Series in Banach Spaces

1 Background material

1.1 Numerical series. Riemann's Theorem

1.2 Main Definitions. Elementary Properties of Vector Series

absolutely convergent $\Leftrightarrow \sum_{k=1}^{\infty} \|x_k\| < \infty$

1.3 Preliminary Material on Rearrangements of Series of Elements of a Banach Space

unconditionally convergent $\Leftrightarrow$ every rearrangements $\sum x_{\pi(k)}$ converges
absolute $\Rightarrow$ unconditionally, the converse is not true

conditionally convergent $\Leftrightarrow$ converges, but not unconditionally, i.e., among its rearrangements there are divergent ones

Theorem. *(1.3.1) If the series $\sum_{k=1}^{\infty} x_k$ in the Banach space X is unconditionally convergent, then all its rearrangements have the same sum.*

perfectly convergent $\Leftrightarrow \sum \alpha_k x_k$ converges for any choice $\alpha_i = \pm 1$

Theorem. *(1.3.2) For series $\sum_{k=1}^{\infty} x_k$ in a Banach space X the following conditions are equivalent:*

(i) *the series converges unconditionally;*

(ii) *all series of the form $x_{n_1} + x_{n_2} + x_{n_3} + \dots$ where $n_1 < n_2 < \dots$ converge; (all subseries)*

(iii) *the series converges perfectly.*

Theorem (Gelfand's theorem)**.** *(1.3.4) Let X be a Banach space and $\sum_{k=1}^{\infty} x_i$ be an unconditionally convergent series in X. Then the collection of the sums $s(\alpha) = \sum_{i=1}^{\infty} \alpha_i x_i$, where $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ runs through all sequences of ± 1, forms a compact set in X.*

2 Series in a finite-dimensional space

2.1 Steinitz's theorem on the sum range of a series

$\mathrm{SR}(\sum x_k)$ sums of all convergent rearrangements $\sum x_{\pi(k)}$

Lemma. *(2.1.1) Let K be a polyhedron in $\mathbb{R}^n$ given by a system of linear equalities and inequalities:*

$$f_i(x) = a_i, \qquad i = 1, 2, \ldots, p$$
$$g_j(x) \le b_j, \qquad j = 1, 2, \ldots, q,$$

where f_i and g_j are linear functionals. Let x_0 be a vertex (extreme point) of K and $A = \{j; g_j(x_0) = b_j\}$. Then the number of elements in A is not smaller than $n - p$.

Lemma (Rounding-off-coefficients lemma)**.** *(2.1.2) Let $(x_i)_{i=1}^n$ be a finite subset of an m-dimensional normed space, $(\lambda_i)_{i=1}^n$ be a set of scalar coefficients, $0 \le \lambda_i \le 1$, and $x = \sum_{i=1}^n \lambda_i x_i$. Then there exists a set of coefficients $\{\theta_i\}_{i=1}^n$, each θ_i equal to 0 or 1 (a set of rounded off coefficients) such that*

$$\left\| x - \sum_{i=1}^n \theta_i x_i \right\| \le \frac{m}{2} \cdot \max_i \|x_i\| \tag{1}$$

Lemma (Rearrangement lemma)**.** *(2.1.3) Suppose that in the m-dimensional normed space X there is given a finite set $\{x_i\}_{i=1}^n$ of vectors, whose sum is denoted by x. Then one can rearrange the elements of this set in such a way that for any natural $k \le n$ the following inequality holds:*

$$\left\| \sum_{i=1}^k x_{\pi(i)} - \frac{k-m}{n} x \right\| \le m \cdot \max_i \|x_i\| \tag{2}$$

$$\left\| \sum_{i=1}^k x_{\pi(i)} \right\| \le m \cdot \max_i \|x_i\| + (m+1) \left\| \sum_{i=1}^n x_i \right\| \tag{3}$$

