Haworth, McCoy: Baire spaces

Various details, problems etc., found when reading [HM].

Introduction

1 Basic properties of Baire spaces

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2.4.1 Proposition 2.11

I have doubts about the proof of [HM, Proposition 2.11].

Proposition 2.4.1. *Let $(X,\mathcal{T})$ be a Baire space and $\mathcal{T}^*$ be a topology on X contained in $\mathcal{T}$. If there exists a $p \in X$ such that $\{U \in \mathcal{T} | p \notin U\} \subseteq \mathcal{T}^*$, then $(X,\mathcal{T}^*)$ is a Baire space.*

Namely I see the following problems in the proof:

- "So $U = \bigcup_{i=1}^{\infty} N_i$ where each N_i is **closed** and nowhere dense in $(X,\mathcal{T}^*)$."

  We only know this for $\overline{U}$. Or, we know that $U \subseteq \bigcup_{i=1}^{\infty} N_i$.
- How do we get contradiction in the second part of the proof if $U - N_n = \emptyset$?

At the moment I neither have a counterexample, nor am I able to correct the proof in the full generality.

However, knowing this for the case that $(X,\mathcal{T})$ is T_1 seems to be sufficient for the rest of this section of this book. Here's the proof for that case. (I've added the assumption that $\{p\}$ is closed in $(X,\mathcal{T})$.)

Proposition 2.4.2. *Let $(X,\mathcal{T})$ be a Baire space and $\mathcal{T}^*$ be a topology on X contained in $\mathcal{T}$. If there exists a $p \in X$ such that $\{p\}$ is closed in $(X,\mathcal{T})$ and $\{U \in \mathcal{T} | p \notin U\} \subseteq \mathcal{T}^*$, then $(X,\mathcal{T}^*)$ is a Baire space.*

Proof. Notice that the induced topology on the subset $X \setminus \{p\}$ is the same for both topologies $(X,\mathcal{T})$ and $(X,\mathcal{T}^*)$. Indeed, let V be open in the topology induced on $X \setminus \{p\}$ by $\mathcal{T}$. If V is open in $\mathcal{T}$, then it is open in $\mathcal{T}^*$, since $p \notin V$. The remaining possibility is that $V = U \setminus \{p\} = U \cap (X \setminus \{p\})$ for some set U which is open in $\mathcal{T}$. But, since $X \setminus \{p\}$ is open, this implies that V is open in $\mathcal{T}$, which is the case we have already solved.[1]

The subspace $X \setminus \{p\}$ is an open subspace of $(X,\mathcal{T})$, thus it is a Baire space by [HM, Proposition 1.14].[2]

If the point p is not isolated in $(X,\mathcal{T}^*)$, then $X \setminus \{p\}$ is dense in $(X,\mathcal{T})$. Consequently, $(X,\mathcal{T}^*)$ is a Baire space by [HM, Theorem 1.15].[3]

If p is isolated in $(X,\mathcal{T}^*)$, then $\mathcal{T} = \mathcal{T}^*$. □

2.4.2 Minimal Hausdorff spaces

The following construction is used in proof of [HM, Proposition 2.14].

Lemma 2.4.3. *Let $(X,\mathcal{T})$ be a topological space, $p \in X$ and $\mathcal{F}$ and open filter in X such that p is unique adherent point of $\mathcal{F}$. Let us define $\mathcal{T}^* = \mathcal{M} \cup \mathcal{N}$, where $\mathcal{M} = \{U \in \mathcal{T}; p \notin U\}$ and $\mathcal{N} = \{U \cup V; p \in U \in \mathcal{T}, V \in \mathcal{F}\}$. (I.e. the neighborhood basis for the new topology is changed only at the point p.)*

[1] Note that we have used the assumption that $\{p\}$ is closed to show that the subspace topology is the same.

[2] Every open subspace of a Baire space is a Baire space.

