Classical descriptive set theory

(Graduate texts in mathematics)

1 Polish spaces

1.1 Topological and metric spaces

1.A Topological spaces The initial topology is called *topology generated by* $(f_i)_{i\in I}$.

1.B Metric spaces

Theorem 1.1 (Urysohn metrization theorem). *(1.1) Let X be a second countable topological space. Then X is metrizable iff X is T_1 and regular.*

[1]

Theorem 1.2 (Urysohn's Lemma). *(1.2) Let X be a metrizable space. If A, B are two disjoint subsets of X, there is a function $f\colon X\to\langle 0,1\rangle$ such that $f(x)=0$ for $x\in A$ and $f(x)=1$ for $x\in B$.*

Theorem 1.3 (Tietze Extension Theorem). *(1.3) Let X be a metrizable space. If $A\subseteq X$ is closed and $f\colon A\to\mathbb{R}$ is continuous, there is $\hat{f}\colon X\to\mathbb{R}$ which is continuous and extends f. Moreover, if f is bounded by M, i.e., $|f(x)|\le M$ for $x\in A$, so is f.*

1.2 Trees

2.A Basic concepts A^n = all sequences $(s_0,\dots,s_{n-1})$ with s_i from the set A

$A^0=\{\emptyset\}$
$\emptyset$ = empty sequence
length(s) = length of a finite sequences

$$A^{<\mathbb{N}}=\bigcup_{n\in\mathbb{N}}A^n$$

$s|m=(s_0,\dots,s_{m-1})$ and $s|0=\emptyset$

We say that $s\in A^n$ is an *initial segment* of $x\in A^{\mathbb{N}}$ if $s=x|n$. We will write $s\subseteq n$ is s is an initial segment of x.

Two finite sequences are *compatible* if one is an initial segment of the other and *incompatible* otherwise. We use $s\perp t$ to indicate that s and t are incompatible.

[1] LH We can replace regular by normal. TODO check

The *concatenation* of two sequences $s = (s_i)_{i<n}$ and $t = (t_j)_{j<m}$ is the sequence $s\hat{\ }t = (s_0, \ldots, s_{n-1}, t_0, \ldots, t_{m-1})$. We write $s\hat{\ }a$ for $s\hat{\ }(a)$ if $a \in A$.

infinite concatenation

Definition 1.4. (2.1) *Tree* = subset $T \subseteq A^{<\mathbb{N}}$ closed under initial segments. (I.e., if $t \in T$ and $s \subseteq t$, then $s \in T$.)

nodes = elements of T

An *infinite branch* of T is a sequence $x \in A^{\mathbb{N}}$ such that $x|n \in T$, for all n. The *body* of T, written as $[T]$, consist of all infinite branches of T, i.e.,

$$[T] = \{x \in A^{\mathbb{N}}; \forall n(x|n \in T)\}.$$

Finally, we call a tree T *pruned* if every $s \in T$ has a proper extension $t \supsetneq s$, $t \in T$.

2.B Trees and Closed Sets If we take A as a topological space with the *discrete topology*, then A is metrizable; $d(a, b) = 1$.

$A^{\mathbb{N}}$ viewed as the product space – metric $d(x, y) = 2^{-n-1}$ if $x \neq y$ and n is the least number with $x_n \neq y_n$.

A metric d is an *ultrametric* if

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

Show that the above metric is an ultrametric.

The *standard basis* for the topology of $A^{\mathbb{N}}$ consists of the sets

$$N_s = \{x \in A^{\mathbb{N}} : s \subseteq x\},$$

where $s \in A^{<\mathbb{N}}$. Note that $s \subseteq t \Leftrightarrow N_s \supseteq N_t$ and $s \perp t \Leftrightarrow N_s \cap N_t = \emptyset$.

Proposition 1.5. *(2.4) The map $T \mapsto [T]$ is a bijection between pruned trees on A and closed subsets of $A^{\mathbb{N}}$. Its inverse is given by*

$$F \mapsto T_F = \{x|n \colon x \in F, n \in \mathbb{N}\}.$$

We call T_F the tree of F.

If T is a tree on A, then for any $s \in A^{\mathbb{N}}$,

$$T_s = \{t \in A^{<\mathbb{N}}; s\hat{\ }t \in T\}$$

and

$$T_{[s]} = \{t \in T; t \text{ is compatible with } s\}.$$

Thus $[T_{[s]}] = [T] \cap N_s$ forms a basis for the topology of $[T]$. Note that $T_{[s]}$ is a subtree of T, but T_s in general is not.

Definition 1.6. (2.5) Let S, T be trees (on sets A, B, resp.). A map $\varphi\colon T \to S$ is called *monotone* if $s \subseteq t$ implies $\varphi(s) \subseteq \varphi(t)$. For such φ let

$$D(\varphi) = \{x \in [S] : \lim_n \operatorname{length}(\varphi(x|n)) = \infty\}.$$

For $x \in D(\varphi)$, let

$$\varphi^*(x) = \bigcup_n \varphi(x|n) \in [T].$$

We call φ *proper* if $D(\varphi) = [S]$.

