Oxtoby: Measure and Category

Notes from [O].

1 Measure and Category on the Line

`Countable sets, sets of first category, nullsets, the theorems of Cantor, Baire and Borel`

Theorem (Cantor). *(1.1) For any sequence $\{a_n\}$ of real numbers and for any interval I there exists a point p in I such that $p \neq a_n$ for every n.*

This proof involves infinitely many choices. ... All choices can be specified, and we have a well defined function of $(I, a_1, a_2, \dots)$ whose value is a point of I different from all the a_n.

A is nowhere dense $\Leftrightarrow$ A' contains a dense open set.

Theorem. *(1.2) Any subset of a nowhere dense set is nowhere dense. The union of two (or any finite number) of nowhere dense sets is nowhere dense. The closure of a nowhere dense set is nowhere dense.*

Theorem (Baire). *(1.3) The complement of any set of first category on the line is dense. No interval in R is of first category. The intersection of any sequence of dense open sets is dense.*

Theorem. *(1.4) Any subset of a set of first category is of first category. The union of any countable family of first category sets is of first category.*

nullset (or a set of *measure zero*) $\forall\, \varepsilon > 0$ there exists a sequence of intervals I_n such that $A \subset \bigcup I_n$ and $\sum |I_n| < \varepsilon$.

Nullsets form a σ-ideal

Theorem (Borel). *(1.5) If a finite or infinite sequence of intervals I_n covers an interval I, then $\sum |I_n| \geq |I|$.*

A is of first category $\not\Rightarrow$ $\overline{A}$ is of first category

$\overline{A}$ is of first category $\Rightarrow$ A is nowhere dense

Cantor set C = numbers in $[0, 1]$ that admit a ternary development in which the digit 1 does not appear. It can be constructed by deleting the open middle third of the interval

C is closed, nowhere dense, first category set, nullset, uncountable

Theorem. *(1.6) The line can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero.*

Corollary. *(1.7) Every subset of the line can be represented as the union of a nullset and a set of first category.*

2 Liouville Numbers

TODO

3 Lebesgue Measure in r-Space

By an *interval* I in Euclidean r-space is meant a rectangular parallepiped with edges parallel to the axes.

outer measure $m^*(I) = \inf\{\sum |I_i|; A \subset \bigcup I_i\}$ (infimum over all *sequences*)

Definition. (3.8) A set A is *measurable* (in the sense of Lebesgue) if for each $\varepsilon > 0$ there exists a closed set F and an open set G such that $F \subset A \subset G$ and $m^*(G - F) < \varepsilon$.

ring of subsets = union, difference
σ-*ring* = countable unions

Theorem. *(3.15) A set A is measurable if and only if it can be represented as an F_σ set plus a nullset (or as a G_δ set minus a nullset).*

σ-additivity, translation

A measurable set $E \subset \mathbb{R}$ is said to have density d at x if

$$\lim_{h \to 0} \frac{m(E \cap \langle x - h, x + h \rangle)}{2h} = d.$$

$\phi(E)$ = all points of $\mathbb{R}$ at which E has density 1

Theorem. *(3.20) For any measurable set $E \subseteq \mathbb{R}$, $m(E \triangle \phi(E)) = 0$.*

Theorem. *(3.21) For any measurable set A, let $\phi(A)$ denote the set of points of $\mathbb{R}$ at which E has density 1. Then ϕ has the following properties, where $A \sim B$ means that $A \triangle B$ is a nullset.*

(i) $\phi(A) \sim A$

(ii) $A \sim B$ *implies* $\phi(A) \sim \phi(B)$

(iii) $\phi(\emptyset) = \emptyset$ *and* $\phi(\mathbb{R}) = \mathbb{R}$

(iv) $\phi(A \cap B) = \phi(A) \cap \phi(B)$

(v) $A \subset B$ *implies* $\phi(A) \subset \phi(B)$

4 The Property of Baire

5 Non-Measurable Sets

Theorem (Ulam)**.** *(5.6) A finite measure μ defined for all subsets of a set X of power $\aleph_1$ vanishes identically if it is equal to zero for every one-element subset.*

