Halbeisen: Combinatorial Set Theory

Notes from [H].

1 The Setting

A reason for its wide range of applications might be that Combinatorics is rather a way of thinking than a homogeneous theory, and consequently Combinatorics is quite difficult to define.

Königs Lemma. Every infinite, finitely branching tree contains an infinite branch.

Infinite Pigeon-Hole Principle. If infinitely many objects are coloured with finitely many colours, then infinitely many objects have the same colour.

Notes

König's Lemma and Ramsey's Theorem. Ramsey's Theorem can also be considered as a proper choice principle which turns out to be even stronger than König's Lemma (see Theorem 5.17).

Part I: Topics in Combinatorial Set Theory

2 Overture: Ramsey's Theorem

The Nucleus of Ramsey Theory

$\omega = \{0, 1, 2, \dots\}$ = the set of natural numbers
$[S]^n$ = all n-elements subsets of S
$[S]^{<\omega}$= the set of all finite subsets of a set S
$[S]^\omega$ denotes the set of all countably infinite subsets of S, in particular, $[\omega]^\omega$ is the set of all infinite subsets of ω.

For sets A and B, let ${}^AB$ denote the set of all functions $f\colon A \to B$.

We would also like to mention that Ramsey's original theorem, which will be discussed later, is somewhat stronger than the theorem stated below but is, like König's Lemma, not provable without assuming some form of the Axiom of Choice (see Proposition 7.8).

Theorem (Ramsey's theorem)**.** *(2.1) For any number $n \in \omega$, for any positive number $r \in \omega$, for any $S \in [\omega]^\omega$, and for any colouring $\pi\colon [S]^n \to r$, there is always an $H \in [S]^o mega$ such that H is homogeneous for π, i.e., the set $[H]^n$ is monochromatic.*

Proposition. *(2.2) For any positive number* $r \in \omega$*, for any* $S \in [\omega]^\omega$*, and for any colouring* $\pi\colon [S]^2 \to r$*, there is always an* $H \in [S]^\omega$ *such that* $[H]^2$ *is monochromatic.*

Corollaries of Ramsey's Theorem

Corollary (Finite Ramsey Theorem)**.** *(2.3) For all* $m, n, r \in \omega$*, where* $r \geq 1$ *and* $n \leq m$*, there exists an* $N \in \omega$*, where* $N \in \omega$*, such that for every colouring of* $[N]^n$ *with* r *colours, there exists a set* $H \in [N]^m$*, all of whose* n*-element subsets have the same colour.*

TODO 2.3,4,5,6

Notes

The Paris-Harrington Result. As mentioned above, Corollary 2.6 is true but unprovable in Peano Arithmetic (also called First-Order Arithmetic). This result was the first natural example of such a statement.

Related results

3. Hindman's Theorem.

11. Applications of Ramsey Theory to Banach Space Theory. see [28], [13], or [1].

3 The Axioms of ZermeloFraenkel Set Theory

A set is called an ordinal number, or just an ordinal, if it is transitive and well-ordered by $\in$. The collection of all ordinal numbers is denoted by Ω.

natural numbers $- \omega =$ the smallest inductive set

3.18 *Cantor products*

More about Cantor products can be found for example in Perron [92, 35]. [1]

Models of ZF

V_α

cumulative hierarchy

transitive closure

3.22

[1] Oskar Perron: Irrationalzahlen, 2nd edn. Chelsea, New York (1951)

Cardinals in ZF

TODO

$$|A| = \{B \in V_{\beta_0} : \text{there exists a bijection between } B \text{ and } A\}$$

aleph
TODO 3.27

4 Cardinal relations in ZF Only

5 Axiom of Choice

König's Lemma and Other Choice Principles

TODO 5.16, 5.17

7 Models of Set Theory with Atoms

The Second Fraenkel Model

The following result shows that in $\mathcal{V}_{F_2}$, König's Lemma fails even for binary trees.

Proposition. *(7.7) In $\mathcal{V}_{F_2}$ there exists an infinite binary tree which does not have an infinite branch.*

In a similar way one can show that Ramsey's original theorem fails in $\mathcal{V}_{F_2}$:

Proposition. *(7.8) In $\mathcal{V}_{F_2}$ there exist an infinite set S and a 2-colouring of $[S]^2$ such that no infinite subset of S is homogeneous.*

8 Twelve Cardinals and Their Relations

The Cardinals ω_1 and $\mathfrak{c}$

The Cardinal $\mathfrak{p}$

Furthermore, a family $\mathscr{F} \subseteq [\omega]^\omega$ has the *strong finite intersection property* (sfip) if every finite subfamily has infinite intersection.
pseudointersection $\Rightarrow$ sfip
sfip $\not\Rightarrow$ preusdointersection: Any free ultrafilter.