Proposition 1.7. *(2.6) The set $D(\varphi)$ is G_δ in $[S]$ and $\varphi^*\colon D(\varphi) \to [T]$ is continuous. Conversely, if $f\colon G \to [T]$ is continuous, with $G \subseteq [S]$ a G_δ set, then there is monotone $\varphi\colon S \to T$ with $f = \varphi^*$.*

A closed set F in a topological space X is a *retract* of X if there is a continuous surjection $f\colon X \to F$ such that $f(x) = x$ for $x \in F$.

Proposition 1.8. *(2.8) Let $F \subseteq H$ be two closed nonempty subsets of $A^{\mathbb{N}}$. Then F is a retract of H.*

2.C Trees on Products A tree on $A = B \times C$ can be understood as a subset of $B^{<\mathbb{N}} \times C^{<\mathbb{N}}$ with the property $(t, u) \in T \Rightarrow \operatorname{length}(t) = \operatorname{length}(u)$.

If T is a tree on $B \times C$ and $x \in B^{\mathbb{N}}$, consider *section tree* $T(x)$ on C defined by

$$T(x) = \{s \in C^{<\mathbb{N}}; (x|\operatorname{length}(s), s) \in T\}.$$

Note that if T is pruned it is not necessarily true that $T(x)$ is also pruned. Also,

$$(x, y) \in [T] \iff y \in [T(x)].$$

Similarly, for $s \in B^{<\mathbb{N}}$, we define $T(s) = \{t \in C^{<\mathbb{N}}; \operatorname{length}(t) \leq \operatorname{length}(s) \wedge (s|\operatorname{length}(t), t) \in T\}$.

2.D Leftmost Branches Let T be a tree on A and let $<$ be a well-ordering of A. If $[T] \neq \emptyset$, then we specify the $<$-*leftmost branch* of T, denoted by a_T, as follows. We define $a_T(n)$ by recursion on n:

$$a_T(n) = \text{ the } < \text{-least element} a \text{ of } A \text{ such that } [T_{(a_T|n)}] \neq \emptyset.$$

lexicographical ordering on $A^{\mathbb{N}}$ or on A^m

When T is pruned, a_T is also characterized by the property that for each m, $a_T|m$ is the lexicographically least element of $T \cap A^m$.

2.E Well-founded Trees and Ranks If a tree T on A has no infinite branches, i.e., $[T] = \emptyset$, then we call T *well-founded.* This is because it is equivalent to saying that the relation $s \prec t \Leftrightarrow s \supsetneqq t$ restricted to T is well-founded. (See appendix B.) On the other hand, if $[T] \neq \emptyset$, we call T *ill-founded.* If T is a well-founded tree, we denote the *rank function* of $\prec$ restricted to T by ρ_T.

$$\rho_T(s) = \sup\{\rho_T(t) + 1; t \in T, t \supsetneqq s\},$$

for $s \in T$. An easy argument shows that we also have

$$\rho_T(s) = \sup\{\rho_T(s\hat{\ }a) + 1 : s\hat{\ }a \in T\}.$$

Also, $\rho_T(s) = 0$ if $s \in T$ is *terminal,* i.e., for no a, $s\hat{\ }a \in T$. We also put $\rho_T(s) = 0$ if $s \notin T$. The *rank* of a well-founded tree is defined by $\rho(T) = \sup\{\rho_T(s) + 1 : s \in T\}$. Thus if $T \neq \emptyset$ $\rho(T) = \rho_T(\emptyset) + 1$. The *rank* of a well-founded tree is defined by $\rho(T) \sup\{\rho_T(s) + 1 : s \in T\}$.

If S, T are trees (on A, B resp.), a map $\varphi \colon S \to T$ is *strictly monotone* if $s \subsetneqq t \Rightarrow \varphi(s) \subsetneqq \varphi)($, i.e, if φ is order preserving for the relation $\supsetneqq$.

Proposition 1.9. *(2.9) Let S, T be trees on A, B respectively. If T is well-founded, then S is well-founded with $\varrho(S) < \varrho(T)$ iff there is a strictly monotone map $\varphi \colon S \to T$.*

2.F The Well-founded Part of a Tree

2.G The Kleene-Brouwer Ordering Now let $(A, <)$ be a linearly ordered set. We define the *Kleene-Brouwer ordering* $<_{KB}$ on $A^{<\mathbb{N}}$ as follows: If $s = (s_0, \dots, s_{m-1})$, $t = (t_0, \dots, t_{n-1})$, then

$$s <_{KB} t \;\Leftrightarrow\; (s \supsetneqq t) \text{ or } [\exists i < \min\{m, n\}(\forall j < i(s_j = t_j) \wedge s_i < t_i)].$$

Proposition 1.10. *(2.12) Assume that $(A, <)$ is a wellordered set. Then for any tree T on A, T is well-founded iff Kleene-Brouwer ordering restricted to T is a wellordered.*

1.3 Polish spaces

Cauchy sequence, completion, complete

Definition 1.11. (3.1) A topological space X is *completely metrizable* if it admits a compatible metric d such that (x, d) is complete. A separable completely metrizable space is called *Polish.*

Open interval $(0, 1)$ is Polish although its usual metric is not complete.