[3] Every space which contains a dense Baire subspace is a Baire space.

a) *The system $\mathcal{T}^*$ is a topology on X.*
b) *If $\mathcal{F}$ does not converge to p, then $\mathcal{T}^*$ is strictly weaker (coarser) than $\mathcal{T}$.*
c) *If $\mathcal{T}$ is Hausdorff, then so is $\mathcal{T}^*$.*
d) *If $\mathcal{F}$ is a regular filter and $\mathcal{T}$ is regular , then so is $\mathcal{T}^*$.*
e) *If $\mathcal{T}$ is Urysohn filter and $\mathcal{T}$ is Urysohn, then so is $\mathcal{T}^*$.*

A system $\mathcal{B} \subseteq \mathcal{T} \setminus \{\emptyset\}$, is an *open filter base,* if, for each $U, V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$.

An open filter base $\mathcal{B}$ is *regular* if, whenever U is a member of $\mathcal{B}$ then there exists a member V of $\mathcal{F}$ with $\operatorname{cl} V \subset U$.

An open filter base $\mathcal{B}$ is *Urysohn* provided that for every $y \in X$, if y is not an adherent point of $\mathcal{B}$, then there is an open set U containing y and a set $V \in \mathcal{B}$ such that $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$.

An Urysohn space, or $T_{2\frac{1}{2}}$ space, is a topological space in which any two distinct points can be separated by closed neighborhoods.

A system $\mathcal{F} \subseteq \mathcal{T}$, is an *open filter base,* if:

- $\emptyset \notin \mathcal{F}$;
- $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;
- $A \in \mathcal{F}$, $A \subseteq B \in \mathcal{T} \Rightarrow B \in \mathcal{F}$.

Clearly, if $\mathcal{B} \subseteq \mathcal{T}$ is an open filter base, then $\mathcal{F} = \{F \in \mathcal{T}; (\exists B \in \mathcal{B}) B \subseteq F\}$ is an open filter such that $\mathcal{B}$ converges to x if and only if $\mathcal{F}$ converges to x. The cluster points of $\mathcal{B}$ and $\mathcal{F}$ are the same, too.

Proof. a) Clearly, $\emptyset \in \mathcal{M}$, $X \in \mathcal{N}$.

Intersection. $U_1, U_2 \in \mathcal{M} \Rightarrow U_1 \cap U_2 \in \mathcal{M}$
$p \in U_1 \in \mathcal{T}$, $p \in U_2 \in \mathcal{T} \Rightarrow p \in U_1 \cap U_2 \in \mathcal{T}$ and also $V_1, V_2 \in \mathcal{F} \Rightarrow V_1, V_2 \in \mathcal{F}$.

Arbitrary union. Suppose $W_i \in \mathcal{T}^*$ for each $i \in I$. If each W_i belongs to $\mathcal{M}$, then so does $\bigcup\limits_{i \in I} W_i$.

Now suppose that there is at least one i_0 such that $W_{i_0} = U_{i_0} \cup V_{i_0}$ with $p \in U_{i_0}$ and $V_{i_0} \in \mathcal{F}$. We can rewrite each W_i as $W_i = U_i \cup V_i$ where $U_i \in \mathcal{T}$ and V_i is either $\emptyset$ or a non-empty open set from $\mathcal{F}$, depending on whether W_i belongs to $\mathcal{M}$ or $\mathcal{N}$.

Then we get $\bigcup\limits_{i \in I} W_i = U \cup V$, where $U = \bigcup\limits_{i \in I} U_i$ and $V = \bigcup\limits_{i \in I} V_i$. We have $p \in U_{i_0} \subseteq U$ and $U \in \mathcal{T}$. Also, since $V_{i_0} \subseteq V$, we have $V \in \mathcal{F}$. Thus $W = U \cap V \in \mathcal{N}$.

b) It is obvious that every set from $\mathcal{T}^*$ is open in $\mathcal{T}$, i.e. $\mathcal{T}^* \subseteq \mathcal{T}$.

