Haworth, McCoy: Baire spaces

Notes from [HM].

Introduction

In some instances one needs to have a "complete" space only to use a theorem such as the Baire Category Theorem, so that being a Baire space all that is really necessary. This is the case in such well-known theorems as the Closed Graph Theorem, the Open Mapping Theorem and the Uniform Boundedness Theorem.

The letters N, Q, and J will represent the natural numbers, the rational numbers, and the irrational numbers, respectively.

1 Basic properties of Baire spaces

1.1 Nowhere dense sets

Let A be a subset of a topological space X. Then A is *dense* in X if $\operatorname{cl} A = X$, and A is *somewhere dense* in X if $\operatorname{Int}\operatorname{cl} A \neq \emptyset$. If A is not somewhere dense in X then it is called *nowhere dense* in X.

Proposition. *(1.2) Let $\mathcal{N}$ be a family of nowhere dense subsets of X. If $\mathcal{N}$ is locally finite at a dense set of points of X, then $\bigcup\mathcal{N}$ is nowhere dense in X.*

Proposition. *(1.3) Let Y be a subspace of X, and let N be a subset of Y. If N is nowhere dense in Y, then N is nowhere dense in X. Conversely, if Y is open (or dense) in X and N is nowhere dense in X, then N is nowhere dense in Y.*

Proposition. *(1.4) For each $a \in A$ let N_a be a subset of the space X_a. Then, $\prod\limits_{a\in A} N_a$ is nowhere dense in $\prod\limits_{a\in A} X_a$ if and only if for some $\beta \in A$, N_β is nowhere dense in X_β or $\operatorname{cl} N_a \neq X_a$ for infinitely many $a \in A$.*

1.2 First and second category sets

first category = *meager* = countable union of nowhere dense sets

second category = *nonmeager*

Proposition 1.3 points out that if Y is an open or dense subset of a space X, and if $A \subset Y$, then the category of A relative to Y is the same as the category of A relative to X.

Proposition. *(1.5) Every dense G_δ-subset of a space of second category is of second category.*

Theorem (Banach category theorem)**.** *(1.6) In a topological space X, the union of any family of open sets of first category is of first category.*

Theorem. *(1.7) Let A be a subset of the space X, and suppose that for every nonempty open set U there exists a nonempty open set V contained in U such that $V \cap A$ is of first category in X. Then A is of first category in X.*

Theorem. *(1.8) A space X is of second category if and only if the intersection of any (monotone decreasing) sequence of dense open subsets is nonempty.*

Theorem. *(1.9) Every T_1-space with no isolated points having a σ-locally finite base has a dense subspace which is of first category.*

A collection $\mathcal{P}$ of nonempty open sets in a space X is a *pseudo-base*[1] for X if every nonempty open subset of X contains at least one member of $\mathcal{P}$. A pseudo-base $\mathcal{P}$ is said to be *locally countable* if each member of $\mathcal{P}$ contains only countably many members of $\mathcal{P}$.

Lemma. *(1.10) Let X and Y be spaces with Y having a countable pseudo-base. If N is nowhere dense (of first category, resp.) in $X \times Y$, then N_x is nowhere dense (of first category, resp.) in Y for all x except a set of first category in X.*

Theorem. *(1.11) Let X and Y be spaces with at least one of them having a locally countable pseudo-base. Let $A \subset X$ and $B \subseteq Y$. Then $A \times B$ is of first category in $X \times Y$ if and only if A is of first category in X or B is of first category in Y.*

Corollary. *(1.12) Let X and Y be spaces with at least one of them having a locally countable pseudo-base. Let $A \subset X$ and $B \subseteq Y$. Then $A \times B$ is of second category in $X \times Y$ if and only if A is of second category in X and B is of second category in Y.*

1.3 Baire spaces

A *Baire space* is a topological space such that every nonempty open subset is of second category.