Definition. (2.1.2) Let X be a Banach space and let $\sum_{k=1}^\infty x_k$ be a conditionally convergent series in X. A linear functional $f \in X^*$ is called a *convergence functional* for this series if $\sum_{k=1}^\infty |f(x_k)| < \infty$. The set of all convergence functionals of a series will be denoted by Γ. Also, we will denote by $\Gamma^\perp \subset X$ the annihilator of the set of convergence functionals:

$$\Gamma^\perp = \{x \in X \colon f(x) = 0 \text{ for all } f \in \Gamma\}.$$

$$\mathcal{P}((x_k)_{k=1}^\infty) = \{x_{i_1} + x_{i_2} + \cdots + x_{i_p}; i_1 < i_2 < \cdots < i_p; p \in \mathbb{N}\}$$
$$\mathcal{Q}((x_k)_{k=1}^\infty) = \left\{ \sum_{i=1}^n \lambda_i x_i; 0 \le \lambda_i \le 1; N = 1, 2, \ldots \right\}$$

Lemma ([F, Lemma 2])**.** *Let C be a convex set in a Banach space X, Γ a subset of X^* and $\Gamma^\perp$ its annihilator. If for every $f \notin \Gamma$ and every $T > 0$ there exist $x', x'' \in C$ such that $f(x') > T$ and $f(x'') < -T$, then $x \in C$ implies $x + \Gamma^\perp \subset C$.*

Lemma. *(2.1.4) Let X be and arbitrary Banach space, and let $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series in X. Then for any $x \in \overline{\mathcal{Q}}$ the set $x + \Gamma^{\perp}$ is contained in $\overline{\mathcal{Q}}$.*

Theorem (Steinitz's theorem)**.** *(2.1.1) Let $\sum_{k=1}^{\infty} x_k$ be a convergent series in an m-dimensional space E, and let $\sum_{k=1}^{\infty} x_k = s$. Then the sum range of the series is the affine subspace $s + \Gamma^{\perp}$, where $\Gamma^{\perp}$ is the annihilator of the set of convergence functionals:* $\mathrm{SR}(\sum_{k=1}^{\infty} x_k) = s + \Gamma^{\perp}$.

From the last theorem it follows that the sum range of a conditionally convergent series in a finite-dimensional space cannot reduce to a single point.

2.2 The Dvoretzky-Hanani Theorem on Perfectly Divergent Series

Definition. (2.2.1) A series $\sum_{k=1}^{\infty} x_k$ is said to be *perfectly divergent* if for any arrangement of signs the series $x_1 \pm x_2 \pm x_3 \pm \dots$ diverges.

Lemma. *(2.2.1) Let $(x_i)_{i=1}^{n}$ be elements of a space X, $\dim X = m$. Then there exist coefficients $\alpha_i = \pm 1$, $i = 1, \dots, n$ such that*

$$\max_{j \le n} \left\| \sum_{i=1}^{j} \alpha_i x_i \right\| \le 2m \cdot \max_{1 \le i \le n} \|x_i\|.$$

Theorem (The Dvoretzky-Hanani Theorem)**.** *(2.2.1) If a series in a finite-dimensional normed space is perfectly divergent, then its general term does not tend to zero.*

2.3 Pecherskii's Theorem

Theorem. *(2.3.1) Let X be an arbitrary Banach space, let $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series in X, and let $\sum_{k=1}^{\infty} x_k = x_0$. Further, suppose that no rearrangement makes the series perfectly divergent. Then $\mathrm{SR}(\sum_{k=1}^{\infty} x_k)$ is the closed affine subspace $x_0 + \Gamma^{\perp}$, where $\Gamma^{\perp}$ is the annihilator in X of the set $\Gamma \subset X^*$ of convergence functionals (the terminology is taken from Definitions 2.1.1 and 2.1.2).* [1]

$$\mathrm{SR}(\sum_{k=1}^{\infty} x_k) = x_0 + \Gamma^{\perp}$$

Analyzing the proof of Steinitz's theorem, one can derive the following assertion, which already holds in any Banach space.