Definition. The *pseudo-intersection number* $\mathfrak{p}$ is the smallest cardinality of any family $\mathscr{F} \subseteq [\omega]^\omega$ which has the *sfip* but which does not have a pseudo-intersection; more formally

$$\mathfrak{p} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq [\omega]^\omega \text{ has the } \textit{sfip} \text{ but no pseudo-intresection}\}.$$

Every ultrafilter on $[\omega]^\omega$ is of cardinality $\mathfrak{c} \Rightarrow \mathfrak{p} \leq \mathfrak{c}$.

Theorem. *(8.1)* $\omega_1 \leq \mathfrak{p}$

The Cardinals $\mathfrak{b}$ and $\mathfrak{d}$

For $f, g \in {}^\omega\omega$ we say that g *dominates* f, denoted $f <^* g$ ...

A family $\mathscr{D} \subseteq {}^\omega\omega$ is *dominating* for each $f \in {}^\omega\omega$ there is a function $g \in \mathscr{D}$ such that $f <^* g$.

Definition. The dominating number $\mathfrak{d}$ is the smallest cardinality of any dominating family; more formally

$$\mathfrak{d} = \min\{|\mathscr{D}|; \mathscr{D} \subseteq {}^\omega\omega \text{ is dominating}\}.$$

A family $\mathscr{B} \subseteq {}^\omega\omega$ is unbounded if there is no single function $f \in {}^\omega\omega$ which dominates all functions of $\mathscr{B}$.

Definition. The bounding number $\mathfrak{b}$ is the smallest cardinality of any unbounded family; more formally

$$\mathfrak{b} = \min\{|\mathscr{B}|; \mathscr{B} \subseteq {}^\omega\omega \text{ is unbounded}\}.$$

Fact 8.2:

$$\mathfrak{b} \leq \mathfrak{d}$$

Every dominating family is unbounded.

Theorem. *(8.3)* $\omega_1 \leq \mathfrak{b}$.

The Cardinals $\mathfrak{s}$ and $\mathfrak{r}$

A set $x \subseteq \omega$ *splits* an infinite set $y \in [\omega]^\omega$ if both $y \cap x$ and $y \setminus x$ are infinite (i.e., $|y \cap x| = |y \setminus x| = \omega$). Notice that any $x \subseteq \omega$ which splits a set $y \in [\omega]^\omega$ must be infinite. A *splitting family* is a family $\mathcal{S} \subseteq [\omega]^\omega$ such that each $y \in [\omega]^\omega$ is split by at least one $x \in \mathcal{S}$.

Definition. The *splitting number* $\mathfrak{s}$ is the smallest cardinality of any splitting family; more formally

$$\mathfrak{S} = \min\{|\mathcal{S}|; \mathcal{S} \subseteq [\omega]^\omega \text{ is splitting.}\}$$

Theorem. *(8.4)* $\mathfrak{s} \leq \mathfrak{d}$

Theorem. *(8.5)* $\mathfrak{b} \leq \mathfrak{r}$

The Cardinals $\mathfrak{a}$ and $\mathfrak{i}$

The Cardinals $\mathfrak{par}$ and $\mathfrak{hom}$

The Cardinal $\mathfrak{h}$

A family $\mathscr{H} = \{\mathscr{A}_\xi; \xi \in \kappa\} \subseteq \mathscr{P}([\omega]^\omega)$ of mad families of cardinality $\mathfrak{c}$ is called *shattering* if for each $x \in [\omega]^\omega$ there is a $\xi \in \kappa$ such that x has infinite intersection with at least two distinct members of $\mathscr{A}_\xi$, i.e., at least two sets of $\mathscr{A}_\xi$ split x.

Definition. The *shattering number* $\mathfrak{h}$ is the smallest cardinality of any shattering family; more formally

$$\mathfrak{h} = \min\{|\mathscr{H}|; \mathscr{H} \text{ is shattering}\}.$$

Theorem. *(8.16)* $\mathfrak{p} \leq \mathfrak{h}$

Notes

[vD], [V]

Related results

52. A family $\mathscr{T} = \{T_\alpha; \alpha \in \kappa\} \subseteq [\omega]^\omega$ is called a *tower* if $\mathscr{T}$ is well-ordered by $^* \supseteq$ (i.e., $T_\beta \subseteq^* T_\alpha \leftrightarrow \alpha < \beta$) and does not have a pseudointersection. The *tower number* $\mathfrak{t}$ is the smallest cardinality (or height) of a tower. Obviously we have $\mathfrak{p} \leq \mathfrak{t}$ and the proof of Theorem 8.16 shows that $\mathfrak{t} \leq \mathfrak{h}$. However, it is open whether $\mathfrak{p} < \mathfrak{t}$.