Theorem. *(1.13) The following are equivalent for a space X:*
- *(i) X is a Baire space.*
- *(ii) The intersection of any (monotone decreasing) sequence of dense open sets is dense in X.*
- *(iii) The complement of any set of first category in X is dense in X.*
- *(iv) Every countable union of closed sets with no interior points in X has no interior point in X.*

Proposition. *(1.14) Every open subspace of a Baire space is a Baire space.*

[1] Some authors use the name π-base for this notion; e.g. [J, p.5]

In contrast, not every closed subspace of a Baire space is a Baire space, as can be seen by taking the space $E^2 - \{(x,0)|x \in J\}$. The closed subspace $\{(x,0)|x \in Q\}$ is clearly of first category.

Theorem. *(1.15) Every space which contains a dense Baire subspace is a Baire space.*

Proposition. *(1.16) X is a Baire space if and only if the complement of every nonempty subset of first category in X is a Baire space.*

Proposition. *(1.17) Let Y be a subspace of the Baire space X. Then Y is of first category in X if and only if $X - Y$ contains a dense G_δ-subset of X.*

Proposition. *(1.18) In a topological space X, the union of any family of open Baire subspaces is a Baire space.*

Proposition. *(1.19) In a topological space X, the union of a finite number of Baire subspaces is a Baire space.*

Actually Proposition 1.19 is true for a locally finite collection of Baire subspaces (see [M]).

Proposition. *(1.20) Every disjoint topological sum of Baire spaces is a Baire space.*

A *pseudo-cover* (also called *almost cover* or *proximate cover*) for a space X is a collection of subsets of X whose union is dense in X. A pseudo-cover $\mathcal{U}$ is said to be open if each member of $\mathcal{U}$ is open in X.

Theorem. *(1.21) If X has an open pseudo-cover, each member of which is a Baire space, then X is a Baire space.*

Corollary. *(1.22) X is a Baire space if and only if each point of X has a neighborhood which is a Baire space.*

Proposition. *(1.23) Every dense G_δ-subspace of a Baire space is a Baire space.*

Theorem. *(1.24) Let X be a dense subspace of the Baire space Y.*
- *(i) X is a Baire space if and only if every somewhere dense G_δ-subset of Y intersects X.*
- *(ii) X is a Baire space if and only if every G_δ-subset of Y contained in $Y - X$ is nowhere dense in Y.*
- *(iii) If every G_δ-subset of Y contained in $Y - X$ is nowhere dense in $Y - X$, then X is a Baire space.*
- *(iv) If $Y - X$ is dense in Y, then X is a Baire space if and only if every G_δ-subset of Y contained in $Y - X$ is nowhere dense in $Y - X$.*

1.4 Isolated points and Baire spaces

Proposition. *(1.28) In every topological space X there are open (possibly empty) subspaces X_B and X_D such that*

a) $X_B \cap X_D = \emptyset$*, and* $X_B \cup X_D$ *is dense in* X*;*
b) X_B *is a Baire space;*
c) *and every singleton subset of* X_D *is nowhere dense in* X_D*.*

Furthermore, X *is a Baire space if and only if* X_D *is a Baire space.*

Proposition. *(1.30) If X is a countable Baire T_1-space, then the set of isolated points of X is dense in X. Furthermore, if X is countably infinite, then the set of isolated points of X is also infinite.*

Proposition. *(1.31) Let X be a Baire T_1-space with no isolated points, and let G be a somewhere dense G_δ-subset of X. If C is a countable subset of G, then $G - C$ is a somewhere dense G_δ-subset of X.*

Proposition. *(1.32) Let X be a Baire T_1-space with no isolated points, and let G be a somewhere dense G_δ-subset of X. If D is a dense first category subset of X, then $G \cap (X - D)$ is uncountable.*

2 Concepts related to Baire spaces

2.1 Baire spaces in the strong sense

A space X is a *Baire space in the strong sense* (also called *totally non-meager*) if every nonempty closed subspace is of second category in itself.

The next proposition: Baire space in the strong sense $\Rightarrow$ a Baire space. The example given after Proposition 1.14 is a metric Baire space that is not a Baire space in the strong sense.

