Classical descriptive set theory

Notes from [Ke]. [1]

1 Polish spaces

1.1 Topological and metric spaces

1.A Topological spaces The initial topology is called *topology generated by* $(f_i)_{i\in I}$.

1.B Metric spaces

Theorem (Urysohn metrization theorem)**.** *(1.1) Let X be a second countable topological space. Then X is metrizable iff X is T_1 and regular.*

[2]

Theorem (Urysohn's Lemma)**.** *(1.2) Let X be a metrizable space. If A, B are two disjoint subsets of X, there is a function $f\colon X\to\langle 0,1\rangle$ such that $f(x)=0$ for $x\in A$ and $f(x)=1$ for $x\in B$.*

Theorem (Tietze Extension Theorem)**.** *(1.3) Let X be a metrizable space. If $A\subseteq X$ is closed and $f\colon A\to\mathbb{R}$ is continuous, there is $\hat{f}\colon X\to\mathbb{R}$ which is continuous and extends f. Moreover, if f is bounded by M, i.e., $|f(x)|\le M$ for $x\in A$, so is f.*

1.2 Trees

2.A Basic concepts A^n = all sequences $(s_0,\dots,s_{n-1})$ with s_i from the set A

$A^0=\{\emptyset\}$

$\emptyset$ = empty sequence

length(s) = length of a finite sequences

$$A^{<\mathbb{N}}=\bigcup_{n\in\mathbb{N}}A^n$$

$s|m=(s_0,\dots,s_{m-1})$ and $s|0=\emptyset$

[1] My note: Supplementary reading: [Mo] and [S]. Also [A, Chapter 3] seems to be a good introduction into the problematic of Polish spaces, Borel and analytic sets. For some details in the topological proofs I have consulted [E] and [AB], both these books have advantage that they are rather exhaustive and contain detailed proofs.

[2] LH We can replace regular by normal. TODO check

We say that $s \in A^n$ is an *initial segment* of $x \in A^{\mathbb{N}}$ if $s = x|n$. We will write $s \subseteq n$ is s is an initial segment of x.

Two finite sequences are *compatible* if one is an initial segment of the other and *incompatible* otherwise. We use $s \perp t$ to indicate that s and t are incompatible.

The *concatenation* of two sequences $s = (s_i)_{i<n}$ and $t = (t_j)_{j<m}$ is the sequence $s\hat{\ }t = (s_0, \ldots, s_{n-1}, t_0, \ldots, t_{m-1})$. We write $s\hat{\ }a$ for $s\hat{\ }(a)$ if $a \in A$.

infinite concatenation

Definition. (2.1) *Tree* = subset $T \subseteq A^{<\mathbb{N}}$ closed under initial segments. (I.e., if $t \in T$ and $s \subseteq t$, then $s \in T$.)

nodes = elements of T

An *infinite branch* of T is a sequence $x \in A^{\mathbb{N}}$ such that $x|n \in T$, for all n. The *body* of T, written as $[T]$, consist of all infinite branches of T, i.e.,

$$[T] = \{x \in A^{\mathbb{N}}; \forall n(x|n \in T)\}.$$

Finally, we call a tree T *pruned* if every $s \in T$ has a proper extension $t \supsetneqq s$, $t \in T$.

2.B Trees and Closed Sets If we take A as a topological space with the *discrete topology*, then A is metrizable; $d(a, b) = 1$.

$A^{\mathbb{N}}$ viewed as the product space – metric $d(x, y) = 2^{-n-1}$ if $x \neq y$ and n is the least number with $x_n \neq y_n$.

A metric d is an *ultrametric* if

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

Show that the above metric is an ultrametric.

The *standard basis* for the topology of $A^{\mathbb{N}}$ consists of the sets

$$N_s = \{x \in A^{\mathbb{N}} : s \subseteq x\},$$

where $s \in A^{<\mathbb{N}}$. Note that $s \subseteq t \Leftrightarrow N_s \supseteq N_t$ and $s \perp t \Leftrightarrow N_s \cap N_t = \emptyset$.