The filter $\mathcal{F}$ does not converge to p in $\mathcal{T}$, but it does in $\mathcal{T}^*$. Hence the topology $\mathcal{T}^*$ is strictly weaker than $\mathcal{T}$.

c) Let $x \in X$ and $x \neq p$. Since x is not a cluster point of $\mathcal{F}$, there is a $V \in \mathcal{F}$ such that $x \notin \overline{V}$. This implies that x has a neighborhood U_1 such that $U_1 \cap V = \emptyset$.

We also know, that $\mathcal{T}$ is Hausdorff, and therefore there are $U_2, U \in \mathcal{T}$ such that $x \in U_2$, $p \in U$ and $U_2 \cap U = \emptyset$. Then the sets $U_1 \cap U_2$ and $U \cup V$ are $\mathcal{T}^*$-open neighborhoods of x and p, respectively. These neighborhoods are disjoint.

d) We want to show that for each point of X and each neighborhood W of this point in $\mathcal{T}^*$ there is a smaller closed neighborhood, which is contained in W.

If $p \in U \cup V$ where $U \in \mathcal{T}$ and $V \in \mathcal{F}$, then there are U', V' such that $\operatorname{cl} U' \subseteq U$ and $\operatorname{cl} V' \subseteq V$. Thus we have $p \in U' \cup V' \subseteq \operatorname{cl} U' \cup \operatorname{cl} V'$. The later set is closed in $\mathcal{T}^*$, too, since the complement is a $\mathcal{T}$-open set which does not contain p.

Now $x \in U \in \mathcal{T}$. Since x is not a cluster point of V then there is a set $V \in \mathcal{F}$ such that $x \notin \operatorname{cl} V$. This implies that there is a neighborhood $U_1 \subseteq U$ of x such that $U_1 \cap V = \emptyset$ and $\operatorname{cl} U_1 \subseteq U$. Then $(X \setminus \operatorname{cl} U_1) \cup V$ is a $\mathcal{T}^*$-open subset, which has empty intersection with U_1. This implies that the $\mathcal{T}^*$-closure of U_1 has empty intersection with this set, too, therefore it is a subset of $\operatorname{cl} U_1$ and, consequently, of U.

TODO Remaining possibility $x \in V \in \mathcal{F}$ (i.e. $x \in U \cup V$); it is similar, since V is $\mathcal{T}$-open

e) TODO □

The following result is mentioned e.g. in [B, p.146, Exercise 9.18], [W, Exercise 17M].

Corollary 2.4.4. *Let X be a Hausdorff space. X is minimal Hausdorff $\Leftrightarrow$ every open filter with unique cluster point converges.*

Recall, that a Hausdorff space $(X, \mathcal{T})$ is minimal Hausdorff if and only if for every Hausdorff topology $\mathcal{T}'$ on X we have $\mathcal{T}' \subseteq \mathcal{T} \Rightarrow \mathcal{T}' = \mathcal{T}$. (Equivalently: Every one-to-one continuous map of X to a Hausdorff space is a homeomorphism.)

Proof. $\boxed{\Rightarrow}$ Suppose that there is an open filter $\mathcal{F}$ on X, which has unique cluster point but does not converge. Then Lemma 2.4.3 yields a topology $\mathcal{T}^*$ which is strictly weaker than $\mathcal{T}$ and Hausdorff.

$\boxed{\Leftarrow}$ Suppose that $(X, \mathcal{T})$ has the property, that every open filter with unique cluster point converges. Suppose that there is a topology $\mathcal{T}'$ which is strictly coarser than $\mathcal{T}$, i.e. $\mathcal{T}' \subsetneqq \mathcal{T}$ and $\mathcal{T}'$ is Hausdorff.

This implies, that there is a set $V \in \mathcal{T} \setminus \mathcal{T}'$. In particular, $\operatorname{Int}' V \subsetneqq V$, where $\operatorname{Int}' V$ denotes the interior of V in the topology $\mathcal{T}'$.

