Series in Banach Spaces

This file contains (attempted) solutions of some problems from [KK] and some comments on some places, which were unclear to me when reading the book.

1 Background material

1.1 Numerical series. Riemann's Theorem

1.2 Main Definitions. Elementary Properties of Vector Series

1.3 Preliminary Material on Rearrangements of Series of Elements of a Banach Space

2 Series in a finite-dimensional space

2.1 Steinitz's theorem on the sum range of a series

Remark 2.1.4 From Steinitz's theorem it follows that the sum range of a conditionally convergent series in a finite-dimensional space cannot reduce to a single point.

If $s + \Gamma^\perp = \{s\}$, then $\Gamma^\perp = \{0\}$. We first notice that this implies that $\Gamma = X^*$.

Since X is finite-dimensional, we have $\dim X = \dim X^* = m$. If Γ is a proper subspace of x, then $\dim\Gamma = k < m$ and $\Gamma = [f_1,\dots,f_k]$. Now $\Gamma^\perp$ is the solution space of the system of k linear equations $f_1(x) = \dots = f_k(x) = 0$ and $\dim \Gamma^\perp = m-k > 0$. Hence $\dim\Gamma + \dim\Gamma^\perp = m$. This equation has infinite-dimensional counterparts under some conditions on Γ, see [Z, p.198,Proposition 3.9.21].

Now, since $\Gamma = X^*$, we get that $\sum f(x_k)$ is absolutely convergent for each $f\in X^*$. This implies that every rearrangement $\sum f(x_{\pi(k)})$ converges to $f(s)$. Again, since X is finite-dimensional, this implies that $\sum x_{\pi(k)} = x$. (Convergence on every coordinate is equivalent to convergence in X.)

2.2 The Dvoretzky-Hanani Theorem on Perfectly Divergent Series

Exercise 2.2.1 For a Banach space X the following two assertions are equivalent:

(1) for any perfectly divergent series in X the general term does not tend to zero;

(2) there exists a constant $C > 0$ such that, for any finite set of elements $\{x_i\}_{i=1}^n \subset X$ there exists a set of coefficients $t_i = \pm 1$ (depending on the set $\{x_i\}_{i=1}^n$) for which

$$\max_{j \le n} \left\| \sum_{i=1}^{j} t_i x_i \right\| \le C \max_i \|x_i\|.$$

Solution. The implication (2)⇒ (1) is basically repeating the same argument as in the proof of Dvoretzky-Hanani Theorem 2.2.1.

We show $\neg(2) \Rightarrow \neg(1)$. Negation of (2) means that, for a given N:
There exists a finite set $\exists x_1^{(N)}, \dots, x_{k(N)}^{(N)}$ such that
for any choice of signs $\forall t_1, \dots, t_{k(N)} = \pm 1$
the inequality

$$\left\| \sum_{i=1}^{k(N)} t_i x_i^{(N)} \right\| \ge N \max\|x_i\|$$

holds.

Using the above finite sets we construct a series y_n which is perfectly divergent and $\lim_{n\to\infty} y_n = 0$

Now we put $y_i^{(N)} = \frac{x_i^{(N)}}{N \max\|x_i^{(N)}\|}$. We construct a series y by putting the segments $y^{(N)}$ one after another: $y^{(1)}, y^{(2)}, \dots$

Now, for any choice of signs t_i, for the N-th segment

$$\left\| \sum_{i=1}^{k(N)} t_i y_i^{(N)} \right\| \ge 1.$$

Hence the series y is not convergent.

On the other hand, we have $\max\|y_i^{(N)}\| \le \frac{1}{N}$, which implies $\lim_{n\to\infty} y_n = 0$.

Exercise 2.2.2 Does the assertion of Exercise 2.2.1 holds true for noncomplete normed space?

Hint from [KK]: In Exercise 2.2.1 each of assertions (1) and (2) is equivalent to the space being finite-dimensional, independently of the completeness assumption. In contrast to Exercise 2.2.1, the present problem is quite delicate. We only know a proof based on Dvoretzky's Theorem 6.2.1.

Solution (P.K.,M.S.). We did not use the completeness of the space in the proof of (1)⇒ (2).

It remains to prove (2)⇒ (1). It suffices to show that if (2) holds in a linear normed space X, then it holds in its completion X^* as well.

