1 Just, Weese: Discovering Modern Set Theory I – Basic Tools

2 Not Entirely Naive Set Theory

2.1 Pairs, Relations, and Functions

2.2 Partial Order Relations

2.3 Cardinality

2.4 Induction

2.4.1 Induction and recursion over the set of natural numbers

5(a): Convince yourself that the set $\{\omega\}$ is finite, but not hereditarily finite.

$\mathrm{TC}(\{\omega\}) = \{\omega\} \cup \mathrm{TC}(\omega) = \{\omega\} \cup \omega$. The transitive closure $\mathrm{TC}(\{\omega\})$ contains ω.

5(b): Find a set x that is countable, but not hereditarily countable.[1]

$x = \mathcal{P}(\omega)$

2.4.2 Induction and recursion over wellfounded sets

2.5 Formal Languages and Models

$$\varphi_2 : \exists x, y, z \forall w (w = x \lor w = y \lor w = z) \to \forall p, q (p * q = q * p).$$

Every group of at most three elements is abelian.

TODO Exercise 6

Exercise 7: Prove the equivalence of Version I and Version II of Gödel's Completeness Theorem.

I⇒II: If T is consistent, then there is a formula φ such that $T \nvdash \varphi$. By version I, this implies $T \not\models \varphi$. For the later to hold, there must be at least one model of T.

II⇒I: If T is not consistent, then T proves any formula, so $T \vdash \varphi$. Suppose that T is consistent and that $T \models \varphi$. Clearly, $T \cup \{\neg\varphi\}$ is inconsistent. (Otherwise there would be a model of $T \cup \{\neg\varphi\}$. Since $T \models \varphi$, both φ and $\neg\varphi$ would be true in this model – a contradiction.) But since $T \cup \{\neg\varphi\}$ is inconsistent, we have $T \vdash \varphi$.

Matt's solution: I⇒II: Suppose T is consistent but has no model. From I we get that T is not consistent

II⇒I: Let T be a theory such that for every sentence φ we have $T \models \varphi$ implies $T \vdash \varphi$. Let T be consistent and assume that it does not have a model. Then

[1] I was thinking about ω_1 at first, but ordinals are introduced much later in the book.

$T \models \neg\varphi$ (vacuously because it doesn't have a model) and hence by assumption we have $T \vdash \neg\varphi$ which would be a contradiction to T being consistent.

Exercise 8: Matt' solution: $\Longleftarrow$ Let T be such that every finite subset S has a model. Assume T does not have a model. Then for all sentences φ in T we have $T \models \varphi$ and $T \models \neg\varphi$. $S = \{\varphi, \neg\varphi\}$ is a finite subset of T hence by assumption has a model. But in any mode M, $M \models \varphi$ and $M \models \neg\varphi$ is impossible hence T has to have a model.

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