Let us choose any point $p \in V \setminus \operatorname{Int}' V$. In particular, the choice of p means that for every $\mathcal{T}'$-neighborhood U of p we have $U \setminus V \neq \emptyset$.

Now, since $\mathcal{T}'$ is Hausdorff, for every $x \neq p$ we have $U_x, V_x \in \mathcal{T}'$ such that $p \in U_x$, $x \in V_x$ and $U_x \cap V_x = \emptyset$. Let $\mathcal{F}$ is the smallest open filter containing $\{U_x; x \in X \setminus \{p\}\}$. (An explicit description of $\mathcal{F}$ would be that $\mathcal{F}$ consists of all open supersets of finite intersections of sets from given sets. Note also that all sets U_x are $\mathcal{T}$-open, too.)

In $(X, \mathcal{T})$, we have $\{p\} = \bigcap_{x \in X} \overline{U}_x$. (It is clear that p belongs to this set and that $x \notin \overline{U}_x$ for $x \neq p$.) This implies that p is unique cluster point of $\mathcal{F}$. Thus $\mathcal{F}$ converges to p, which implies $V \in \mathcal{F}$. This contradicts the fact, that $\mathcal{F}$ contains no $\mathcal{T}'$-open neighborhoods of p. □

2.4.3 More on minimal Hausdorff spaces

compact Hausdorff $\Rightarrow$ minimal Hausdorff $\Rightarrow$ H-closed $\Rightarrow$ Hausdorff

H-closed spaces. A Hausdorff topological space X is called *H-closed* or *absolutely closed* if it is closed in any Hausdorff space, which contains X as a subspace. Similarly, *P-closed* spaces can be defined for any topological property P.

E.g., it is well-known that every compact Hausdorff space is H-closed.

Every minimal Hausdorff space is H-closed. To show this we will use a characterization of H-closed spaces using open filters (see e.g. [W, Problem 17K]. Lemma 2.4.7 and Lemma 2.4.8 are take from [W, Problem 17K], too.).

Lemma 2.4.5. *Let X be a Hausdorff space. X is H-closed if and only if every open filter in X has a cluster point.*

Proof. $\boxed{\Rightarrow}$ Suppose that $(X,\mathcal{T})$ is a H-closed space and that $\mathcal{F}$ is an open filter on X with no cluster point.

For $p\notin X$ we define topology $\mathcal{T}^*=\mathcal{M}\cup\mathcal{N}$ on $X\cup\{p\}$, where

$$\mathcal{M}=\{U\subseteq X; U\in\mathcal{T}\}$$
$$\mathcal{N}=\{\{p\}\cup F; F\in\mathcal{F}\}$$

It is easy to show that:

- $\mathcal{T}^*$ is a topology on X. (The proof is similar to the proof given in Lemma 2.4.3).
- X is a subspace of $X\cup\{p\}$.
- X is not closed in $X\cup\{p\}$. (Since every set in $\mathcal{F}$ is non-empty, we see that every $\mathcal{T}^*$/neighborhood $\{p\}\cup F$ of the point p has a non-empty intersection with X.)

If we show that $(X\cup\{p\},\mathcal{T}^*)$ is Hausdorff, we obtain a contradiction and we are done.

If $x,y\in X$ are two distinct points, they can be separated in $X\cup\{p\}$ by the same open sets as in X.

Now let $x\in X$. We want to show that x and p can be separated by sets from $\mathcal{T}^*$. Since x is not a cluster point of F, there exists a set $F\in\mathcal{F}$ such that $x\notin \operatorname{cl} F$. This means that x has an open neighborhood U such that $U\cap F=\emptyset$. Then the sets U and $F\cup\{p\}$ are $\mathcal{T}^*$-neighborhoods of x and p which separate these two points.