Notice that the condition (2) can be reformulated as:

$$\min_{t_i \in \{\pm 1\}^n} \max_{j \le n} \left\| \sum_{i=1}^{n} t_i x_i \right\| \le C \max_i \|x_i\|.$$

Let us define: $f\colon X^n \to \mathbb{R}$ by

$$f(x_1,\dots,x_n) = \min_{t_i\in\{\pm1\}^n} \max_{j\leq n} \left\| \sum_{i=1}^{n} t_i x_i \right\|.$$

We will show that this function is continuous (with respect to the product topology on X^n).

Suppose we are given a fixed finite set $(t_1,\dots,t_n)\in\{\pm1\}^n$. It is easy to see that the assignment

$$(x_1,\dots,x_n)\mapsto(t_1x_1, t_1x_1+t_2x_2,\dots,t_1x_1+\dots+t_nx_n)$$

is a continuous linear functional. Now, using the facts that $x\mapsto\|x\|$ is continuous on X and the assignment $(a_1,\dots,a_n)\mapsto\max\{a_1,\dots,a_n\}$ is continuous as a map from $\mathbb{R}^n$ to $\mathbb{R}$, we see that, for any fixed $t_1\dots t_n$, the map

$$(x_1,\dots,x_n)\mapsto \max_{j\leq n}\left\|\sum_{i=1}^{j} t_i x_i\right\|$$

is a continuous function from X^n to $\mathbb{R}$.

Now, taking minimum is also a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$, and to obtain the function f we just take the minimum over the finite set $\{\pm1\}^n$.

Hence the function f is continuous, too. We will use this fact to prove the validity of (2) in X^*.

Suppose we are given a finite subset $\{x_1\dots x_n\}$ of X^*. From the density of X in X^* we have a sequence x_{ik} of elements of X converging to x_i. Now, as the condition (2) holds in X, for each k we have

$$f(x_{1k},\dots,x_{nk})\leq C\max_i\|x_{ik}\|.$$

Since both sides of this inequality are continuous functions $X^n\to\mathbb{R}$ and the convergence in X^n with the product topology is equivalent to the pointwise convergence, we get

$$f(x_1,\dots,x_n)\leq C\max\|x_i\|$$

by taking the limit $k\to\infty$.

Notice that the hint from [KK] claims even more – that these conditions are equivalent to finite-dimensionality of the space X. We did not show this fact.

3 Conditional convergence in an infinite-dimensional space

3.1 Basic counterexamples

3.2 A series whose sum range consists of two points

Constant – in the proof of Theorem 3.2.1 The reasoning why the limit of any convergent rearrangement of the given series is a constant function was not clear to me.

[KK, p.33]: "For any $k \in \mathbb{N}$, starting with one of them, will not depend on the coordinate t_k. Since all terms take only integer values, h_0 is an integer constant."

Proof of [W, Proposition 2.2]: "Thus for some integer K_0 the function $\sum_{k=1}^{K} c_{\sigma(k)}$ is constant with regard to x_l for $K \geq K_0$, and thus the limit of the series also has to be constant with regard to x_l. As this applied to an arbitrary l, the limit simply has to be constant."

The notions "not depending on t_k" and "constant with regard to t_k" means (if I understand them correctly) that, if I change the value of t_k and all remaining coordinates are unchanged, the value of the function h_0 is unchanged as well. (This is equivalent to saying that level sets are tail sets – see bellow.) There are non-constant functions having this property: just take $f_{\mathcal{F}}(x) = \mathcal{F}\text{-}\lim x_n$ for any free ultrafilter $\mathcal{F}$. (This function has also integer values if x_n is integer-valued.)

Another thing, which makes this claim suspicious, is that in L_p (which consists of equivalence classes of functions) I would expect result like "is constant almost everywhere" instead of "is constant". (Although probably when author works in L_p, this is what he means under "constant function" – but what I mean is that if the same argument would prove that it is constant everywhere, then it looks suspiciously.)

Proof using 0-1 law. (M.S.) I'll give my proof that the limit h_0 of any convergent rearrangement of the series in this proof is a constant function. I did not use the special form of the series. I have only used the fact that, starting from some index, the partial sums of the series do not depend on k-th coordinate and the fact that the limit is integer valued. So maybe it is possible to find a simpler proof for this particular situation.