Lemma. *(2.3.1) Suppose the series $\sum_{n=1}^{\infty} x_n$ in a Banach space X has the following two properties: $\sum_{n=1}^{\infty} x_n = x$ and*

[1] An equivalent formulation from [C]: $y \in \mathrm{SR} \sum_{n=1}^{\infty} x_n \Leftrightarrow (\forall g \in X^*)\ (\exists \sigma)\ g(s) = g(\sum_{k=1}^{\infty}) x_{\sigma(k)}$

(A) *for any* $\varepsilon > 0$ *there exists* $N = N(\varepsilon)$ *and* $\delta > 0$ *such that if* $\{y_i\}_{i=1}^n$ *is a finite set of terms of series* $\{y_i\}_{i=1}^n \subset \{x_i\}_{i=N}^\infty$, $\|\sum_{i=1}^n y_i\| \le \delta$, *then one can find a permutation* π *of the first* n *natural numbers for which*

$$\max_{j\le n} \left\| \sum_{i=1}^{j} y_{\pi(j)} \right\| \le \varepsilon;$$

(B) *for any* $\varepsilon > 0$ *there exists a number* $M = M(\varepsilon)$ *such that if* $\{y_i\}_{i=1}^n$ *is a finite set of terms of series* $\{y_i\}_{i=1}^n \subset \{x_i\}_{i=M}^\infty$, *and if* $0 \le \lambda_i \le 1$, $i = 1, \dots, n$, *then one can find a set of coefficients* $\{\theta_i\}_{i=1}^n$, $\theta_i \in \{0, 1\}$, *for which*

$$\left\| \sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \theta_i y_i \right\| \le \varepsilon.$$

Then for any $x \in \mathrm{SR}(\sum_{n=1}^\infty x_n)$ *the equality* $\mathrm{SR}(\sum_{n=1}^\infty x_n) = x + \Gamma^\perp$ *holds.*

The aim of the following lemmas is to show that under the assumptions of Theorem 2.3.1 the series $\sum_{n=1}^\infty x_n$ satisfies the conditions (A) and (B) of Lemma 2.3.1.

Lemma. *(2.3.2) Suppose that there is no rearrangement that makes the series* $\sum_{n=1}^\infty x_n$ *perfectly divergent. Then for any* $\varepsilon > 0$ *there exists a number* $N = N(\varepsilon)$ *such that, for any finite collection, written in arbitrary order, of elements* $\{y_1, y_2, \dots, y_n\}$ *of the set* $\{x_N, x_{N+1}, x_{N+2}, \dots\}$ *one can find a collection of signs* $\alpha_i = \pm 1$ *for which*

$$\max_{j\le n} \left\| \sum_{i=1}^{j} \alpha_i y_i \right\| \le \varepsilon.$$

Lemma. *(2.3.3) Let* ε *be an arbitrary positive number, and let* $\{x_i\}_{i=1}^N$ *be a set of elements of the normed space* X, *with the property that for any finite subset* $\{y_i\}_{i=1}^m \subset \{x_i\}_{i=1}^N$ *there exist signs* $\alpha_i = \pm 1$ *such that* $\|\sum_{i=1}^m \alpha_i y_i\| \le \varepsilon$. *Then for any set of coefficients* $\{\lambda_i\}_{i=1}^N$, $0 \le \lambda_i \le 1$ *there exist "rounded off" coefficients* $\theta_i \in \{0, 1\}$ *for which*

$$\left\| \sum_{i=1}^{N} \lambda_i x_i - \sum_{i=1}^{N} \theta_i x_i \right\| \le \varepsilon.$$