$\boxed{\Leftarrow}$ Let X be a space which is not H-closed. This means that there is some Hausdorff space Y such that X is a subspace of Y and X is not closed in Y. Let $p\in\overline{X}\setminus X$ and let $\mathcal{N}_p$ denotes the set of all neighborhoods of p in Y. Then $\mathcal{F}=\{U\cap X; U\in\mathcal{N}_p\}$ is an open filter on X. (Every set $U\cap X$ is non-empty, since $p\in\overline{X}$.)

We will show that this open filter has no cluster point in X. Indeed, let $x\in X$. In the Hausdorff space Y we have neighborhoods $V\ni x$ and $U\ni p$ such that $V\cap Y=\emptyset$. Then $V\cap X$ is a neighborhood of x in X such that

$(V\cap X)\cap(U\cap X)=\emptyset$, which shows that $x\notin\overline{U\cap X}$ and x is not a cluster point of $\mathcal{F}$. $\square$

Lemma 2.4.6. *Let X be a Hausdorff space. X is H-closed if and only if every open cover $\mathcal{C}$ of X contains a finite subsystem $\mathcal{D}$ such that $\bigcup\{\overline{D}; D\in\mathcal{D}\}=X$, i.e., the closures of the sets from $\mathcal{D}$ cover X.*

Proof. $\boxed{\Rightarrow}$ We assume that there is an open cover $\mathcal{C}$ such that $\bigcup\{\overline{D}; D\in\mathcal{D}\}\underset{\neq}{\subseteq} X$ for every finite incollection $\mathcal{D}$ of $\mathcal{C}$. We will show that

$$\mathcal{B}=\{X\setminus\overline{U}; U\in\mathcal{C}\}$$

is an open filter base with no cluster point.

The fact that closures of no finite incollection of $\mathcal{C}$ cover X means that $\mathcal{B}$ has finite intersection property. All sets in $\mathcal{B}$ are open, hence $\mathcal{B}$ is an open filter base.

If $x\in X$ then there exists $U\in\mathcal{C}$ such that $x\in U$. We have $U\cap(X\setminus\overline{U})=\emptyset$. This shows that x is not a cluster point of $\mathcal{B}$.

$\boxed{\Leftarrow}$ Suppose that X is not H-closed. This implies the existence of an open filter $\mathcal{F}$ which has no cluster point. Let

$$\mathcal{C}=\{X\setminus\overline{F}; F\in\mathcal{F}\}.$$

Let us show first that $\mathcal{C}$ is a cover. Indeed, any point $x\in X$ is not a cluster point of $\mathcal{F}$. This implies existence of an $F\in\mathcal{F}$ such that $x\notin\overline{F}$, i.e. $x\in X\setminus\overline{F}$.

Now if we have a finite subsystem

$$\mathcal{D}=\{X\setminus\overline{F_i}; i=1,\dots,n\}$$

then

$$\bigcup_{D\in\mathcal{D}}\overline{D}=\bigcup_{i=1}^{n}\overline{X\setminus\overline{F_i}}=\bigcup_{i=1}^{n}(X\setminus\operatorname{Int}\overline{F_i})\subseteq\bigcup_{i=1}^{n}(X\setminus F_i)=X\setminus\bigcap_{i=1}^{n}\overline{F_i}\subseteq X\setminus\bigcap_{i=1}^{n}F_i\underset{\neq}{\subseteq} X.$$

(The last inclusion follows from the fact that $\bigcap\limits_{i=1}^{n}F_i$ belongs to $\mathcal{F}$, hence it is non-empty.)

We have shown that closures for no finite subsystem $\mathcal{D}\subseteq\mathcal{C}$ cover the space X. $\square$

We have already mentioned that every compact space is H-closed. Both 2.4.6 and 2.4.7 show that H-closedness is, in some sense, similar to compactness.

It is very natural to ask whether every H-closed space is compact.

Lemma 2.4.7. *Let X be an H-closed space. X is compact if and only if X is regular.*

Proof. $\boxed{\Rightarrow}$ Every compact Hausdorff space is completely regular.