Proposition. *(2.1) X is a Baire space in the strong sense if and only if every nonempty closed subspace is a Baire space.*

Proposition. *(2.2) Every G_δ-subspace of a Baire space in the strong sense X is a Baire space in the strong sense.*

2.2 Baire Category Theorem

Theorem. *(2.3) Every locally compact Hausdorff space X is a Baire space, and hence is a Baire space in the strong sense.*

Theorem (Baire Category)**.** *(2.4) Every complete metric space X is a Baire space, and hence is a Baire space in the strong sense.*

Van Doren [67] has shown that the closed continuous image of a complete metric space contains a dense completely metrizable subspace. Thus, the Baire Category Theorem is valid for every closed continuous image of a complete metric space (see Theorem 4.10).

The example given after Proposition 1.14 is a metrizable Baire space mm is neither topologically complete nor locally compact.

2.3 Complete type properties which imply Baire

A Tychonoff space X is *complete in the sense of Čech* if there exists a sequence $\{\mathcal{U}_i\}$ of open coverings of X such that for every family of closed sets $\{F_a | a \in A\}$ which has the finite intersection property, and which has the property that for each i, there exists an F_a contained in some $U \in \mathcal{U}_i$; $\bigcap\limits_{a\in A} F_a \neq \emptyset$.

It is well known that for a metric space complete in the sense of Čech is equivalent to being topologically complete. The above definition was chosen so that one could easily see that the Čech complete spaces contain the almost countably complete spaces.

A space is *quasi-regular* if every nonempty open set contains the closure of some nonempty open set. A quasi-regular space X is *almost countably complete* if there exists a sequence $\{\mathcal{P}_n\}$ of pseudo-bases for X such that for every sequence of sets $\{U_i\}$ which has the finite intersection property $U_{n_k} \in \mathcal{P}_{n_k}$; $\bigcap\limits_{k=1}^{\infty} \operatorname{cl} U_{n_k} \neq 0$. A space X is *pseudo-complete* if it is quasi-regular and if there exists a sequence $\{\mathcal{P}_i\}$ of pseudo-bases for X such that for every sequence of sets $\{U_i\}$ with $U_i \in \mathcal{P}_i$ and $\operatorname{cl} U_{i+1} \subset U_i$ for each i; $\bigcap\limits_{i=1}^{\infty} U_i \neq \emptyset$.

Pseudo-completeness has an interesting generalization called *weakly α-favorable* which utilizes ideas from game theory. [2]

Also see the section on *Banach-Mazur game.*

It is easy to see that every pseudo-complete space is weakly α-favorable. White [70] shows that the concepts of weakly α-favorable and pseudo-complete coincide for the class of quasi-regular spaces which have dense metrizable subspaces.

Theorem. *(2.5) Every weakly α-favorable spaces X is a Baire space.*

The example given after Proposition 1.14 is pseudo-complete and, thus, weakly α-favorable. However, it is not a Baire space in the strong sense.

Lemma ([Ku, p.514-515, Lemma III.40.2])**.** *Let R be a set of cardinality $\mathfrak{c}$ and $\mathbf{M}$ be a family of subsets of R such that $|\mathbf{M}| \leq \mathfrak{c}$ and each element of $\mathbf{M}$ has cardinality $\mathfrak{c}$. Then R contains a subset Z such that both Z and $R - Z$ have cardinality $\mathfrak{c}$ and both contain at least one element from each set belonging to the family $\mathbf{M}$.*

Theorem. *(2.6) If $(X, \mathcal{T})$ is a separable completely metrizable space*[3] *with no isolated points, then there exists a subset Z of X with the following properties:*

(i) Both Z and $X - Z$ are dense in X, have cardinality $\mathfrak{c}$, and are Baire spaces in the strong sense;

(ii) If Y is a subspace of Z or $X - Z$ that does not have an isolated point, then Y is not weakly α-favorable.