Lemma (Chobanyan's lemma)**.** *(2.3.4) Let* $\{x_i\}_{i=1}^n$ *be elements of the space* X, *with* $\sum_{i=1}^n x_i = 0$. *Then there exists a permutation* σ *such that, for any choice of signs* $\alpha_i = \pm 1$,

$$\max_{k\le n} \left\| \sum_{i=1}^{k} \alpha_i x_{\sigma(i)} \right\| \ge \max_{k\le n} \left\| \sum_{i=1}^{k} x_{\sigma(i)} \right\|. \tag{4}$$

Lemma. *(2.3.5) Let* $\{x_i\}_{i=1}^n \subset X$*,* $\|\sum_{i=1}^n x_i\| \leq \varepsilon$*, and assume that for any permutation* ν *there exists a choice of signs* $\alpha_i = \pm 1$ *for which*

$$\max_{k \leq n} \left\| \sum_{i=1}^k \alpha_i x_{\nu(i)} \right\| \leq \varepsilon.$$

Then there exists a permutation σ *such that* $\max_{k \leq n} \|\sum_{i=1}^k x_{\sigma(i)}\| \leq 3\varepsilon$*.*

3 Conditional convergence in an infinite-dimensional space

3.1 Basic counterexamples

There is no known example of a series in a Hilbert space whose sum range is a linear, but nonclosed subspace.

3.2 A series whose sum range consists of two points

Based on papers of Kornilov [K] and Kadets-Wozniakowski.[2]

Construction of this example is done in $L_p(Q)$, where $Q = [0,1]^\omega$.

Theorem. *(3.2.1) The series* $\sum_{n=1}^\infty h_n$ *converges to* 0 *in the metric of any spaces* $L_p(Q)$*,* $1 \leq p < \infty$*. There exists a permutation* π *such that the rearrange series* $\sum_{n=1}^\infty h_{\pi(n)}$ *converges to* 1*. The sum of any convergent rearrangement of the original series,* $h_0 = \sum_{n=1}^\infty h_{\sigma(n)}$*, is equal to either* 0 *or* 1

Lemma. *(3.2.1)*[3] *Let* (X, μ) *and* (Y, ν) *be measures spaces with probability measures. Let* $f(x,y)$ *and* $g(x,y)$ *be functions in* $L_1(X \times Y)$*, each of which depends on only one variable:* $f(x,y) = \tilde{f}(x)$*,* $g(x,y) = \tilde{g}(y)$*. Then*

$$\|f + g\| \geq \|f\| + \|g\|[1 - 2(\mu \times \nu)(\operatorname{supp} f)].$$

It is not known whether any set in a Hilbert space for any $n \in \mathbb{N}$ can serve as the sum range of the series.[4]

3.3 Chobanyan's Theorem

$r_1, r_2, \ldots$ denotes a sequence of independent random variables that take the values ± 1 with equal probabilities.

Consider the random variables $S_k = \|\sum_{i=1}^k r_i x_i\|$.

[2] This paper was not available to me.

[3] This lemma is given without proof in [KK]. An easy proof can be found in [W, Lemma 4.1].

[4] Is seems that this problem has been solved since the publication of the book – see [W].

Lemma. *(3.3.1) For any $t > 0$,*

$$P\left[\sup_{k\leq n} S_k > t\right] \leq 2P[S_n > t].$$

Lemma. *(3.3.2) $E(\sup_{k<n} S_k) \leq 2E(S_n)$.*

Lemma (Chobanyan's inequality)**.** *(3.3.3) Let $\{x_i\}_{i=1}^n$ be elements of a normed space with $\sum_{i=1}^n x_i = 0$. Then there exists a permutation σ for which*

$$\sup_{k\leq n}\left\|\sum_{i=1}^k x_{\sigma(i)}\right\| \leq 2E\left(\left\|\sum_{i=1}^n r_i x_i\right\|\right).$$

Theorem. *(3.3.1) Let $\{x_i\}_{i=1}^n$ be elements of a Banach space with the property that $\lim_{m>n\to\infty} E(\|\sum_{i=n}^m r_i x_i\|) = 0$. Then the assertion of Steinitz's theorem holds for the series $\sum_{i=1}^\infty x_i$.*