$\boxed{\Leftarrow}$ If we have open cover of X, then for each $x \in X$ there is (at least one) set U_x from this cover such that $x \in X$. I.e. we have an open cover $\mathcal{C} = \{U_x; x \in X\}$. Since X is regular, for every x we have an open set V_x such that

$$x \in V_x \subseteq \overline{V}_x \subseteq U_x.$$

This yields an open cover $\mathcal{C}' = \{V_x; x \in X\}$.

By Lemma 2.4.7 there exist a finite set $x_1 \dots x_n$ such that the closures $\overline{V}_{x_i}$, $i = 1, \dots, n$ cover X. We get

$$X \subseteq \bigcup_{i=1}^{n} \overline{V_{x_i}} \subseteq \bigcup_{i=1}^{n} U_{x_i},$$

hence $\{U_{x_i}; i = 1, \dots, n\}$ is a finite subcover. □

Lemma 2.4.8. *Every minimal Hausdorff space is H-closed.*

Proof. TODO □

Lemma 2.4.9. *Every minimal Hausdorff space is semiregular.*

Proof. TODO ???

□

Proposition 2.4.10. *X is minimal Hausdorff $\Leftrightarrow$ X is H-closed and semiregular.*

Dôkaz. $\boxed{\Rightarrow}$

$\boxed{\Leftarrow}$ TODO

□

3 Characterizations of Baire spaces

3.1 Blumberg type theorems

3.2 Covering and filter characterizations

3.3 Characterizations of Baire spaces involving pseudo-complete spaces

3.4 The Banach-Mazur game

3.5 Countably-Baire spaces

4 The dynamics of Baire spaces

4.1 Images and inverse images of Baire spaces

4.2 Baire spaces extensions

$\mathcal{B} = \{U^*; U$ is open in $X\}$ is a base for a topology on $X(F)$. Since $X^* = X(F)$, we see that $\mathcal{B}$ covers $X(F)$. If $x \in U^* \cap V^*$, then also $x \in (U \cap V)^*$. Since $(U \cap V)^* = U^* \cap V^*$, we see that $(U \cap V)^* \subseteq U^*$ and $(U \cap V)^* \subseteq V^*$.

X is a dense subspace of $X(F)$. Since $\emptyset^* = \emptyset$, every non-empty basic set has the form U^* for some $U \neq \emptyset$. Then $U^* \cap X = U$ is non-empty.

Every open filter is contained in an open ultrafilter. TODO finite intersection property should be sufficient

TODO Zorn

If A is an open dense subset in X and $\mathcal{F}$ is an open ultrafilter on X, then $A \in \mathcal{F}$. This follows from the fact that the system $\mathcal{F} \cup \{A\}$ has f.i.p.

More generally: If A is an open set which intersects every element of $\mathcal{F}$ (where $\mathcal{F}$ is an open ultrafilter), then $A \in \mathcal{F}$.

If f is an open continuous function from X into Y, then there is an open continuous function g from $X(F)$ into $X(G)$ such that $g|_X = f$. Let $f\colon X \to Y$ be any map. If we are given a set F of open filters on X, we can define the corresponding set of open filters on Y as

$$G := \{f[\mathcal{F}]; \mathcal{F} \in F\},$$

where

$$f[\mathcal{F}] = \{B \subseteq Y; B \text{ is open and } f^{-1}(B) \in \mathcal{F}.\}$$

If f is continuous then each $f[\mathcal{F}]$ is indeed an open filter.

Now we can define $g\colon X(F) \to Y(G)$ as $g(x) = f(x)$ for $x \in X$ and $g(\mathcal{F}) = f[\mathcal{F}]$ for $\mathcal{F} \in F$. It is clear that $g|_X = f$.

This map is also *continuous.* It suffices to notice that for an open subset $V \subseteq Y$ we have

$$g(\mathcal{F}) \in V^* \Leftrightarrow \mathcal{F} \in f^{-1}(V)$$

which implies that

$$g^{-1}(V^*) = (f^{-1}(V))^*.$$

TODO open map ??? !!!