[2] weakly α-favorable space is called *Choquet space* in [Ke] $\Leftrightarrow$ Player II has a winning strategy in Choquet game.

[3] =Polish space

[4]

Proposition. *(2.7) Every complete pseudo-semi-metric quasi-regular space* $(X, \mathcal{T})$ *is pseudo-complete.*

closed base, *cospace*, *cocompact* [5]

An *open filter base* $\mathcal{F}$ on a space X is a nonempty collection of nonempty open subsets of X such that whenever U and V are members of $\mathcal{F}$, then there exists a member W of $\mathcal{F}$ with $W \subseteq U \cap V$. $\mathcal{F}$ is *regular* if, whenever U is a member of $\mathcal{F}$ then there exists a member V of $\mathcal{F}$ with $\operatorname{cl} V \subset U$. A space $(X, \mathcal{T})$ is *countably subcompact* if there exists an open base $\mathcal{B}$ for $(X, \mathcal{T})$ such that whenever $\{U_i\}$ is a countable regular filter base contained in $\mathcal{B}$, $\bigcap_{i=1}^{\infty} U_i \neq \emptyset$.

Theorem. *(2.8) Every quasi-regular countably subcompact space* X *is a Baire space.*

Theorem. *(2.9) Every quasi-regular countably cocompact space is a Baire space.*

2.4 Minimal spaces

Proposition. *(2.10) A minimal* T_1*-space* $(X, \mathcal{T})$ *is a Baire space if and only if* X *is finite or uncountable.*

Proposition. *(2.11) Let* $(X, \mathcal{T})$ *be a Baire space and* $\mathcal{T}^*$ *be a topology on* X *contained in* $\mathcal{T}$*. If there exists a* $p \in X$ *such that* $\{U \in \mathcal{T} | p \notin U\} \subseteq \mathcal{T}^*$*, then* $(X, \mathcal{T}^*)$ *is a Baire space.*[6]

Proposition. *(2.12) Every minimal (*T_1 *Baire)-space is finite or uncountable.*

Theorem. *(2.13).* $(X, \mathcal{T})$ *is a minimal (*T_1 *Baire)-space if and only if* $(X, \mathcal{T})$ *is a Baire minimal* T_1*-space.*

Let $\mathcal{F}$ be an open filter base on a space X. *adherent point* = cluster point, *convergent* to x = contains all open neighborhoods

$\mathcal{F}$ is *Urysohn* provided that for every $y \in X$, if y is not an adherent point of $\mathcal{F}$, then there is an open set U containing y and a set $V \in \mathcal{F}$ such that $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$.

A space $(X, \mathcal{T})$ is a minimal Hausdorff (resp. Urysohn,[7] regular Hausdorff) space if and only if every open (resp. Urysohn, regular) filter base with a unique adherent point is convergent [V].

[4] TODO Definition of pseudo-semi-metric space

[5] TODO

[6] My note: Does this condition imply that the subspace of $(X, \mathcal{T})$ and $(X, \mathcal{T}^*)$ on the subset $X \setminus \{p\}$ is the same?

[7] An Urysohn space, or $T_{2\frac{1}{2}}$ space, is a topological space in which any two distinct points can be separated by closed neighborhoods.

Proposition. *(2.14) Let* $(X, \mathcal{T})$ *be a Hausdorff (resp. Urysohn, regular Hausdorff) Baire space, and let* $\mathcal{F}$ *be a nonconvergent open (resp. Urysohn, regular) filter base with a unique adherent point* p*. If* $\mathcal{T}* = \mathcal{M} \cup \mathcal{N}$*, where* $\mathcal{M} = \{U \in \mathcal{T}; p \notin U\}$ *and* $\mathcal{N} = \{U \cup V; p \in U \in \mathcal{T}$ *and* $V \in \mathcal{F}\}$*, then* $\mathcal{T}^*$ *is a Hausdorff (resp, Urysohn, regular Hausdorff) Baire topology on* X *that is properly contained in* $\mathcal{T}$*.*

