2 Just, Weese: Discovering Modern Set Theory II – Set-Theoretic Tools for Every Mathematician

2.13 Filters and ideals in partial orders

2.13.1 The general concept of a filter

$\mathcal{F}^*$ = dual ideal

The sets in $\mathcal{P}(X) \setminus \mathcal{F}^*$ are called the *stationary sets w.r.t. $\mathcal{F}$*, or simply the *$\mathcal{F}$-stationary sets.*

Let $\varkappa$ be a regular uncountable cardinal, and let $\mathcal{F}$ be a filter on a nonempty set X. We say that $\mathcal{F}$ is *$\varkappa$-complete* or *$\varkappa$-closed* if $\bigcap \mathcal{A} \in \mathcal{F}$ for every $\mathcal{A} \in [\mathcal{F}]^{<\varkappa}$. An $\aleph_1$-complete filter is also called *countably complete.*

If the neighborhood filter $\mathcal{N}_x$ is countably complete, then x is said to be a *P-point* in X.

If there exists a nonprincipal $\varkappa$-complete filter on a set of size $\varkappa$, then $\varkappa$ is called a *measurable cardinal.*

Definition. (13.3) Let $\langle P, \leq \rangle$ be a p.o., and let $A \subseteq P$. An element p of P is a *lower bound* for A if $p \leq q$ for every $q \in A$. We say that p and q are *compatible in A* if A contains a lower bound for the set $\{p, q\}$, we write $p \not\perp q$. Otherwise we say that p and q are incompactible and write $p \perp q$. A subset F of P is called a *filter in* $\langle P, \leq \rangle$ if:

(i) F is closed upwords, i.e., $\forall p \in F\ \forall q \in P\ (p \leq q \to q \in F)$;

(ii) Every finite subset of F has a lower bound in F. [1]

Note that we speak about filters *in* P to avoid confusion with filters *on* X.

$\mathcal{K}(X)$ = filter of closed sets

Ideal = (I)* closed downwards and (II)* every finite subset has an upper bound in I.

Exercise. Let $\langle P, \leq \rangle$ be a p.o. Then $\{I \subseteq P; I$ satisfies (I)$^*\}$ is a topology on P.

A subset B of P is a *filter base* in $\langle P, \leq \rangle$ if it satisfies condition (II) of the definition of a filter.

$A \subseteq P$ is *centered* if every finite subset of A has a lower bound in P. Unfortunately, it is not always true that every centered subset of a p.o. is contained in a filter.

Counterexample: $P = \{\{n\}; n \in \omega\} \cup \{\omega \setminus \{n\}; n \in \omega\}$. The set $A = \{\omega \setminus \{n\}; n \in \omega\}$ is a centered system which is not contained in any filter in $\langle P, \subseteq \rangle$.

[1] V*? Is this not the condition that F is down-directed???

A p.o. is called *well-met* if every two compatible elements have a greatest lower bound in P.

Theorem. *(13.5) Let $\langle P, \leq \rangle$ be a well-met p.o., and let A be a centered subset of P. Then there eixsts a smallest filter F in $\langle P, \leq \rangle$ such that $A \subseteq F$.*

Theorem. *(13.6) Let X be a topological space. The following are equivalent:*

(i) X is compact;

(ii) Every filter of closed subsets of X has a nonempty intersection.

(iii) Every ultrafilter of closed subsets of X is fixed.

Remark 13.8: In chapter 9 we showed that Tychonoff's Theorem implies the Axiom of Choice. Note that (AC) is necessary in the proof of Theorem 13.7 in order to guarantee the existence of a function f as in Exercise 13.13. But if each of the spaces X_i is Hausdorff, then the function f is uniquely determined.

However, this does not mean that the restriction of Tychonoff's Theorem to the class of Hausdorff spaces is provable in ZF alone. We also used the fact taht there exists an ultrafilter of closed sets in $\prod_{i \in I} X_i$, and this fact is not provable in ZF alone, althought it follows from the so-called *Prime Ideal Theorem* which is weaker than the full Axiom of Choice.

Let $\varkappa$ be a regular uncountable cardinal. A p.o. is *$\varkappa$-closed* if every decreasing sequence $\langle p_\xi : \xi < \lambda \rangle$ of elements of P of length $\lambda < \varkappa$ has a lower bound in P. A p.o. P is *$\varkappa$-directed closed* if every filter base $B \subseteq P$ of size less than $\varkappa$ has a lower bound in P.