Set of all open ultrafilters is admissible. TODO filter base

Theorem 4.14. If F is an admissible set of open filters on X, then $X(F)$ is a Baire space (in fact $X(F)$ is α-favorable).

We will describe the tactic (stationary strategy) for the second player. Suppose that the player I has chosen a non-empty open set U_n. This open set contains some set of the form W_n^*, where W_n is a non-empty open subset of X. The player II can simply choose $V_n = W_n^*$.

For any run of the game we get a system $\{W_n; n \in \mathbb{N}\}$ of open subsets of X such that $W_{n+1} \subseteq W_n$. There are two possibilities.

Either $\bigcap\limits_{n\in\mathbb{N}} W_n \neq \emptyset$. Since $W_n \subseteq W_n^* \subseteq U_n$, this implies that $\bigcap\limits_{n\in\mathbb{N}} U_n \neq \emptyset$ and player II wins.

The second possibility is that $\bigcap\limits_{n\in\mathbb{N}} W_n = \emptyset$. In this case, there exists an open filter $\mathcal{F} \in F$ such that $\{W_n; n \in \mathbb{N}\} \subseteq \mathcal{F}$. (This follows from the fact that F is admissible.) So we get for each $n \in \mathbb{N}$ that $W_n \in \mathcal{F} \Rightarrow \mathcal{F} \in W_n^* \Rightarrow \mathcal{F} \in U_n$. Hence $\mathcal{F} \in \bigcap\limits_{n\in\mathbb{N}} U_n$ and the intersection $\bigcap\limits_{n\in\mathbb{N}} U_n$ is non-empty.

Almost compact spaces

if F is the set of all open filters or the set of all open ultrafilters on a space X, then $X(F)$ is a generalized absolutely closed space. Let $\mathcal{F}$ be any open ultrafilter on X. First we show that $\{A \cap X; A \in \mathcal{F}\}$ has the finite intersection property.

Suppose that A, B are two open subsets of $X(F)$ which belong to $\mathcal{F}$. Then $A \cap B \neq \emptyset$ and there exists a point $x \in A \cap B$. (The point x belongs to $X(F)$, so it is an ultrafilter on x.) From this we have existence of sets U_A, U_B, which are open in X, such that
$x \in U_A^* \subseteq A$, $x \in U_B^* \subseteq B$
$\Rightarrow U_A^* \cap U_B^* = (U_A \cap U_B)^* \neq \emptyset$
$\Rightarrow U_A \cap U_B \neq \emptyset$
$\Rightarrow A \cap B \cap X \neq \emptyset$

The same argument works for finitely many sets instead of two sets.

This shows that $\{A \cap X; A \in \mathcal{F}\}$ is a system of open subset of X that has finite intersection property. Therefore there exists an open ultrafilter $y \in X(F)$ such that $\{A \cap X; A \in \mathcal{F}\} \subseteq y$.

Now if $y \in U^*$, then U belongs to y, which implies $U \cap A \cap X \neq \emptyset$ for every $A \in \mathcal{F}$. From this we get $U^* \cap A \neq \emptyset$ for every $A \in \mathcal{F}$. This means that y is a cluster point of the open ultrafilter $\mathcal{F}$.

Filters generated by the countably infinite point finite monotone open bases. TODO

The product $\prod_a sX_a$ is a Baire space. TODO

If X is first countable, then sX is first countable. TODO

Contents

References

[B] N. Bourbaki. *Elements of Mathematics. General Topology. Chapters I-IV.* Springer-Verlag, Berlin, 1989.

[HM] R.C. Haworth and R.A. McCoy. *Baire spaces.* PWN, Warszawa, 1977. Dissertationes Mathematicae CXLI.

[SS] L. A. Steen and J. A. Seebach. *Counterexamples in Topology.* Springer Verlag, New York, 1978.

[W] S. Willard. *General topology.* Addison-Wesley, Massachussets, 1970.