Proposition 1.12. *(3.3)*

(i) *The completion of a separable metric space is Polish.*

(ii) *A closed subspace of a Polish space is Polish.*

(iii) *The product of a sequence of completely metrizable (resp. Polish)spaces is completely metrizable (resp. Polish). The sum of a family of completely metrizable spaces is completely metrizable. The sum of a sequence of Polish spaces is Polish.*

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^{\mathbb{N}}$, unit interval $\mathbb{I}$ and unit circle $\mathbb{T}$ are Polish. The n-dimensional cube $\mathbb{I}^n$, the *Hilbert cube* $\mathbb{I}^{\mathbb{N}}$, the n-dimensional torus $\mathbb{T}^n$ and the infinite dimensional torus $\mathbb{T}^{\mathbb{N}}$ are Polish.

A= countable discrete $\Rightarrow A^{\mathbb{N}}$ is Polish

the *Cantor space* $\mathcal{C} = 2^{\mathbb{N}}$

the *Baire space* $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$

3.B Extensions of Continuous Function and Homeomorphisms Let X be a topological space, (Y, d) a metric space and $f \colon A \to Y$.

$\mathrm{osc}_f = \inf\{\mathrm{diam}(f(U)); U \text{ is an open nbhd of } X\}$

1.4 Compact metrizable spaces

4.C A Universality Property of the Hilbert Cube

Theorem 1.13. *(4.14) Every separable metrizable space is homeomorphic to a subspace of the Hilbert cube $I^{\mathbb{N}}$. In particular, the Polish spaces are, up to homeomorphism, exactly the G_δ subspaces of the Hilbert cube.*

4.D Continuous Images of the Cantor Space A point X in a topological space is called *condensation point* if every open neighborhood of x in uncountable.

Theorem 1.14. *(4.18) Every non-empty compact metrizable space is a continuous image of $\mathcal{C}$.*

1.5 Locally compact spaces

4.A Basic facts

Proposition 1.15. *(4.6) Let X be a compact topological space. Then X is metrizable iff X is Hausdorff and second countable.*[2]

4.B Examples

Theorem 1.16 (Banach). *(4.7) The unit ball $B_1(X^*)$ of a separable Banach space is compact in the weak*-topology. A compatible metric is given by*

$$d(x^*, y^*) = \sum_{n=0}^{\infty} 2^{-n-1} |\langle x_n, x^* \rangle - \langle x_n, y^* \rangle|$$

[2]See Urysohn's metrization theorem 1.1

for (x_n) dense in the unit ball.[3] [4]

1.6 Perfect Polish spaces

6.A Embedding the Cantor space in a perfect Polish space

Definition 1.17. (6.1) A *Cantor scheme* TODO

Theorem 1.18. *(6.2) Let X be a nonempty perfect Polish space. Then there is an embedding of $\mathcal{C}$ into X.*

The proof is by constructing a Cantor scheme of open subsets of X with decreasing diameter.

Corollary 1.19. *(6.3) If X is a nonempty perfect Polish space, then* $\operatorname{card} X = 2^{\aleph_0}$*. In particular, a nonempty perfect subset of a Polish space has the cardinality of the continuum.*

6.B The Cantor-Bendixson theorem

Theorem 1.20 (Cantor-Bendixson). *Let X be a Polish space. Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C countable open.*

1.7 Zero-dimensional spaces

1.8 Baire category

1.9 Polish groups

9.D Universal Polish groups

Theorem 1.21 (Uspenskiĭ). *(9.18) Every Polish group is isomorphic to a (necessarily closed) subgroup of $H(\mathbb{I}^{\mathbb{N}})$.* [5]

We use now the following result in infinite-dimensional topology (see [BP]).

Theorem 1.22 (Keller's Theorem). *(9.19) If X is a separable infinite-dimensional Banach space, $B_1(X^*)$ with the weak*-topology is homeomorphic to the Hilbert cube $\mathbb{I}^{\mathbb{N}}$.*

[3] My note: This shows Keller's Theorem 9.19.

[4] This result often appears in functional-analysis texts under the name Alaoglu's theorem. Usually only the *compact* part is shown. See [M] for the history.

[5] The proof uses Theorem 4.7 Banach-Alaoglu.

Contents

References

[BP] Czesław Bessaga and Alexander Pełczyński. *Selected Topics in Infinite-Dimensional Topology.* PWN, Warszawa, 1975.

[M] Robert E. Megginson. *An Introduction to Banach Space Theory.* Springer, New York, 1998. GTM 193.