First of all, it is known that if a sequence f_n converges to f in L_p, then the exists a subsequence which converges to f almost everywhere (e.g. [DM, Theorem 2.8.2]). [1] This can be used to show that the limit h_0 does not depend on the k-th coordinate.

We say that a subset A of $Q = \langle 0,1\rangle^\omega$ is a *tail set* if $x \in A$ and $x =^* y$ implies $y \in A$, where $x =^* y$ denotes the fact that the sequences x and y differ only in finitely many terms.

We will use Kolmogorov's 0-1 law, which is usually formulated in terms of random variables [S, p.381,Theorem 1] [H, p.201]. For us, the reformulation using probability measures is more appropriate (like in [K, p.104,Exercise 17.1]). For our situation:

Theorem (Zero-one law)**.** *If $A \subseteq Q$ is a measurable tail set, then either $\mu(A) = 0$ or $\mu(A) = 1$.*

Recall, that we are working on the space $\langle 0,1\rangle^\omega$ with the standard product measure obtained from Lebesgue measure on $\langle 0,1\rangle$. In particular, $\mu(Q) = 1$.

Now, the fact that h_0 does not depend on k-th coordinate for any k means that the *level sets* $L_c = \{x \in Q; h_0(x) = c\}$ (where c can be any real number) are

[1] TODO reference from [ŠŠN]

tail set. We assume that $h_0 \in L_p(Q)$, therefore it is measurable. Hence all level sets are measurable. Then for each c we have either $\mu(L_c) = 1$ or $\mu(L_c) = 0$.

Now, as h is integer valued, we have $Q = \bigcup_{c\in\mathbb{Z}} L_c$ and $1 = \sum_{c\in\mathbb{Z}} \mu(L_c)$ by σ-additivity. Therefore there exists $c \in \mathbb{Z}$ such that $\mu(L_c) = 1$, i.e., $f(x) = c$ almost everywhere. □

Notice, that the example with $f_{\mathcal{F}}$ shows that the same result is not true true without the assumption that the function in question is integer valued. I.e., not every function $f\colon Q \to \mathbb{R}$ such that the level sets are tail set is constant almost everywhere.

3.3 Chobanyan's Theorem

Proof of Theorem 3.3.1:

Let $A \subseteq G \subseteq \mathbb{N}$. Then

$$E\left(\left\|\sum_{i\in A} r_i x_i\right\|\right) \overset{(*)}{\leq} \frac{1}{2}\left(E\left(\left\|\sum_{i\in A} r_i x_i + \sum_{i\in G\setminus A} r_i x_i\right\|\right) + \right.$$
$$\left. + E\left(\left\|\sum_{i\in A} r_i x_i - \sum_{i\in G\setminus A} r_i x_i\right\|\right)\right) \overset{(\triangle)}{=} E\left(\left\|\sum_{i\in G} r_i x_i\right\|\right) \quad (1)$$

The inequality $(*)$ follows from the triangle inequality. The equality $(\triangle)$ is a consequence of $E\left(\left\|\sum_{i\in G} r_i x_i\right\|\right) = E\left(\left\|\sum_{i\in A} r_i x_i + \sum_{i\in G\setminus A} r_i x_i\right\|\right) = E\left(\left\|\sum_{i\in A} r_i x_i - \sum_{i\in G\setminus A} r_i x_i\right\|\right)$. (Modifying some r_i's to the opposite sign does not change the random variable we are working with.)[2]

Using this inequality, the assumption $\lim_{m>n\to\infty} E(\|\sum_{i=n}^m r_i x_i\|) = 0$ and Chobanyan's inequality we are going to verify the conditions (A) and (B) of Lemma 2.3.1.

