corollary (Rainwater)**.** *(3.137) Let* X *be a Banach space, let* $\{x_n\}$ *be a bounded sequence in* X *and* $x \in X$*. If* $f(x_n) \to f(x)$ *for every* $f \in \mathrm{Ext}(B_{X^*})$*, then* $x_n \overset{w}{\to} x$*.*

3.12 Remarks and Open Problems

Contents

References

[FHHMZ] Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos, and Vaclav Zizler. *Banach Space Theory. The Basis for Linear and Nonlinear Analysis.* Springer, New York, 2011. CMS books in mathematics.