Proposition. *(2.4) The map $T \mapsto [T]$ is a bijection between pruned trees on A and closed subsets of $A^{\mathbb{N}}$. Its inverse is given by*

$$F \mapsto T_F = \{x|n \colon x \in F, n \in \mathbb{N}\}.$$

We call T_F the tree of F.

If T is a tree on A, then for any $s \in A^{\mathbb{N}}$,

$$T_s = \{t \in A^{<\mathbb{N}}; s\hat{\ }t \in T\}$$

and

$$T_{[s]} = \{t \in T; t \text{ is compatible with } s\}.$$

Thus $[T_{[s]}] = [T] \cap N_s$ forms a basis for the topology of $[T]$. Note that $T_{[s]}$ is a subtree of T, but T_s in general is not.

Definition. (2.5) Let S, T be trees (on sets A, B, resp.). A map $\varphi\colon T \to S$ is called *monotone* if $s \subseteq t$ implies $\varphi(s) \subseteq \varphi(t)$. For such φ let

$$D(\varphi) = \{x \in [S] : \lim_n \text{length}(\varphi(x|n)) = \infty\}.$$

For $x \in D(\varphi)$, let

$$\varphi^*(x) = \bigcup_n \varphi(x|n) \in [T].$$

We call φ *proper* if $D(\varphi) = [S]$.

Proposition. *(2.6) The set $D(\varphi)$ is G_δ in $[S]$ and $\varphi^*\colon D(\varphi) \to [T]$ is continuous. Conversely, if $f\colon G \to [T]$ is continuous, with $G \subseteq [S]$ a G_δ set, then there is monotone $\varphi\colon S \to T$ with $f = \varphi^*$.*

A closed set F in a topological space X is a *retract* of X if there is a continuous surjection $f\colon X \to F$ such that $f(x) = x$ for $x \in F$.

Proposition. *(2.8) Let $F \subseteq H$ be two closed nonempty subsets of $A^{\mathbb{N}}$. Then F is a retract of H.*

2.C Trees on Products A tree on $A = B \times C$ can be understood as a subset of $B^{<\mathbb{N}} \times C^{<\mathbb{N}}$ with the property $(t, u) \in T \Rightarrow \text{length}(t) = \text{length}(u)$.

If T is a tree on $B \times C$ and $x \in B^{\mathbb{N}}$, consider *section tree* $T(x)$ on C defined by

$$T(x) = \{s \in C^{<\mathbb{N}}; (x|\,\text{length}(s), s) \in T\}.$$

Note that if T is pruned it is not necessarily true that $T(x)$ is also pruned. Also,

$$(x, y) \in [T] \;\Leftrightarrow\; y \in [T(x)].$$

Similarly, for $s \in B^{<\mathbb{N}}$, we define $T(s) = \{t \in C^{<\mathbb{N}}; \text{length}(t) \le \text{length}(s) \wedge (s|\,\text{length}(t), t) \in T\}$.

2.D Leftmost Branches Let T be a tree on A and let $<$ be a well-ordering of A. If $[T] \neq \emptyset$, then we specify the $<$-*leftmost branch* of T, denoted by a_T, as follows. We define $a_T(n)$ by recursion on n:

$$a_T(n) = \text{ the } < \text{-least element} a \text{ of } A \text{ such that } [T_{(a_T|n)}] \neq \emptyset.$$

lexicographical ordering on $A^{\mathbb{N}}$ or on A^m

When T is pruned, a_T is also characterized by the property that for each m, $a_T|m$ is the lexicographically least element of $T \cap A^m$.