(A) for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ and $\delta > 0$ such that if $\{y_i\}_{i=1}^n$ is a finite set of terms of series $\{y_i\}_{i=1}^n \subset \{x_i\}_{i=N}^\infty$, $\|\sum_{i=1}^n y_i\| \leq \delta$, then one can find a permutation π of the first n natural numbers for which

$$\max_{j\leq n} \left\|\sum_{i=1}^{j} y_{\pi(j)}\right\| \leq \varepsilon;$$

(B) for any $\varepsilon > 0$ there exists a number $M = M(\varepsilon)$ such that if $\{y_i\}_{i=1}^n$ is a finite set of terms of series $\{y_i\}_{i=1}^n \subset \{x_i\}_{i=M}^\infty$, and if $0 \leq \lambda_i \leq 1$, $i = 1, \dots, n$, then one can find a set of coefficients $\{\theta_i\}_{i=1}^n$, $\theta_i \in \{0, 1\}$, for which

$$\left\|\sum_{i=1}^{n} \lambda_i y_i - \sum_{i=1}^{n} \theta_i y_i\right\| \leq \varepsilon.$$

[2] Observation of P.L. If $x_i = 1$ for each i, then this is the well-known "drunken sailor problem". This inequality then says that the average distance from the beginning is larger, if we allow him to walk longer. Which is intuitively clear.

Proof of condition (A). Given $\varepsilon > 0$ choose N such that for $m > n > N$ the inequality $E(\|\sum_{i=n}^{m} r_i x_i\|) < \frac{\varepsilon}{5}$ holds and put $\delta = \frac{\varepsilon}{5}$.

Let $\{y_i\}_{i=1}^{n} \subset \{x_i\}_{i=N}^{\infty}$ and $\|\sum_{i=1}^{n} y_i\| \leq \delta$. Let us denote $z_1 = y_1, \ldots, z_n = y_n$ and $z_{n+1} = -\sum_{i=1}^{n} y_i$. Now $\sum_{i=1}^{n+1} z_i = 0$, so we can use Chobanyan's inequality for the elements $z_1, \ldots, z_{n+1}$.

We get the existence of a permutation $\sigma \in S_{n+1}$

$$\max_{k \leq n+1} \left\| \sum_{i=1}^{k} z_{\sigma(i)} \right\| \leq 2E\left(\left\| \sum_{i=1}^{n+1} r_i z_i \right\| \right) \leq 2E\left(\left\| \sum_{i=1}^{n} r_i y_i \right\| \right) + 2E\left(\|r_{n+1} z_{n+1}\|\right)$$

We have $E\left(\|r_{n+1} z_{n+1}\|\right) = \|z_{n+1}\| = \|\sum_{i=1}^{n} y_i\| \leq \frac{\varepsilon}{5}$. Using (1) we get[3] $E\left(\|\sum_{i=1}^{n} r_i y_i\|\right) \leq \frac{\varepsilon}{5}$. So together we have

$$\max_{k \leq n+1} \left\| \sum_{i=1}^{k} z_{\sigma(i)} \right\| \leq \frac{4}{5}\varepsilon$$

Next we modify the permutation $\sigma \in S_{n+1}$ to a permutation $\pi \in S_n$ in a natural way, by omitting the element $n+1$. (More precisely, if $\sigma(k_0) = n+1$, then $\pi(i) = \sigma(i)$ for $i \leq k_0$ and $\pi(i) = \sigma(i+1)$ for $i \geq k_0$.) [4] For any k we have either

$$\sum_{i=1}^{k} y_{\pi(i)} = \sum_{i=1}^{k} z_{\sigma(i)}$$

or

$$\sum_{i=1}^{k} y_{\pi(i)} + z_{n+1} = \sum_{i=1}^{k+1} z_{\sigma(i)}.$$

Using triangle inequality we get

$$\max_{k \leq n} \left\| \sum_{i=1}^{k} y_{\pi(i)} \right\| \leq \max_{k \leq n+1} \left\| \sum_{i=1}^{k} z_{\sigma(i)} \right\| + \|z_{n+1}\| \leq \frac{4}{5}\varepsilon + \frac{\varepsilon}{5} = \varepsilon.$$

Proof of condition (B). We can choose N and δ as in the first part. By Lemma 2.3.3 it suffices to show the existence of $\alpha_i = \pm 1$ with $\|\sum_{i=1}^{n} \alpha_i y_i\| \leq \varepsilon$.

But since (using (1) in the same way as above)

$$E\left(\left\| \sum_{i=1}^{n} r_i y_i \right\| \right) \leq \frac{\varepsilon}{5} < \varepsilon,$$

there exists some $\alpha_i = \pm 1$ with $\|\sum_{i=1}^{n} \alpha_i y_i\| \leq \varepsilon$.