Corollary (Chobanyan's theorem)**.** *(3.3.1) Let X be a Banach space. Then for the assertion of Steinitz's theorem to hold for a series $\sum_{n=1}^\infty x_n$ in X it is sufficient that the series $\sum_{n=1}^\infty r_n x_n$ converge almost everywhere, or, in the other words, that the series $\sum_{n=1}^\infty \pm x_n$ converge for almost all choices of signs.*

3.4 The Khinchin Inequalities and the Theorem of M. I. Kadets on Conditionally Convergent Series in L_p

4 Unconditionally convergent series

4.1 The Dvoretzky-Rogers Theorem

Lemma (The Dvoretzky-Rogers Lemma)**.** *(4.1.1) Let X be an n-dimensional normed space. Then there exist elements $\{x_i\}_{i=1}^n \subset S(X)$ such that*

$$\left\|\sum_{i=1}^m t_i x_i\right\| \leq \left(1 + \sqrt{\frac{m(m-1)}{n}}\right) \cdot \sqrt{\sum_{i=1}^m t_i^2} \tag{5}$$

5 Orlicz's theorem and the structure of finite-dimensional subspaces

5.1 Finite Representability

Definition. (5.1.1) Let X and Y be Banach Spaces. The *Banach-Mazur distance* between X and Y is the number $d(X,Y) = \inf_T\{\|T\|, \|T^{-1}\|\}$, where the infimum is taken over all isomorphisms $T\colon X \to Y$. If X and Y are not isomorphic, we put $d(X,Y) = +\infty$.

$$d(X,Y) \cdot d(Y,Z) \geq d(X,Z)$$

6 Some results from the general theory of Banach spaces

6.1 Fréchet Differentiability of Convex Functions

6.2 Dvoretzky's theorem

Theorem (Dvoretzky's theorem)**.** *(6.2.1) Let k be an arbitrary natural number and let $\varepsilon > 0$. Then there exists a number $n(k, \varepsilon)$ such that, for any normed space X with $\dim X > n(k, \varepsilon)$ there is a k-dimensional subspace $Y \subset X$ such that $d\left(Y, l_2^{(k)}\right) < 1 + \varepsilon$.*

6.3 Basic sequences

A sequence $(e_k)_{k=1}^{\infty}$ in an infinite-dimensional Banach space is called a *basis* if every element $x \in X$ has a unique representation as a series in the elements e_k:

$$x = \sum_{k=1}^{\infty} a_k(x) e_k, \qquad a_k(x) \in \mathbb{R}.$$

A sequence of elements $(e_k)_{k=1}^{\infty}$ of a Banach space is called a *basic sequence* if $(e_k)_{k=1}^{\infty}$ is a basis in $\overline{\mathrm{Lin}}(e_k)_{k=1}^{\infty}$.

Theorem. *(6.3.3) Let X be an infinite-dimensional Banach space. Then for any $\varepsilon > 0$ there exists a basic sequence $(e_n)_{n=1}^{\infty} \subset X$ whose basic constant is not smaller than $1 - \varepsilon$.*

6.4 Some Applications to Conditionally Convergent Series

Theorem. *(6.4.1) In any infinite-dimensional Banach space there are series whose sum range reduces to a single point, but which are not unconditionally convergent.*[5]

Theorem (Bessage-Pelczyński theorem)**.** *(6.4.3) The following assertions are equivalent:*

(a) *The Banach space x contains no subspaces isomorphic to c_0.*

(b) *Every weakly absolutely convergent series in X is weakly convergent.*

(c) *Every weakly absolutely convergent series in X is unconditionally convergent.*

(d) *Every weakly absolutely convergent series in X is norm convergent.*

[5]My note: I think they use the fact that the projections (coefficient functionals) are continuous, (i.e., every basis is a Schauder basis). I am not sure that they have shown this result in the text.

References

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