Some exercises from [Ke]

Polish spaces

$L_1(X,Y)$ with the strong topology is Polish for X, Y separable (14/Exercise 3.4.5). Let X, Y be separable Banach spaces. We denote by $L(X,Y)$ the (generally non-separable) Banach space of bounded linear operators $T\colon X \to Y$ with norm $\|T\| = \sup\{\|TX\|; x \in X, \|x\| \leq 1\}$. If $X = Y$ we let $L(X,Y) = L(X)$. Denote by $L_1(X,Y)$ the unit ball

$$L_1(X,Y) = \{T \in L(X,Y); \|T\| \leq 1\}$$

of $L(X,Y)$. The *strong topology* on $L(X,Y)$ is the topology generated by the family of functions $f_x(T) = Tx$, $f_x\colon L(X,Y) \to Y$, for $x \in X$.

We will use the term initial topology rather than "topology generated by a family of functions" (this term is used in both [E, Ke]). Initial topology on X w.r.t. family $\{f_i, i \in I\}$ is characterized by the property that $f\colon Y \to X$ is continuous if and only if all compositions $f_i \circ f$ are continuous. E.g., product space has the initial topology w.r.t. the projections and a subspace has the initial topology w.r.t the embedding.

We want to show that $L_1(X,Y)$ with the strong topology is a Polish space. We choose a countable subset $D \subseteq X$ which is dense in X and closed under the rational linear combinations. The space Y^D is Polish since D is countable and Y is separable. The map $T \mapsto T|_D$ from $L_1(X,Y)$ into Y^D is injective (since D is dense in X). The range of this map is the following closed subset of Y^D

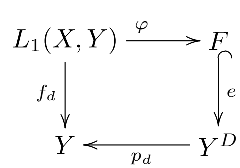
$$\begin{aligned} F = \{f \in Y^D; &\forall x, y \in D \forall p, q \in \mathbb{Q}[f(px+qy) = pf(x) + qf(y)]\} \\ &\wedge \forall x \in D(\|f(x)\| \leq \|x\|). \end{aligned}$$

(To see this we just need to note that only maps of this form can be obtained by a restriction of a map from $L_1(X,Y)$ and, on the other hand, every such a map can by uniquely extended to a map from $L_1(X,Y)$. The proof of this claim is standard – we define $\tilde{f}(x)$ as the limit of $f(x_n)$ for any $x_n \to x$ and we show that this map is well-defined and belongs to $L_1(X,Y)$.)

It remains to show that the map $\varphi\colon L_1(X,Y) \to F$ given by $T \mapsto T|_D$ is a homeomorphism. We have seen above that it is a bijection.

φ is continuous Let us denote by $e\colon F \hookrightarrow Y^D$ the embedding of E into Y^D. The topological space F has the initial topology w.r.t. the family $p_d \circ e$, $d \in D$. (Note that Y_D has the initial topology w.r.t. p_d's and F has the initial topology w.r.t. to e.)

We have the following commutative diagram.



The continuity of φ now follows from the continuity of $p_d \circ e \circ \varphi = f_d$.

(Since F is a compact space – we will show this later – the continuity of φ follows easily from the continuity of φ^{-1}. Nevertheless, it was good to include this prove, since now we will use similar method, but the case of φ is the easier one.)

φ^{-1} is continuous

The map $\varphi^{-1}\colon F \to L_1(X,Y)$ assigns to a function $f\colon D \to Y$ which belongs to F (i.e., it is $\mathbb{Q}$-linear and has norm at most 1) a linear and continuous extension $\tilde{f}\colon X \to Y$ (belonging to $L_1(X,Y)$).

Now, $L_1(X,Y)$ has the initial topology w.r.t the family f_x, $x \in X$. Thus we need to show that $f_x \circ \varphi$ is continuous for any given $x \in X$.

Let $x \in X$ be fixed. Choose any sequence $x_n \in D$ such that $\lim\limits_{n\to\infty} x_n = x$. The map $f_x \circ \varphi^{-1}$ maps f to $\lim\limits_{n\to\infty} f(x_n)$. Now
$(\exists n_0)(\forall n \geq n_0)\|x_n - x\| \leq \frac{\varepsilon}{3}$ (by the convergence of the sequence x_n) Directly by the definition of F this implies $(\forall h \in F)(\forall n \geq n_0)\|h(x_n) - h(x)\| \leq \frac{\varepsilon}{3}$.
$g \in U \Rightarrow \|g(x_{n_0}) - f(x_{n_0})\| \leq \frac{\varepsilon}{3}$ for some open neighborhood U of f.
Using these 2 facts we get for every $g \in U$

$$\|g(x) - f(x)\| \leq \|g(x) - g(x_{n_0})\| + \|g(x_{n_0}) - f(x_{n_0})\| + \|f(x) - f(x_{n_0})\| \leq \varepsilon.$$

This shows the continuity of φ^{-1}.