A limit cardinal is said to be *weakly inaccessible* if (i) it is greater than $\aleph_0$, and (ii) it cannot be represented as a sum of fewer smaller cardinals. It is called inaccessible if, in addition, (iii) it exceeds the number of subsets of any set of smaller cardinality. By a continuation of the above reasoning, Ulam showed that in Theorem 5.6 it is sufficient to assume that no cardinal less than or equal to that of X is weakly inaccessible.

6 The Banach-Mazur Game

`Winning strategies, category and local category, indeterminate games`

7 Functions of First Class

`Oscillation, the limit of a sequence of continuous functions, Riemann integrability`

oscillation

Theorem. *(7.1) If f is a real-valued function on $\mathbb{R}$, then the set of points of discontinuity of f is an F_σ.*

Theorem. *(7.2) For any F_σ set E there exists a bounded function f having E for its set of points of discontinuity.*

Theorem. *(7.3) If f can be represented as the limit of an everywhere convergent sequence of continuous functions, then f is continuous except at a set of points of first category.*

Theorem. *(7.4) Let f be a real-valued function on $\mathbb{R}$. The set of points of discontinuity of f is of first category if and only if f is continuous at a dense set of points.*

Theorem. *(7.5) In order that a function f be Riemann-inteqrable on every finite intert:al it is necessary and sufficient that f be bounded on every finite interval and that its set of points of discontinuity be a nullset.*

Corollary. *(7.7) Any continuous function on a closed interval is integrable.*

Theorem. *(7.8) 8. The set of points of discontinuity of any monotone function f is countable. Any countable set is the set of points of discontinuity of some monotone function.*

8 The Theorems of Lusin and Egoroff

`Continuity of measurable functions and of functions having the property of Baire, uniform convergence on subsets`

A real-valued function f on $\mathbb{R}$ is called measurable if $f^{-1}(U)$ is measurable for every open set U in $\mathbb{R}$. f is said to have the property of Baire if $f^{-1}(U)$ has the property of Baire for every open set U in $\mathbb{R}$. [1]

Theorem. *(8.1) A real-valued function f on $\mathbb{R}$ has the property of Baire if and only if there exists a set P of first category such that the restriction of f to $\mathbb{R} - P$ is continuous.*[2]

Theorem (Lusin)**.** *(8.2) A real-valued function f on $\mathbb{R}$ is measurable if and only if for each $\varepsilon > 0$ there exists a set E with $m(E) < \varepsilon$ such that the restriction of f to $\mathbb{R} - E$ is continuous.*

Theorem (Egoroff)**.** *(8.3) If a sequence of measurable functions f_n converges to f at each point of a set E of finite measure, then for each $\varepsilon > 0$ there is a set $F \subset E$ with $m(F) < \varepsilon$ such that f_n converges to f uniformly on $E - F$.*

9 Metric and topological spaces

`Definitions, complete and topologically complete spaces, the Baire category theorem`

A metric space (X, ρ) is *topologically complete* if it is homeomorphic to some complete space. An important property of such spaces is that the Baire category theorem still holds.

Theorem. *(9.1) If X is a topologically complete metric space, and if A is of first category in X, then $X - A$ is dense in X.*

A topological space X is called a *Baire space* if every non-empty open set in X is of second category, or equivalently, if the complement of every set of first category is dense. In a Baire space, the complement of any set of first category is called a *residual set.*

Theorem. *(9.2) In a Baire space X, a set E is residual if and only if E contains a dense G_δ subset of X.*

15 The Kuratowski-Ulam Theorem

`Sections of plane sets having the property of Baire, product sets, reducibility to Fubini's theorem by means of a product transformation`

Theorem (Kuratowski-Ulam)**.** *(15.1) If E is a plane set of first category, then E_x is a linear set of first category for all x except a set of first category. If E is a nowhere dense subset of the plane $X \times Y$, then E_x is a nowhere dense subset of Y for all x except a set of first category in X.*

[1] Question: Is this related to Baire hierarchy of functions? Answer: Functions belonging to some Baire class are precisely Borel measurable functions. Every Borel set has property of Baire, hence every Borel measurable function is Baire function.