2.E Well-founded Trees and Ranks If a tree T on A has no infinite branches, i.e., $[T] = \emptyset$, then we call T *well-founded.* This is because it is equivalent to saying that the relation $s \prec t \Leftrightarrow s \supsetneqq t$ restricted to T is well-founded. (See appendix B.) On the other hand, if $[T] \neq \emptyset$, we call T *ill-founded.* If T is a well-founded tree, we denote the *rank function* of $\prec$ restricted to T by ρ_T.

$$\rho_T(s) = \sup\{\rho_T(t) + 1; t \in T, t \supsetneqq s\},$$

for $s \in T$. An easy argument shows that we also have

$$\rho_T(s) = \sup\{\rho_T(s\hat{\ }a) + 1 : s\hat{\ }a \in T\}.$$

Also, $\rho_T(s) = 0$ if $s \in T$ is *terminal,* i.e., for no a, $s\hat{\ }a \in T$. We also put $\rho_T(s) = 0$ if $s \notin T$. The *rank* of a well-founded tree is defined by $\rho(T) = \sup\{\rho_T(s) + 1 : s \in T\}$. Thus if $T \neq \emptyset$ $\rho(T) = \rho_T(\emptyset) + 1$. The *rank* of a well-founded tree is defined by $\rho(T) \sup\{\rho_T(s) + 1 : s \in T\}$.

If S, T are trees (on A, B resp.), a map $\varphi\colon S \to T$ is *strictly monotone* if $s \subsetneqq t \Rightarrow \varphi(s) \subsetneqq \varphi)($, i.e, if φ is order preserving for the relation $\supsetneqq$.

Proposition. *(2.9) Let S, T be trees on A, B respectively. If T is well-founded, then S is well-founded with $\varrho(S) < \varrho(T)$ iff there is a strictly monotone map $\varphi\colon S \to T$.*

2.F The Well-founded Part of a Tree

2.G The Kleene-Brouwer Ordering Now let $(A, <)$ be a linearly ordered set. We define the *Kleene-Brouwer ordering* $<_{KB}$ on $A^{<\mathbb{N}}$ as follows: If $s = (s_0, \ldots, s_{m-1})$, $t = (t_0, \ldots, t_{n-1})$, then

$$s <_{KB} t \;\Leftrightarrow\; (s \supsetneqq t) \text{ or } [\exists i < \min\{m, n\}(\forall j < i(s_j = t_j) \wedge s_i < t_i)].$$

Proposition. *(2.12) Assume that $(A, <)$ is a wellordered set. Then for any tree T on A, T is well-founded iff Kleene-Brouwer ordering restricted to T is a wellordered.*

1.3 Polish spaces

Cauchy sequence, completion, complete

Given any metric space (X, d), there is a complete metric space $(\hat{X}, \hat{d})$ such that (X, d) is a subspace of $(\hat{X}, \hat{d})$ and X is dense in $\hat{X}$. This space is unique up to isometry and is called the *completion* of (X, d). Clearly, $\hat{X}$ is separable if and only if X is separable.

Definition. (3.1) A topological space X is *completely metrizable* if it admits a compatible metric d such that (X, d) is complete. A separable completely metrizable space is called *Polish.*

Open interval $(0, 1)$ is Polish although its usual metric is not complete.

Proposition. *(3.3)*

(i) The completion of a separable metric space is Polish.

(ii) A closed subspace of a Polish space is Polish.

(iii) The product of a sequence of completely metrizable (resp. Polish)spaces is completely metrizable (resp. Polish). The sum of a family of completely metrizable spaces is completely metrizable. The sum of a sequence of Polish spaces is Polish.

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^{\mathbb{N}}$, unit interval $\mathbb{I}$ and unit circle $\mathbb{T}$ are Polish. The n-dimensional cube $\mathbb{I}^n$, the *Hilbert cube* $\mathbb{I}^{\mathbb{N}}$, the n-dimensional torus $\mathbb{T}^n$ and the infinite dimensional torus $\mathbb{T}^{\mathbb{N}}$ are Polish.