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[3]If $y_i = x_{n_i}$ and $n = \min n_i$, $m = \max n_i$ then we can choose $G = \{n, \ldots, m\}$ and $A = \{i_1, \ldots, i_n\}$.

[4]This can be displayed graphically as follows:

$$\begin{pmatrix} 1 & \ldots & k-1 & k & k+1 & \ldots & n & n+1 \\ \sigma(1) & \ldots & \sigma(k-1) & n+1 & \sigma(k+1) & \ldots & \sigma(n) & \sigma(n+1) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \ldots & k-1 & k & k+1 & \ldots & n & n+1 \\ \sigma(1) & \ldots & \sigma(k-1) & \cancel{n+1} & \sigma(k+1) & \ldots & \sigma(n) & \sigma(n+1) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \ldots & k-1 & k & k+1 & \ldots & n-1 & n \\ \sigma(1) & \ldots & \sigma(k-1) & \sigma(k+1) & \sigma(k+2) & \ldots & \sigma(n) & \sigma(n+1) \end{pmatrix}$$

3.4 Khintchin inequalities. Theorem of M. I. Kadets

Estimates in the proof of Khinchin inequalities.

I have posted my solutions here:
`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=372367`
`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=372373`
`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=372375`
`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=52&t=372376`
(With the hope that someone will comment on them or post a better solution.)

$\min\left\{1-\prod_{i=1}^n \cos t_i; \sum t_i^2 = 1\right\} = 1 - \left(\cos\frac{1}{\sqrt{n}}\right)^n$

$$1 - \left(\cos\frac{1}{\sqrt{n}}\right)^n \geq 1 - e^{-1/2}$$

[5] [6]

The inequality

$$\cosh\frac{1}{\sqrt{n}} \leq e^{1/2n}$$

follows from comparing Taylor expansions:
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$
$e^x = 1 + x + \frac{x^2}{2!} + \dots$
$\cosh\frac{1}{\sqrt{n}} = 1 + \frac{1}{2!n} + \frac{1}{4!n^2} + \frac{1}{6!n^3}$

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[5] This was incorrect:
First let us recall the well-known inequality $(1+\frac{1}{n})^n \leq e$. (The sequence $\left(1+\frac{1}{n}\right)^n$ is increasing, see [VN, Veta V.1.3])

From this inequality we get that $\left(1-\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)^n = \left(1-\frac{1}{n^2}\right)^n \leq 1$ and

$$\left(1-\frac{1}{n}\right)^n \leq \frac{1}{\left(1+\frac{1}{n}\right)^n} \geq \frac{1}{e}.$$

This did not work either:
First, let us recall the well-known inequality (see [KN, 2.1.38])

$$\left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}.$$

We have that $\left(1-\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)^{n+1} = \left(1-\frac{1}{n^2}\right)^n\left(1+\frac{1}{n}\right) \geq \left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right) = 1-\frac{1}{n^2}$

[6] Another unsuccessful trial:
The sequence $\left(1-\frac{1}{n}\right)^n$ is increasing – by AM-GM inequality for $a_0 = 1$, $a_1 = \dots = a_n = 1+\frac{1}{n}$ we get $\left(1-\frac{1}{n}\right)^n \leq \left(\frac{1+(n-1)}{n+1}\right)^{n+1} = \left(1-\frac{1}{n+1}\right)^{n+1}$

Hence

$$\left(1-\frac{1}{n}\right)^n \leq \frac{1}{e}$$

We know that $\cos x \geq 1 - \frac{x^2}{2}$. Hence

$$\left(\cos\frac{1}{\sqrt{n}}\right) \geq \left(1-\frac{1}{2n}\right)^n.$$

$e^{1/2n} = 1 + \frac{1}{2n} + \frac{1}{2!(2n)^2} + \frac{1}{3!(2n)^3}$

$$\frac{1}{(2k)!n^k} \leq \frac{1}{k!2^k n^k}$$

$\cosh \frac{1}{\sqrt{n}} \leq e^{1/2n} \Rightarrow \cosh \left(\frac{1}{\sqrt{n}}\right)^n \leq e^{1/2}$

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