Just, Weese: Discovering Modern Set Theory I – Basic Tools

How to read this books

Everybody should attempt the exercises rated G (general audience). Beginners are encouraged to also attempt exercises rated PG (parental guidance), but may sometimes want to consult their instructor for a hint.

1 Not Entirely Naive Set Theory

1.1 Pairs, Relations, and Functions

1.2 Partial Order Relations

Definition. (2.1) Let R be a binary relation on a set X. We say that R is *wellfounded,* if for every nonempty subset $Y \subseteq X$ there exists a $z \in Y$ such that $\langle y, z\rangle \notin R$ for all $y \in Y \setminus \{z\}$. A relation R is *strictly wellfounded* if it is wellfounded and irreflexive.

TODO p.25 simple product

$$\langle a, b\rangle \preceq_s \langle c, d\rangle \Leftrightarrow a \preceq_A c \wedge b \preceq_B d$$

TODO infinite version

1.3 Cardinality

Definition. (3.1) We say that two sets A and B are equipotent and write $A \approx B$ if there exists a one-to-one function $f\colon A \to B$ from A onto B.

1.4 Induction

1.4.1 Induction and recursion over the set of natural numbers

Definition. (4.3) For each natural number we define recursively a set that we shall henceforth identify with this number:

$$0 = \emptyset;$$
$$k + 1 = k \cup \{k\}.$$

In the framework of axiomatic set theory that we start developing in Chapter 7, there is no way to eliminate the circularity of Definition 3.

Definition. (4.4) Let x be any set.

(a) For each $n \in \omega$ we define recursively a set $\bigcup^{(n)} x$ as follows:

$$\bigcup{}^{(0)} x = x$$
$$\bigcup{}^{(k+1)} x = \bigcup(\bigcup{}^{(k)} x)$$

(b) We define a set $\mathrm{TC}(x)$, called the transitive closure of x, by $\mathrm{TC}(x) = \bigcup\{\bigcup^{(n)} x : n \in \omega\}$.
(c) x is *transitive* iff $TC(x) = x$. We say that x is *hereditarily finite* (*hereditarily countable*) iff $\mathrm{TC}(x)$ is countable.

[1]

Theorem (Cantor-Schröder-Bernstein)**.** *(4.5) Let a, b be arbitrary sets. If $|a| \leq |b|$ and $|b| \leq |a|$, then $|a| = |b|$*

1.4.2 Induction and recursion over wellfounded sets

Definition. (4.20) Let X be a nonempty set, let $x \in X$, and $Y \subseteq X$. Assume that W is a wellfounded relation on X. The set $\{z \in X : \langle z, x\rangle \in W \wedge z \neq x\}$ is called the *initial segment* of $\langle X, W\rangle$ determined by x, and denoted by $I_W(x)$ (or simply $I(x)$ if the relation W is implied by context). We say that x is W-minimal for Y if $x \in Y$ and $I(x) \cap Y = \emptyset$. If x is W-minimal for X, then we simply say that x is W minimal.

Theorem (Principle of Induction over a Wellfounded Set)**.** *(4.21) Let X be a nonempty set, let $Y \subseteq X$, and let W be a wellfounded relation on X. If the implication*

$$I_W(x) \subseteq Y \to x \in Y \qquad (\mathrm{Ind}_W)$$

holds for all $x \in X$, then $X = Y$.

Definition. (4.29) TODO The set $DC(Y)$ is called the *downward closure* of Y or the *initial segment* of $\langle X, \preceq\rangle$ generated by Y.

If $\langle X, \preceq\rangle$ is a w.o., and $Y \subseteq X$ is such that $DC(Y) \neq X$, then the minimum element of $X \setminus DC(Y)$ is denoted by $\sup^+ Y$.

Theorem. *(4.30) Let $\langle X, \preceq_X\rangle$ and $\langle Z, \preceq_Z\rangle$ be w.o.'s. Then exactly one of the following holds:*
(a) *$\langle X, \preceq_X\rangle \cong \langle Z, \preceq_Z\rangle$; or*
(b) *There exists an $x^* \in X$ such that $\langle I(x^*), \preceq_X\rangle \cong \langle Z, \preceq_Z\rangle$; or*
(c) *There exists a $z^* \in X$ such that $\langle X, \preceq_X\rangle \cong \langle I(z^*), \preceq_Z\rangle$.*

[1] My note: x is transitive $\Leftrightarrow \bigcup x \subseteq x \Leftrightarrow (y \in x \to y \subseteq x)$

2 An Axiomatic Foundation of Set Theory

2.5 Formal Languages and Models

$\tau_0 \colon I \to \omega \setminus \{0\}$, $\tau_1 \colon J \to \omega \setminus \{0\}$; assign to relational/functional symbol its *arity.*

The language is supposed to enable us to describe certain mathematical structures called *models* of the language L or L*-structures.*

The relational, functional, and constant symbols are called *nonlogical* symbols, the remaining ones are the *logical* symbols of the language. Depending on the particular language L, the sets I, J and K may be empty, countable or uncountable. If $I \cup J \cup K$ is a countable set, then we say that the language L is countable.

TODO L_S language of set theory

We shall also consider a language L_G that has no relational symbols, one binary functional symbol, and one constant symbol In the case of L_c, the functional symbol f_0 will be denoted by $*$, and the constant symbol c_0 will be rendered as e. We call L_G the language of group theory.

A formula without free variables is called a *sentence.*

$\Sigma \vdash \varphi$ if there is a formal proof of φ from the formulas in Σ (Σ is some set of formulas.)

The following two characterizations of consistency will be frequently used in this text without explicit reference.

- A set Σ of formulas of L is consistent iff $\Sigma \nvdash \varphi$ for some sentence φ of L.
- Let Σ be a set of formulas of L and let φ be a formula of L. Then $\Sigma + \{\varphi\}$ is consistent iff $\Sigma \nvdash \not\varphi$.

TODO valuation p.77

A theory in a language L is any set of sentences of this language. Let T be a theory. Then $\mathfrak{A}$ is called a model of T if $\mathfrak{A} \models \varphi$ for each $\varphi \in T$. We denote this property by $\mathfrak{A} \models T$.

Theorem (Gödel's Completeness Theorem, Version I)**.** *(5.1) Let φ be a sentence of a first-order language L, and let T be a theory in the language L. If $T \models \varphi$, then $T \vdash \varphi$.*

Theorem (Gödel's Completeness Theorem, Version II)**.** *(5.2) Every consistent first-order theory has a model.*

Theorem (Compactness theorem)**.** *(5.3) If T is a theory in a first-order language L, then T has a model iff every finite subset S of T has a model.*

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References