$A=$ countable discrete $\Rightarrow A^{\mathbb{N}}$ is Polish

the *Cantor space* $\mathcal{C} = 2^{\mathbb{N}}$

the *Baire space* $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$

Example. (3.4.5)Let X, Y be separable Banach spaces. We denote by $L(X,Y)$ the (generally non-separable) Banach space of bounded linear operators $T\colon X \to Y$ with norm $\|T\| = \sup\{\|TX\|; x \in X, \|x\| \leq 1\}$. If $X = Y$ we let $L(X,Y) = L(X)$. Denote by $L_1(X,Y)$ the unit ball

$$L_1(X,Y) = \{T \in L(X,Y); \|T\| \leq 1\}$$

of $L(X,Y)$. The *strong topology* on $L(X,Y)$ is the topology generated by the family of functions $f_x(T) = Tx$, $f_x\colon L(X,Y) \to Y$, for $x \in X$.

$L_1(X,Y)$ is a Polish space.

3.B Extensions of Continuous Function and Homeomorphisms Let X be a topological space, (Y,d) a metric space and $f\colon A \to Y$.

$\mathrm{osc}_f = \inf\{\mathrm{diam}(f(U)); U$ is an open nbhd of $X\}$

Theorem (Kuratowski)**.** *(3.8) Let X be metrizable, Y be completely metrizable, $A \subseteq X$, and $f\colon A \to Y$ be continuous. Then there is a G_δ set G with $A \subseteq G \subseteq \overline{A}$ and a continuous extension $g\colon G \to Y$ of f.*

3.C Polish Subspaces of Polish Spaces

Theorem. *(3.11) If X is metrizable and $Y \subseteq X$ is completely metrizable, then Y is a G_δ in X. Conversely, if X is completely metrizable and $Y \subseteq X$ is a G_δ, then Y is completely metrizable.*

In particular, a subspace of a Polish space is Polish iff it is a G_δ.

1.4 Compact metrizable spaces

4.A Basic facts

Proposition. *(4.6) Let X be a compact topological space. Then X is metrizable iff X is Hausdorff and second countable.*[3]

[3] See Urysohn's metrization theorem 1.1

4.B Examples

Theorem (Banach)**.** *(4.7) The unit ball $B_1(X^*)$ of a separable Banach space is compact in the weak*-topology. A compatible metric is given by*

$$d(x^*, y^*) = \sum_{n=0}^{\infty} 2^{-n-1} |\langle x_n, x^* \rangle - \langle x_n, y^* \rangle|$$

for (x_n) dense in the unit ball.[4] [5]

$B_1(\ell_\infty) = [-1, 1]^{\mathbb{N}}$

Example. (4.9) Let X, Y be separable Banach spaces. The *weak topology* on $L(X, Y)$ is the one generated by the functions (from $L(X, Y)$ into the scalar field)

$$T \mapsto \langle Tx, y^* \rangle; \qquad x \in X, y^* \in Y^*.$$

Show that if Y is reflexive, $L_1(X, Y)$ with the weak topology is compact metrizable. Find compatible metric. [6] [7]

Example. (4.10) *Extreme points* in a topological vector space.

If K is a compact metrizable (in the relative topology) convex subset of a topological vector space, then the set $\partial_e K$ is G_δ in K and thus Polish.[8]

Example (König's Lemma)**.** (4.12) Let T be a tree on A. If T is finite splitting then $[T] \neq \emptyset$ iff T is infinite.

4.C A Universality Property of the Hilbert Cube

Theorem. *(4.14) Every separable metrizable space is homeomorphic to a subspace of the Hilbert cube $I^{\mathbb{N}}$. In particular, the Polish spaces are, up to homeomorphism, exactly the G_δ subspaces of the Hilbert cube.*

Theorem. *(4.17) Every Polish space is homeomorphic to to a closed subspace of $\mathbb{R}^{\mathbb{N}}$.* [9]

4.D Continuous Images of the Cantor Space A point X in a topological space is called *condensation point* if every open neighborhood of x in uncountable.