Metric on $L_1(X,Y)$ Since we have a homeomorphism between F and $L_1(X,Y)$ we can simply transfer the metric from F to this spaces. On F we have the product metric

$$d(f,g) = \sum_{d\in D} 2^{-n-1}\|f(d_n) - g(d_n)\|.$$

Since the set D corresponds to a countable dense subset of X closed under $\mathbb{Q}$-linear combinations, this yields a metric on $L_1(X,Y)$

$$d_1(S,T) = \sum_{n=1}^{\infty} 2^{-n-1}\|S(d_n) - T(d_n)\|$$

for any subset $D = \{d_n; n \in \mathbb{N}\}$ with the above properties. Now we claim that we obtain an equivalent metric if we take any dense countable set $\{x_n; n \in \mathbb{N}\}$ in the unit ball of X

$$d(S,T) = \sum_{n=1}^{\infty} 2^{-n-1}\|S(x_n) - T(x_n)\|.$$

If we are given some such set $A = \{x_n; n \in \mathbb{N}\}$ we can construct the corresponding set $D = \{\sum_{i=1}^{k} c_{i_k} x_{i_k}; k \in \mathbb{N}, c_{i_k} \in \mathbb{Q}\}$ of all $\mathbb{Q}$-linear combinations.

To see that these two metrics on $L_1(X,Y)$ are equivalent it suffices to observe that for a linear map T the sequence $T(d)$ converges to 0 for each $d \in D$ if and only if it converges to 0 for each $x_n \in A$.

$L_1(X,\mathbb{R})$ is compact Since in this case F is a closed subset of $\prod_{d\in D}\langle -\|d\|, \|d\|\rangle$, we see that F is compact. The same metric as above can be in this case rewritten as

$$d(x^*, y^*) = \sum_{n=0}^{\infty} 2^{-n-1} |\langle x_n, x^*\rangle - \langle y_n, y^*\rangle|$$

for some dense sequence (x_n) in the unit ball of X.

Note that the fact that $L_1(X,\mathbb{R})$ is compact can be shown without using the separability. This claim is usually called Banach-Alaoglu theorem. It was proved in 1940 by Leonidas Alaoglu. Stefan Banach has proved this theorem in his 1932's book in the separable case. [M]

I think it's worth noting we have in fact proved Theorem 9.19, which will be needed in some of the later chapters:

Theorem (Keller's Theorem). *If X is a separable infinite-dimensional Banach space, $B_1(X^*)$ with the weak*-topology is homeomorphic to the Hilbert cube $\mathbb{I}^{\mathbb{N}}$.*

1 Zero-dimensional spaces

Zero-dimensional spaces and retracts The following theorem is included in the book without proof

Theorem. *Let X be separable metrizable. Then X is zero-dimensional iff every non-empty closed subset of X is a retract of X.* [1]

$\boxed{\Rightarrow}$ By Proposition 2.8 this is true for the Cantor space $\mathcal{C} = 2^{\mathbb{N}}$. Since every separable zero-dimensional space is a subspace of $2^{\mathbb{N}}$ we get this part of theorem. (If F is closed in X and X is embedded in $\mathcal{C}$, we have a retraction from $\mathbb{C}$ to F. The restriction is a retraction from X to F.)

$\boxed{\Leftarrow}$ It suffices to show that X is a subspace of 2^{α} for some cardinal α. By diagonal theorem ([E, Theorem 2.3.20]) we only need to find a family of maps from X to 2 which separates points and closed sets. Now if $x \notin F$, where $x \in X$ and F is a closed subset of F, then we have a retraction $r\colon X \to F \cup \{x\}$ (the set $F \cup \{x\}$ is closed). Now the map $f\colon F \cup \{x\} \to 2$ given by $f(x) = 0$ and $f[F] = \{1\}$ is continuous (both $\{x\}$ and F are clopen in the subspace $\{x\} \cup F$). Thus $f \circ r$ is the required map from X to 2 separating the point x and the closed subset F.

References

[E] R. Engelking. *General Topology.* PWN, Warsaw, 1977.

[Ke] A. S. Kechris. *Classical descriptive set theory.* Springer-Verlag, Berlin, 1995. Graduate Texts in Mathematics 156.

[1] This book does not include the proof, it refers to [Ku, Ch.II,§6,Cor.2]

[Ku] K. Kuratowski. *Topology, Vol. I.* Academic Press, New York, 1966.

[M] Robert E. Megginson. *An Introduction to Banach Space Theory.* Springer, New York, 1998. GTM 193.