Oxtoby: Measure and Category

Some notes made when reading [O]. (E.g. problems in some proofs, some steps in the proofs that were not obvious (for me) immediately.)

16 The Banach Category Theorem

Theorem (Banach Category Theorem)**.** *(16.1) In a topological space X, the union of any family of open sets of first category is of first category.*

In the proof of Theorem 16.1:

$\overline{G} - \bigcup \mathcal{F}$ is nowhere dense: If $U \subseteq \overline{G}$, $U \neq \emptyset$ open, then $U \cap G \neq \emptyset$. (Since if $x \in U$ then $x \in \overline{G}$, which means that every neighborhood of x has a non-empty intersection with G. In particular, $U \cap G \neq \emptyset$.) Then $U \cap G \subseteq G$ would be an open set disjoint from all sets belonging to $\mathcal{F}$, contradicting the maximality of $\mathcal{F}$.

Similar argument can be used to show that $U \cap \overline{G} \neq \emptyset$ (U open) $\Rightarrow U \cap G \neq \emptyset$.

Every space is union of open (closed) Baire subspace and a set of first category. (See [BK, Lemma 4.1])

Let $\mathcal{G}$ be a system of all meager open subsets of X. Then $G = \bigcup \mathcal{G}$ is an open subset of first category an $Y := X \setminus G$ is a Baire space.

(Since Y is closed, we have $\overline{A}^Y = \overline{A}^X$ for any $A \subseteq Y$. Also $\operatorname{Int}_X(A) \subseteq \operatorname{Int}_Y(A)$. Hence every nowhere dense subset of Y is nowhere dense in X and every meager subset of Y is meager in X. If $U \neq \emptyset$ would be an open meager subset of Y, then $U \cup G$ would be an open meager in X, contradicting the definition of G.)

Now if we put $G' = G \cup \partial G = G \cup (\overline{G} \cap \overline{X \setminus G})$, then G' is meager (since boundary of any open set is nowhere-dense) and closed (since for open set $\partial G = \overline{G} \setminus \operatorname{Int} G = \overline{G} \setminus G$ and $G \cup \partial G = \overline{G}$). The subspace $X \setminus G'$ is a Baire space.

(Now $Y := X \setminus G'$ is open, thus we have $\operatorname{Int}_X(A) = \operatorname{Int}_Y(A)$ and $\overline{A}^Y = \overline{A}^X \cap X \subseteq \overline{A}^X$ for any $A \subseteq Y$. If A is nowhere dense in Y, then $\operatorname{Int}_Y \overline{A}^Y = \operatorname{Int}_X(\overline{A}^X \cap Y) = \emptyset \Rightarrow A \cup G \subseteq A \cup G' \subseteq \overline{A}^X \cup G'$ is meager in $X \Rightarrow A \cup G$ is meager in X. Consequently, if A is meager in Y, then $A \cup G$ is meager in X. Thus for any open meager subset U of Y the set $U \cap G$ is meager in X. Again, $U \neq \emptyset$ would contradict the maximality of G.)

64_6 The class of Borel sets that have an F_σ subset and a G_δ superset of equal measure is a σ-algebra that includes all closed sets. (For a finite Borel measure in a metric spaces.)

We are dealing with a metric space $\Rightarrow$ closed subsets are $G_\delta \Rightarrow$ this σ-algebra contains all closed sets.

[1] We will show that

$$\mathcal{S} = \{A \subseteq X; (\exists F \subseteq A \subseteq G)\mu(G \setminus F) = 0, F \text{ is } F_\sigma, G \text{ is } G_\delta\}$$

is a σ-algebra.

Complements. $F \subseteq A \subseteq G \Rightarrow X \setminus G \subseteq X \setminus A \subseteq X \setminus F$.

Countable unions. There are several equivalent characterizations of $\mathcal{S}$. (Note that here we are using that μ is finite measure.)

$$\mathcal{S} = \{A \subseteq X; (\forall \varepsilon > 0)(\exists V \subseteq A \subseteq U)\mu(U \setminus V) < \varepsilon, V \text{ is closed}, U \text{ is open}\}$$
$$\mathcal{S} = \{A \subseteq X; (\forall \varepsilon > 0)(\exists F \subseteq A \subseteq U)\mu(U \setminus F) < \varepsilon, F \text{ is } F_\sigma, U \text{ is open}\}$$

For a system $\{A_i; i \in \mathbb{N}\}$ of sets from $\mathcal{S}$ we choose $F_i \subseteq A_i \subseteq U_i$ such that $\mu(U_i \setminus F_i) < \varepsilon.2^{-i}$, U_i is open, F_i is F_σ. Let $U := \bigcup_{i=1}^\infty U_i$ and $F := \bigcup_{i=1}^\infty F_i$. Then U is open, F is F_σ and

$$U \setminus F = (\bigcup_{i=1}^\infty U_i) \setminus (\bigcup_{i=1}^\infty F_i) \subseteq \bigcup_{i=1}^\infty (U_i \setminus F_i),$$

hence $\mu(U \setminus F) < \varepsilon$. We also have

$$F = \bigcup_{i=1}^\infty F_i \subseteq \bigcup_{i=1}^\infty A_i \subseteq \bigcup_{i=1}^\infty U_i = U,$$

which implies $\bigcup_{i=1}^\infty A_i \in \mathcal{S}$.

References

[BK] M. Balcerzak and A. Kharazishvili. On uncountable unions and intersections of measurable sets. *Georgian Mathematical Journal*, 6(3):201–212, 1999.

[O] John C. Oxtoby. *Measure and Category.* Springer-Verlag, New York, 2nd edition, 1980. Graduate Texts in mathematics 2.

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16 The Banach Category Theorem **1**

[1] The following is taken from `http://math.stackexchange.com/questions/41462/sigma-algebra-of-well-approximated-borel-sets`