Theorem. *(4.18) Every non-empty compact metrizable space is a continuous image of $\mathcal{C}$.*

[4] My note: This shows Keller's Theorem 9.19.

[5] This result often appears in functional-analysis texts under the name Alaoglu's theorem. Usually only the *compact* part is shown. See [Me] for the history.

[6] See [AB, Theorem 6.25] for the proof of the case $Y = \mathbb{R}$.

[7] MS: $L(X, Y^{**}) = L(X, L(Y^*, \mathbb{R}))$. This would be naturally isomorphic to $L(X \otimes Y, \mathbb{R})$ if we had an appropriate tensor product for Banach spaces, but here the situation is not so simple.

[8] It can be shown that every Polish space is homeomorphic to some $\partial_e K$. Result of R. Haydon, see [Ke, 33.L]. An introduction into this topic can be found in [AB, Section 7.12].

[9] Different proof [E, Lemma 4.3.22–Theorem 4.3.24].

1.5 Locally compact spaces

4.E The space of continuous functions on a compact space

4.F The Hyperspace of compact sets [10]

topological upper limit $\overline{\mathrm{Tlim}}_n K_n$ is the set

$$\{x \in X; \text{Every open nbhd of } x \text{ meets } K_n \text{ for infinitely many } n\}$$

topological lower limit $\underline{\mathrm{Tlim}}_n K_n$ is the set

$$\{x \in X; \text{Every open nbhd of } x \text{ meets } K_n \text{ for all but finitely many } n\}$$

1.6 Perfect Polish spaces

6.A Embedding the Cantor space in a perfect Polish space

Definition. (6.1) A *Cantor scheme* on a set X is a family $(A_s)_{s\in 2^{<\mathbb{N}}}$ such that:

(i) $A_{s^0} \cap A_{s^1} = \emptyset$ for $s \in 2^{<\mathbb{N}}$,

(ii) $A_{s^i} \subseteq A_s$, for $s \in 2^{<\mathbb{N}}$, $i \in \{0,1\}$.

Theorem. *(6.2) Let X be a nonempty perfect Polish space. Then there is an embedding of $\mathcal{C}$ into X.*

The proof is by constructing a Cantor scheme of open subsets of X with decreasing diameter.

Corollary. *(6.3) If X is a nonempty perfect Polish space, then* $\operatorname{card} X = 2^{\aleph_0}$. *In particular, a nonempty perfect subset of a Polish space has the cardinality of the continuum.*

6.B The Cantor-Bendixson theorem

Theorem (Cantor-Bendixson)**.** *(6.4) Let X be a Polish space. Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C countable open.*

Corollary. *(6.5) Any uncountable Polish space contains a homeomorphic copy of $\mathcal{C}$ and in particular has cardinality $2^{\aleph_0}$.*

1.7 Zero-dimensional spaces

7.A Basic facts

Theorem. *(7.3) Let X be separable metrizable. Then X is zero-dimensional iff every non-empty closed subset of X is a retract of X.* [11]

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[10] My note: Proof that the Hausdorff metric and Vietoris topology coincide on $K(X)$ can be found in [AB, Theorem 3.91]

[11] This book does not include the proof, it refers to [Ku, Ch.II,§6,Cor.2]

7.B A Topological Characterization of the Cantor Space

Theorem (Brouwer)**.** *(7.4) The Cantor space $\mathcal{C}$ is unique, up to homeomorphism, perfect nonempty, compact metrizable, zero-dimensional space.*

7.C A Topological Characterization of the Baire Space

Definition. (7.5) A *Lusin scheme* on a set X is a family $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ such that:

(i) $A_{s^i} \cap A_{s^j} = \emptyset$ for $s \in \mathbb{N}^{<\mathbb{N}}$,

(ii) $A_{s^i} \subseteq A_s$, for $s \in \mathbb{N}^{<\mathbb{N}}$, $i \in \{0, 1\}$.