Proposition. *(2.15) Let* $(X, \mathcal{T})$ *be a regular Hausdorff Baire space, and let* $\mathcal{F}$ *be a nonconvergent open (resp. Urysohn, regular) filter base with a unique adherent point* p*. If* $\mathcal{B} = \mathcal{M} \cup \mathcal{N}$*, where* $\mathcal{M} = \{U \in \mathcal{T}; U \subseteq X - \operatorname{cl} V$ *for each* $V \in \mathcal{F}\}$ *and* $\mathcal{N} = \{U \in \mathcal{T}; p \in \operatorname{cl} U\}$*, then the topology on* X *generated by the subbase* $\{X - \operatorname{cl} U | U \in \mathcal{B}\}$ *is a Hausdorff (resp. Urysohn, regular Hausdorff) Baire topology properly contained in* $\mathcal{T}$*.* [8]

Theorem. *(2.16)* $(X, \mathcal{T})$ *is a minimal (Hausdorff Baire)-space if and only if* $(X, \mathcal{T})$ *is a Baire minimal Hausdorff space.*

Theorem. *(2.17)* $(X, \mathcal{T})$ *is a minimal (Urysohn Baire)-space if and only if* $(X, \mathcal{T})$ *is a Baire minimal Urysohn space.*

Proposition. *(2.18) Every minimal (regular Hausdorff)-space* $(X, \mathcal{T})$ *is countably subcompact.*

Theorem. *(2.19)* $(X, \mathcal{T})$ *is a minimal (regular Hausdorff Baire)-space if and only if* $(X, \mathcal{T})$ *is a minimal (regular Hausdorff)-space.*

Proposition. *(2.20) If* $(X, \mathcal{T})$ *is a regular Hausdorff Baire space, then the following are equivalent:*

(i) $(X, \mathcal{T})$ *is a minimal Hausdorff (resp. Urysohn, regular Hausdorff)-space.*

(ii) Let $\mathcal{B}$ *be any open base for* $\mathcal{T}$ *and let* $\mathcal{T}^*$ *be the topology on* X *generated by the subbase* $\{x - \operatorname{cl} U | U \in \mathcal{B}\}$*. If* $\mathcal{T}^*$ *is a Hausdorff (resp. Urysohn, regular Hausdorff) Baire topology, then* $\mathcal{T}^* = \mathcal{T}$*.*

Given a topological property P, a P-space $(X, \mathcal{T})$ is P-closed if its image is closed in every P-space in which it can be embedded.

Proposition. *(2.21) Every Baire-closed space is finite.*

3 Characterizations of Baire spaces

3.1 Blumberg type theorems

We will say that space X has *Blumberg's property with respect to* Y if for every function $f\colon X \to Y$, there exists a dense subset D of X such that $f|_D$ is continuous.

[8](Probably) a typo in the book: $U \in X - \operatorname{cl} V$ should be $U \subseteq X - \operatorname{cl} V$; "for each" should be "for some"

Theorem. *(3.1) Let Y contain an infinite discrete subset. Then if X has Blumberg's property with respect to Y, X is a Baire space.*

Theorem. *(3.2) Let X be a pseudo-semi-metrizable Baire space, let Y be a second countable space, and let $f\colon X \to Y$ be a function. Then there exists a dense metrizable subspace D of X such that $f|_D$ is continuous.*

Corollary. *(3.3) Let X be a pseudo-semi-metrizable space, and let Y be a second countable space which contains an infinite discrete subset. Then X is a Baire space if and only if it has Blumberg's property with respect to Y.*

Corollary. *(3.4) Every pseudo-semi-metrizable Baire space contains a dense metrizable subspace.*

Theorem. *(3.5) Let X have a σ-disjoint pseudo-base, and let Y be a second countable space which contains an infinite discrete subset. Then X is a Baire space if and only if it has Blumberg's property with respect to Y.*

3.2 Covering and filter characterizations

Theorem. *(3.10) The following are equivalent for a space X:*

(i) X is a Baire space;
(ii) Every point finite open cover of X is locally finite at a dense set of points.
(iii) Every countable point finite open cover of X is locally finite at a dense set of points.