$\varkappa$-directed closed $\Rightarrow$ $\varkappa$-closed

A subset $A \subseteq P$ is an *antichain* in P if every two elements of A are incompatible.[2]

A subset $D \subseteq P$ is a *dense* subset in P if $\forall p \in P\ \exists q \in D\ (q \leq p)$.

A subset $C \subseteq P$ is a *predense* subset in P if $\forall p \in P\ \exists q \in C\ (q \not\perp p)$.

A p.o. $\langle P, \leq \rangle$ satisfies *$\varkappa$-chain condition* if every antichain in P has cardinality less than $\varkappa$. The $\aleph_1$-c.c. is also called the *countable chain condition.*

A topological space (X, τ) has the c.c.c. if and only if the p.o. $\langle \tau \setminus \{\emptyset\}, \subseteq \rangle$ has the c.c.c.

2.13.2 Ultraproducts

SKIPPED

2.13.3 A first look at Boolean algebras

$Finco(A)$ = finite + cofinite subsets of A

$F_{<\varkappa}(A = \{X \subseteq A; |X| < \varkappa \vee |-X| < \varkappa\})$

TODO p.17–19 Stone duality

[2] V*?

2.14 Trees

A *tree* is a p.o. $\langle T, \leq_T\rangle$ such that for every $t \in T$ the initial segment $\hat{t} = \{s \in T : s <_T t\}$ is wellordered by the relation $\leq_T$.

Let $\varkappa$, λ be cardinals > 0, and ${}^{<\varkappa}\lambda = \bigcup\{{}^{\alpha}\lambda : \alpha < \varkappa\}$. Then $\langle {}^{<\varkappa}, \subseteq\rangle$ is a tree. If $\lambda = 2$, this tree is called the *full binary tree of height $\varkappa$.*

A node r of a tree T is *splitting* if there are nodes $s, t \in T$ such that $r <_T t$, $r <_T s$ and s, t are incomparable in $\langle T, \leq_T\rangle$.

The *height* $ht(t)$ of a node $t \in T$ is the ordert type of $\hat{t}$. The *height of the tree* T is the ordinal $ht(T) = \sup\{ht(t) + 1; t \in T\}$. For an ordinal α we define:
$T(\alpha) = \{t \in T; ht(t) = \alpha\}$;
$T_{(\alpha)} = \{t \in T; ht(t) < \alpha\}$.

A subset P of T is a *path* in T if P is a chain such that $\hat{t} \subseteq P$ for all $t \in P$. A *branch* is a maximal path. A branch B is *cofinal* in T if B intersects every nonempty level of T, i.e., if $ot(\langle B, \leq_T\rangle) = ht(T)$.

recursion over tree - as an example they prove:
Every closed subset of a Polish space is either countable or of cardinality $2^{\aleph_0}$.

For a cardinal $\varkappa$, let us call T a *$\varkappa$-Kurepa tree* if T has height $\varkappa$, every level of T is of cardinality $< \varkappa$, and T has $> \varkappa$ cofinal branches.

$\aleph_1$-Kurepa tree = *Kurepa tree*

Kurepa hypothesis (KH) There exists a Kurepa tree.

KH is relatively consistent with ZFC. On the other hand, if the existence of an inaccessible cardinal is consistent with ZFC, then so is the negation of KH.

Lemma (König). *(14.2) If T is a tree of height ω such that $T(n)$ is finite for all $n \in \omega$. Then T contains a cofinal branch.*

Lemma. *(14.3) If $|X| = \varkappa$, $|\mathcal{Y}| < \mathrm{cf}(\varkappa)$ and $X = \bigcup \mathcal{Y} \Rightarrow |Y| = \varkappa$ for some $Y \in \mathcal{Y}$*

König's lemma and AC - SKIPPED

Consider the following statements:

(WTY) Every product $\prod_{n\in\omega} X_n$ of finite non empty Hausdorff spaces X_n is nonempty and compact.

(WCT) Let L be a first-order language without functional symbols, and let $(S_n)_{n\in\omega}$ be an increasing sequence of finite sets of quantifier-free sentences of L. If each S_n has a model, then the theory $S = \bigcup_{n\in\omega} S_n$ also has a model.

Theorem (ZF). *(14.4) The following statments are equivalent:*
(i) König's Lemma;
(ii) (WTY);
(iii) (WCT).