TODO vanishing diameter, associated map

1.8 Baire category

8.A Meager sets meager = first category = countable union of nowhere dense sets

The complement of a meager set is called *comeager* (or *residual*). So a set is comeager iff it contains the intersection of a countable family of dense open sets.

8.B Baire Spaces

Proposition. *(8.1) Let X be a topological space. The following statements are equivalent:*

(i) Every nonempty open set in X is non-meager.

(ii) Every comeager set in X is dense.

(iii) The intersection of countably many dense open sets in X is dense.

Definition. (8.2) A topological space is called a *Baire space* if it satisfies the equivalent conditions of 8.1.

Proposition. *(8.3) If X is a Baire space and $U \subseteq X$ is open, U is a Baire space.*

Theorem (The Baire Category theorem)**.** *(8.4) Every completely metrizable space is Baire. Every locally compact Hausdorff space is Baire.*

Definition. (8.5) Let X be a topological space and $P \subseteq X$. If P is comeager, we say that P *holds generically* or that the *generic element of* X is in P. (Sometimes the word *typical* is used instead of generic.)

[12]

[12] Exercise 8.6 – see [S, Proposition 2.5.11]

8.C Choquet Games and Spaces

Definition. (8.10) *Choquet game*: Players I and II take turns in playing nonempty open subsets of X so that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$.

We say that II wins this run of game if $\bigcap_n V_n (= \bigcap_n U_n) \neq \emptyset$.

Theorem (Oxtoby). *(8.11) A nonempty topological space X is a Baire space iff player I has no winning strategy in the Choquet game G_X.*

[13]

Definition. (8.12) A nonempty topological space is a *Choquet space* if player II has a winning strategy in G_X.

Every Choquet space is Baire. (The converse fails even for nonempty separable metrizable spaces.)

Products of Choquet spaces are Choquet.[14] Also, open nonempty subspaces of Choquet spaces are Choquet. (It is not true that product of Baire spaces are Baire. See, however, 8.44.)

8.D Strong Choquet Games and Spaces

Definition. (8.14) *strong Choquet game*

A nonempty space is called a *strong Choquet space* if player II has a winning strategy in G^s_X.

Any strong Choquet space is Choquet. (The converse turns out to be false.)

8.E A Characterization of Polish Spaces

Theorem. *(8.17) Let X be a nonempty separable metrizable space and $\hat{X}$ a Polish space in which X is dense. Then*

i) *(Oxtoby) X is Choquet $\Leftrightarrow$ X is comeager in $\hat{X}$;*

ii) *(Choquet) X is strong Choquet $\Leftrightarrow$ X is G_δ in $\hat{X}$ $\Leftrightarrow$ X is Polish*

Theorem (Choquet). *(8.18) A nonempty, second countable topological space is Polish iff it is T_1, regular and strong Choquet.*

Theorem (Sierpiński). *(8.19) Let X be Polish and Y separable metrizable. If there is a continuous open surjection of X onto Y, then Y is Polish.*

[13] In fact the proof shows that: Baire $\Rightarrow$ II has a *stationary* winning strategy. (Stationary strategies are called tactics by some authors.)

[14] My note: I think that almost the same proof will work for the box product.

8.F Sets with the Baire Property Let $\mathcal{I}$ be a σ-ideal on a set X.

$A =_{\mathcal{I}} B \Leftrightarrow A \triangle B \in \mathcal{I}$

$A =^* B \Leftrightarrow A$, B are equal modulo meager sets

Definition. (8.21) Let X be a topological space. A set $A \subseteq X$ has the *Baire property* (BP) if $A =^* U$ for some open set $U \subseteq X$.