Theorem. *(3.11) The following are equivalent for a space X:*

(i) X is of second category.
(ii) Every point finite open cover of X is locally finite somewhere.
(iii) Every countable point finite open cover of X is locally finite somewhere.

A space X is *lightly compact* (also called *feebly compact* or *weakly compact*) if every locally finite collection of open sets of X is finite. Iséki [[32] has shown that a space is lightly compact if and only if for every decreasing sequence of nonempty open sets $\{U_i\}$, $\bigcap\limits_{i=1}^{\infty} \operatorname{cl} U_i \neq \emptyset$.

quasi-regular lightly compact $\Rightarrow$ pseudocomplete

Theorem. *(3.12) If X is a quasi-regular space, then the following are equivalent:*

(i) X is Baire space.
(ii) X is a Baire space.
(iii) Every point finite open filter base $\mathcal{F}$ on X is locally finite at a dense set of points of $\bigcup \mathcal{F}$.
(iv) Every countable, point finite, regular open filter base $\mathcal{F}$ on X is locally finite at a dense set of points of $\bigcup \mathcal{F}$.
(v) Every countable, point finite, regular open filter base $\mathcal{F}$ which is not locally finite at any point of $\bigcup \mathcal{F}$ has an adherent point.

3.3 Characterizations of Baire spaces involving pseudocomplete spaces

Proposition. *(3.13) In every quasi-regular space X there are open (possibly empty) subspaces X_P and X_A such that*

(i) $X_P \cap X_A = \emptyset$, and $X_P \cup X_A$ is dense in X;

(ii) X_P is pseudocomplete;

(iii) and every pseudocomplete subspace of X_A is nowhere dense in X_A.

Furthermore, X is a Baire space if and only if X_A is a Baire space.

3.4 The Banach-Mazur game

$\mathcal{U}$ = all subsets of X such that $\operatorname{Int} U \neq \emptyset$

A, B disjoint and $A \cup B = X$

Game $G(A, B)$: Two players (A) and (B) alternately choose sets U_i from $\mathcal{U}$ such that $U_{i+1} \subset U_i$ for each i. Player (A) wins if $A \cap (\bigcap_{i=1}^{\infty} U_i) \neq \emptyset$; otherwise player (B) wins.

Proposition. *A space X is of second category if and only if player $(\emptyset)$ does not have a winning strategy for the game $G(X, \emptyset)$ with (X) playing first.*

X is a Baire space if and only if player $(\emptyset)$ does not have a winning strategy for the game $G(X, \emptyset)$ with $(\emptyset)$ playing first.

We might point out that X is α-favorable if and only if player (X) has a winning strategy for the game $G(X, \emptyset)$ with $(\emptyset)$ playing first. Thus, being α-favorable obviously implies being a Baire space in this setting.

Theorem. *(3.16) TODO*

3.5 Countably-Baire spaces

A space is said to have the *countable chain condition* if every disjoint family of nonempty open subsets of X is countable.

Proposition. *(3.17) If every pseudo-base for X contains a countable pseudo-cover, then X has the countable chain condition.*

4 The dynamics of Baire spaces

4.1 Images and inverse images of Baire spaces

TODO *feebly continuous*, *feebly open*, *feeble homomorphism*

Theorem. *(4.1) If f is an almost continuous feebly open function from a Baire space X onto a space Y, then Y is a Baire space.*

4.2 Baire spaces extensions

If X is a topological space, an open filter base $\mathcal{F}$ on X is an *open filter on* X if whenever $U \in \mathcal{F}$ and V is an open subset of X containing U, then $V \in \mathcal{F}$. Also $\mathcal{F}$ will be said to be free if $\bigcap \mathcal{F} = \emptyset$. An open ultrafilter is an open filter which is maximal in the collection of all open filters. [9]

Let F be any set of open filters on X, and let $X(F)$ be the disjoint union of X and F. For each set U, open in X, let $U^* = U \cup \{\mathcal{F} \in F; U \in \mathcal{F}\}$. Note that $(U \cap V)^* = U^* \cap V^*$ for every open U and V in X. Let $X(F)$ have the topology generated by the base $\{U^*; U \text{ is open in } X\}$. Now X is a dense subspace of $X(F)$.