[2] I think that almost the same proof would work for: f is (Lebesgue) measurable $\Leftrightarrow$ $f|_{\mathbb{R}\setminus P}$ is Borel measurable for some nullset P
Have a look at [M, 2.H.10]

16 The Banach Category Theorem

`Open sets of first category or measure zero, Montgomery's lemma, the theorems of Marczewski and Sikorski, cardinals of measure zero, decomposition into a nullset and a set of first category`

Theorem (Banach Category Theorem)**.** *(16.1) In a topological space X, the union of any family of open sets of first category is of first category.*

To discuss the analogue of Theorem 16.1 for open sets of measure zero, we need the following lemma, which is due to Montgomery [K, §30.10].

Lemma (Montgomery)**.** *(16.2) Let $\{G_\alpha; \alpha \in A\}$ be a well-ordered family of open subsets of a metric spaceX, and for each $\alpha \in A$ let F_α be a closed subset of*

$$H_\alpha = G_\alpha - \bigcup_{\beta<\alpha} G_\beta.$$

Then the set $E = \bigcup_{\alpha\in A} F_\alpha$ is an F_σ.

A cardinal is said to have *measure zero* if every finite measure defined for all subsets of a set of that cardinality vanishes identically if it is zero for points.

A measure μ defined on the class of Borel subsets of a space X is called a *Borel measure*. It is *normalized* if $\mu(X) = 1$, and *nonatomic* if it is zero for points.

Theorem. *(16.3) Let μ be a finite Borel measure in a metric space X. If G is the union of a family $\mathcal{G}$ of open sets of measure zero, and if* $\operatorname{card}\mathcal{G}$ *has measure zero, then $\mu(G) = 0$.*

Theorem. *(16.4) If X is a metric space with a base whose cardinal has measure zero, and if μ is a finite Borel measure in X, then the union of any family of open sets of measure zero has measure zero.*

Metrizability cannot be omitted – using example from [KM].

The following theorem is perhaps the ultimate generalization of Theorem 1.6.

Theorem. *(16.5) Let X be a metric space with a base whose cardinal has measure zero. Let μ be a nonatomic Borel measure in X such that*

(i) every set of infinite measure has a subset with positive finite measure, and

(ii) every set of measure zero is contained in a G_δ set of measure zero.

Then X can be represented as the union of a G_δ, set of measure zero and a set of first category.

17 The Poincaré Recurrence Theorem

18 Transitive Transformations

19 The Sierpinski-Erdös Duality Theorem

20 Examples of Duality

21 The Extended Principle of Duality

22 Category Measure Spaces

`Spaces in which measure and category agree, topologies generated by lower densities, the Lebesgue density topology`

TODO

Supplementary notes

In this edition, a set of Supplementary Notes and Remarks has been added at the end, group according to chapter.

Chapter 3

For an alternative proof of the Lebesgue density theorem (3.20), see [Z].

References

[K] K. Kuratowski. *Topology, Vol. I.* Academic Press, New York, 1966.

[KM] J. H. B. Kemperman and Dorothy Maharam. R^c is not almost Lindelöf. *Proc. Amer. Math. Soc.*, 24(4):772–773, 1970.

[M] Y. N. Moschovakis. *Descriptive set theory.* North-Holland Publishing Company, Amsterdam, 1980.

[O] John C. Oxtoby. *Measure and Category.* Springer-Verlag, New York, 2nd edition, 1980. Graduate Texts in mathematics 2.

[Z] L. Zajíček. An elementary proof of one-dimensional density theorem. *Amer. Math. Monthly*, 86:297–298, 1979.

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