Proposition. *(8.22) Let X be a topological space. The class of sets having the BP is a σ-algebra on X. It is the smallest σ-algebra containing all open sets and all meager sets.*

Proposition. *(8.23) Let X be a topological space and $A \subseteq X$. Then the following statements are equivalent:*

i) *A has the BP;*

ii) *$A = G \cup M$, where G is G_δ nad M is meager;*

iii) *$A = F \setminus M$ where F is F_σ and M is meager.*

Example. (8.24) There exists a subset $A \subseteq \mathbb{R}$ not having BP.

8.G Localization

Definition. (8.25) If A is comeager in U, we say that A *holds generically* in U or that U *forces* A, in symbols

$$U \models A.$$

Thus A is comeager if $X \models A$.

Note that

$$U \subseteq V, A \subseteq B \Rightarrow (V \models A \Rightarrow U \models B).$$

Proposition. *(8.26) Let X be a topological space and suppose that $A \subseteq X$ has the BP. Then either A is meager or there is a nonempty open set $U \subseteq X$ on which A is comeager (i.e., $X \models (X \setminus A)$ or there is nonempty open $U \subseteq X$, with $U \models A$). If X is a Baire space, exactly one of these alternatives holds.*

A *weak basis* for a topological space X is a collection of nonempty open sets such that every nonempty open set contains one of them. It is clear that in the previous result U can be chosen in any given weak basis.

$$\sim A = X \setminus A$$

Proposition. *(8.27) Let X be a topological space.*

(i) If $A_n \subseteq X$, then for any open $U \subseteq X$,

$$U \models \bigcap_n A_n \Leftrightarrow \forall n (U \models A_n)$$

(ii) *If X is a Baire space, A has BP in X and U varies below over nonempty open sets in X, and V over a weak basis, then*

$$U \models \sim A \Leftrightarrow \forall V \subseteq U(V \not\models A).$$

Theorem. *(8.29) Let X be a topological space and $A \subseteq X$. Put*

$$U(A) = \bigcup \{U \, open \, ; U \models A\}.$$

Then $U(A) \setminus A$ is meager, and if A has the BP, $A \setminus U(A)$, and thus $A \Delta U(A)$ is meager, so $A =^* U(A)$.

8.H The Banach-Mazur game *Banach-Mazur game* (or **-*game*) $G^{**}(A)$
Player II wins the run if the game if $\bigcap_n V_n (= \bigcap_n U_n) \subseteq A$.

Theorem (Banach-Mazur, Oxtoby)**.** *(8.33) Let X be a nonempty topological space. Then*

i) *A is comeager $\Leftrightarrow$ II has a winning strategy in $G^{**}(A)$.*

ii) *If X is Choquet and there is a metric d on X whose open balls are open in X, then A is meager in a nonempty open set $\Leftrightarrow$ I has winning strategy in $G^{**}(A)$.*

Definition. (8.34) A game is *determined* if at least one of the two players has a winning strategy.

8.K The Kuratowski-Ulam Theorem Exercise 8.44: If X, Y are second countable Baire spaces, so is $X \times Y$.

1.9 Polish groups

9.D Universal Polish groups

Theorem (Uspenskiĭ)**.** *(9.18) Every Polish group is isomorphic to a* (*necessarily closed*) *subgroup of $H(\mathbb{I}^{\mathbb{N}})$.* [15]

We use now the following result in infinite-dimensional topology (see [BP]).

Theorem (Keller's Theorem)**.** *(9.19) If X is a separable infinite-dimensional Banach space, $B_1(X^*)$ with the weak*-topology is homeomorphic to the Hilbert cube $\mathbb{I}^{\mathbb{N}}$.*

[15] The proof uses Theorem 4.7 Banach-Alaoglu.