Proposition. *(4.13) If there are no free open ultrafilters on a space X, then X is a Baire space.*

We will say that a set of open filters F on a space X is *admissible* if for every collection $\mathscr{U} = \{U_i\}_{i=1}^{\infty}$ of non-empty[10] open subsets of X with $U_{i+1} \subset U_i$ and $\bigcap_{i=1}^{\infty} U_i = \emptyset$ there exists and $\mathcal{F} \in F$ such that $\mathscr{U} \subset F$. Admissible sets of filters include all open filters, all open ultrafilters, and all free open ultrafilters.

Theorem. *(4.14) If F is an admissible set of open filters on X, then $X(F)$ is a Baire space (in fact $X(F)$ is α-favorable).*

Theorem. *(4.15) If F is the set of all open filters on the space X, then $X(F)$ is a Baire space in the strong sense.*

Corollary. *(4.16) Every topological space X is a dense subspace of some compact Baire space in the strong sense.*

It is easy to see that if F is the set of all open filters or the set of all open ultrafilters on a space X, then $X(F)$ is a *generalized absolutely closed space;* that is, every open filter on $X(F)$ has an adherent point.

Theorem. *(4.17) Every topological space X is a closed nowhere dense subset of some generalized absolutely closed Baire space.*

Herrlich [H] gives an example of a Hausdorff-closed space which is not a Baire space.

Theorem. *(4.18) If F is an admissible set of open ultrafilters on the quasi-regular space X, then $X(F)$ is pseudo-complete.*

Corollary. *(4.19) Every quasi-regular space is a dense subspace of some pseudo-complete space.*

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[9] Typo: In the books the authors write: "whenever $U \in \mathcal{F}$ and V is an subset of X containing U"; it should be "whenever $U \in \mathcal{F}$ and V is an *open* subset of X containing U"

[10] The authors did not include the condition $U_i \neq \emptyset$ in the book. But I think that without this condition no non-empty set of open filters would be admissible.

Corollary. *(4.20) Every quasi-regular space is a closed nowhere dense subset of some pseudo-complete space.*

If X is a topological space, define $sX = X(F)$, where F is the smallest set of admissible open filters on X, or equivalently, F is the set of open filters on X generated by the countably infinite point finite monotone decreasing open filter bases on X.

Let X be a subset of the topological space Y. We say that Y is first countable outside of X it for each point $x \in Y \setminus X$ there is a countable collection $\mathcal{B}$ of open subsets of Y containing x such that every open subset of Y containing x contains some element of $\mathcal{B}$.

Let A and B be subsets of the topological space X. We say that A is point separated from B if for each $a \in A$ and $b \in B$ there is an open set containing a that does not contain b.

Theorem. *(4.21) For any topological space X the following are true:*
(i) sX is first countable outside of X. (ii) $sX - X$ is T_0 and point separated from X.

Corollary. *(4.22) Every first countable space is a dense subspace of some first countable Baire space.*

A subset U of the topological space X is *regular-open* if $\operatorname{Int}\operatorname{cl} U = U$. X is *semiregular* if the collection of all regular-open sets is a base for the topology on X.

Theorem. *(4.23) If Y is a semiregular extension of the space X such that*
(i) Y is first countable outside of X,
(ii) $Y \setminus X$ is T_0 and point separated from X,
then Y can be embedded as a subspace of sX containing X.

4.3 Hyperspaces and function spaces

5 Products of Baire spaces

5.1 Finite products

5.2 Infinite products

5.3 k-Baire products

5.4 Product counterexamples

Contents

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