If $\varkappa$ is an infinite cardinal, then a *$\varkappa$-tree* is a tree T of height $\varkappa$ such that $|T(\alpha)| > \varkappa$ for every $\varkappa < \alpha$. In particular, a $\varkappa$-Kurepa tree is a $\varkappa$-tree. A *$\varkappa$-Aronszajn tree* is a $\varkappa$-tree without cofinal branches. An $\aleph_1$-Aronszajn tree is simply called an *Aronszajn tree.* An infinite cardinal has the *tree propperty* if there are no $\varkappa$-Aronszajn trees.

König's Lemma asserts that $\aleph_0$ has the tree property.

Theorem. *(14.8) There exists an Aronszajn tree.*

Theorem. *(14.11) If $\varkappa$ is a regular infinite cardinal such that $2^{<\varkappa} = \varkappa$, then there exists a $\varkappa^+$-Aronszajn tree.*

GCH $\Rightarrow$ existence of $\varkappa^+$-Aronszajn trees for every regular infinite cardinal $\varkappa$

Strongly inaccessible cardinals with the tree property are called *weakly compact cardinals.* See Chapters 15 and 24.

An *antichain* in a tree $\langle T, \leq_T\rangle$ is a subset $A \subseteq T$ of pairwise *incomparable* elements of T. This notion of "antichain" is different from the one in Chapter 13, but there is a closed connection between the two concepts.

Trees if height ω_1 without uncountable chains or antichains are called *Suslin trees.*

Suslin Hypothesis (SH) There are no Suslin trees.

MA $\Rightarrow$ SH (Chapter 19)

Chapter 22: SH is relatively consistent

The rest of the chapter SKIPPED

2.15 A little Ramsey Theory

Theorem (Pigeonhole Principle)**.** *(15.1) Assume $\varkappa$ is an infinite cardinal, A is a set of cardinality $\varkappa$ and $A = \bigcup_{i<\sigma} A_i$ where $\sigma < \mathrm{cf}(\varkappa)$. Then there exists $i < \sigma$ such that $|A_i| = \varkappa$.*

$[X]^\rho$ = family of all subsets of X with cardinality ρ

$\mathcal{P} = \{P_i; in \in I\}$ is a *partition* of $[X]^\rho$ if

$$\bigcup\{\{P_i; in \in I\}\} = [X]^\rho \text{ and } P_i \cap P_k = \emptyset.$$

$f_{\mathcal{P}}\colon [X]^\rho \to I$ *canonical coloring associated with* $\mathcal{P}$

A set Y is *homogeneous* for the partition $\mathcal{P}$ if there is an $i_0 \in I$ such that $[Y]^\rho \subseteq P_{i_0}$.

Let $\varkappa, \lambda, \rho, \sigma$ be cardinals. The symbol[3]

$$\varkappa \to (\lambda)^\rho_\sigma$$

stands for the following statement:

Whenever X is a set of cardinality $\varkappa$, I is a set of cardinality σ and $\mathcal{P} = \{P_i; i \in I\}$ is a partition of $[X]^\rho$, there exists $Y \subseteq X$ with $|Y| = \lambda$ thait is homogeneous for $\mathcal{P}$.

Theorem 15.1 can be expressed as

Theorem. *(15.2) For all infinite cardinals $\varkappa$ and for all $\sigma < \mathrm{cf}(\varkappa)$, the relation $\varkappa \to (\varkappa)^1_\sigma$ holds.*

[3]Read: $\varkappa$ arrows λ super ρ sub σ

Exercise 15.2: Let $\varkappa, \lambda, \rho, \sigma, \varkappa', \lambda', \rho', \sigma'$ be cardinals such that $\varkappa' \geq \varkappa$, $\lambda' \leq \lambda$, $\rho' \leq \rho$ and $\sigma' \leq \sigma$. Show that if $\varkappa \to (\lambda)^{\rho}_{\sigma}$ then also $\varkappa' \to (\lambda')^{\rho'}_{\sigma'}$.

Theorem (Ramsey's Theorem)**.** *(15.3) For positive natural numbers k, l:*

$$\omega \to (\omega)^k_l.$$

Theorem. *(15.4)Let m, k, l be positive natural nubmers. There exists a natural number n such that*

$$n \to (m)^k_l.$$

A subset $A \subseteq \omega$ is called *relatively large* if $|A| \geq \min A$. We write $n \stackrel{*}{\to} (m)^k_l$ if for every coloring $f\colon [n]^k \to l$ there exists a relatively large homogeneous set of size at least m.

Contents