2 Borel sets

2.10 Measurable Spaces and Functions

2.11 Borel Sets and Functions

11.B The Borel Hierarchy

$$\mathbf{\Sigma}^0_1(X) = \{U \subseteq X; U \text{ is open}\}$$
$$\mathbf{\Pi}^0_\xi(X) = \sim \mathbf{\Sigma}^0_\xi(X)$$
$$\mathbf{\Sigma}^0_\xi(X) = \{\bigcup_n A_n; A_n \in \mathbf{\Pi}^0_{\xi_n}, \xi_n < \xi, n \in \mathbb{N}\}, \text{ if } \xi > 1$$

11.C Borel functions Let X, Y be topological spaces. A map $f\colon X \to Y$ is Borel (measurable) if the inverse image of a Borel (equivalently: open or closed) set is Borel.

If Y is metrizable and Borel functions $f_n\colon X \to Y$ converge to $f\colon X \to Y$ pointwise, then the limit function f is Borel. Exercise 11.2i.

Proposition. *(11.5) Every Borel set has the Baire property, and every Borel function is Baire measurable.*

Theorem (Lebesgue, Hausdorff)**.** *(11.6) Let X be a metrizable space. The class of Borel functions $f\colon X \to \mathbb{R}$ is the smallest class of functions from X into $\mathbb{R}$ which contains all the continuous functions and is closed under taking pointwise limits of sequences of functions.*

2.12 Standard Borel Spaces

2.13 Borel Sets as Clopen Sets

2.14 Analytic Sets and the Separation Theorem

14.A Basic Facts about Analytic Sets

Definition. (14.1) Let X be a Polish space. A set $A \subseteq X$ is called analytic if there is a Polish space Y and a continuous function $f\colon Y \to X$ with $f(Y) = A$. (The empty set is analytic, by taking $Y = \emptyset$.)

14.B The Lusin Separation Theorem

2.15 Borel Injections and Isomorphisms

2.16 Borel Sets and Baire Category

2.17 Borel Sets and Measures

17.D Lusin's Theorem on Measurable Functions

Theorem (Lusin)**.** *(17.12) Let X be a metrizable space and μ a finite Borel measure on X. Let Y be a second countable topological space and $f\colon X \to Y$ a μ-measurable function. For every $\varepsilon > 0$, there is a closed set $F \subseteq X$ with $\mu(X \setminus F) < \varepsilon$ such that $f|F$ is continuous. Moreover, if X is Polish, we can take F to be compact.*

In particular, if $Y = \mathbb{R}$, there is a continuous $g\colon X \to \mathbb{R}$ with $\mu(\{x; f(x) \neq g(x)\}) < \varepsilon$.

17.E The Space of Probability Borel Measures

2.22 The Borel Hierarchy

TODO 22.14 *reduction property*

2.23 Some Examples

2.24 The Baire Hierarchy

24.A The Baire Classes of Functions

Definition. (24.1) Let X, Y be metrizable spaces. A function $f\colon X \to Y$ is of *Baire class 1* if $f^{-1}(U) \in \mathbf{\Sigma}^0_2(X)$ for every open set $U \subseteq Y$. If Y is separable, it is clearly enough in this definition to restrict U to any countable subbasis for Y. Recursively, for $1 < \xi < \omega_1$ we define now a function $f\colon X \to Y$ to be of *Baire class ξ* if it is the pointwise limit of a sequence of functions $f_n\colon X \to Y$, where f_n is of Baire class $\xi_n < \xi$.

Notation: $\mathcal{B}_\xi(X, Y)$

As usual, $B_\xi(X) = B_\xi(X, \mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (the context should make clear which case we are considering).

Thus $\mathbf{\Sigma}^0_1$-measurable = continuous and $\mathbf{\Sigma}^0_2$-measurable = Baire class 1. The following

Theorem (Lebesgue, Hausdorff, Banach)**.** *(24.3) Let X, Y be metrizable spaces, with Y separable. Then for $1 < \xi < \omega_i$, $f\colon X \to Y$ is in B_ξ iff f is $\mathbf{\Sigma}^0_{\xi+1}$-measurable. In particular, $\bigcup_\xi \mathcal{B}_\xi$ is the class of Borel functions.*

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