56. The ultrafilter number $\mathfrak{u}$ is the smallest cardinality of any ultrafilter base. We leave it as an exercise to the reader to show that $\mathfrak{r} < \mathfrak{u}$.

9 The Shattering Number Revisited

10 Happy Families and Their Relatives

Happy Families

Ramsey Ultrafilters

Ramsey ultrafilter TODO

Proposition. *(10.7) TODO*

Proposition. *(10.9) If* $\mathfrak{p} = \mathfrak{c}$*, then there exists a Ramsey ultrafilter.*

P-points and Q-points

A free ultrafilter $\mathscr{U}$ is a *P-point* if for each partition $\{u_n \subseteq \omega; n \in \omega\}$ of ω either $u_n \in \mathscr{U}$ for a (unique) $n \in \omega$, or there exists an $x \in \mathscr{U}$ such that for each $n \in \omega$, $x \cap u_n$ is finite.

Furthermore, a free ultrafilter $\mathscr{U}$ is a *Q-point* if for each partition of ω info finite pieces $\{I_n \subseteq \omega; n \in \omega\}$ (i.e., for each $n \in \omega$, I_n is finite), there exists an $x \in \mathscr{U}$ such that for each $n \in \omega$, $x \cap I_n$ has at most one element.

Fact. (10.10) $\mathscr{U}$ is a Ramsey ultrafilter if and only if $\mathscr{U}$ is a P-point and a Q-point.

Fact. (10.11) For every free ultrafilter $\mathscr{U}$, the following conditions are equivalent:

(a) $\mathscr{U}$ is a P-point.
(b) For every family $\{x_n; n \in \omega\} \subseteq \mathscr{U}$ there is an $x \in \mathscr{U}$ such that for all $n \in \omega$, $x \subseteq^* x_n$ (i.e., $x \setminus x_n$ is finite).
(c) For every family $\{x_n; n \in \omega\} \subseteq \mathscr{U}$ there is a function $f \in {}^\omega\omega$ and a set $x \in \mathscr{U}$ such that for all $n \in \omega$, $x \setminus f(n) \subseteq x_n$.

Ramsey Families and P-families

Notes

Related Results

70. *Rapid and unbounded filters.* A free filter $\mathscr{F} \subseteq [\omega]^\omega$ is called a *rapid filter* if for each $f \in {}^\omega\omega$ there exists an $x \in \mathscr{F}$ such that for all $n \in \omega$, $|x \cap f(n)| \leq n$.

Q-points $\Rightarrow$ rapid (see Fact 25.10), but the converse does not hold (see for example [BJ, Lemma 4.6.3])

However, like for P-points or Q-points, the existence of rapid filter is independent of ZFC (see Proposition 25.11).

References

11 Coda: A Dual Form of Ramsey's Theorem

The Hales–Jewett Theorem

Theorem (van der Waerden)**.** *(11.1) For any positive integers r and n, there is a positive integer N such that for every r-colouring of the set $\{0, 1, \ldots, N\}$ we find always a monochromatic (non-constant) arithmetic progression of length n.*

Theorem (Hales-Jewett Theorem)**.** *(11.2) For all positive integers $n, r \in \omega$ there exists a positive integer $l \in \omega$ such that for any r-colouring of ${}^l n$ there is always a non-empty set $s \subseteq l$ and a function $g \colon l \setminus s \to n$ such that $\{f \in {}^l n : f|_{l \setminus s} = g \wedge f \text{ is constant}\}$ is monochromatic.*

Families of Partitions

Carlson's Lemma and the Partition Ramsey Theorem

A Weak Form of the HalpernLäuchli Theorem

Part II: From Martins Axiom to Cohens Forcing

Part III: Combinatorics of Forcing Extensions

12 Properties of Forcing Extensions

13 Cohen Forcing Revisited

14 Silver-Like Forcing Notions

A Model in Which $\mathfrak{d} < \mathfrak{r}$

Proposition. *(22.4)* $\omega_1 = \mathfrak{d} < \mathfrak{r} = \mathfrak{c}$ *is consistent with ZFC.*

15 Miller Forcing

Contents

References

[BJ] Tomek Bartoszyński and Haim Judah. *Set Theory. On the Structure of Real Line.* A.K.Peters, Wellesely, Massachusetts, 1995.

[H] Lorenz J. Halbeisen. *Combinatorial Set Theory. With a Gentle Introduction to Forcing.* Springer-Verlag, London, 2012. Springer Monographs in Mathematics.

[V] J. Vaughan. Small uncountable cardinals and topology. In J. van Mill and G. M. Reed, editors, *Open Problems in Topology*, pages 197–218. North-Holland, Amsterdam, 1990.

[vD] E. K. van Douwen. The integers and topology. In K. Kunen and J. E. Vaughan, editors, *Handbook of set-theoretic topology*, pages 111–167. North-Holland, Amsterdam, 1984.