Bhaskara Rao, Bhaskara Rao: Theory of Charges

Notation: $I_A = \chi_A$, $\mathcal{L}(\mathcal{F})$ = Def 3.1.1, field on Ω = algebra of sets

1 Preliminaries

Let Ω be a set and $\mathcal{F}$ a collection of subsets of Ω.

$\mathcal{F}$ is a *lattice* if $A, B \in \mathcal{F} \Rightarrow A \cup B, A \cap B \in \mathcal{F}$.

$\mathcal{F}$ is a *semi-ring* if $\emptyset \in \mathcal{F}$; $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$ and if $A, B \in \mathcal{F}$ and $A \subset B$, then there exists a finite number $A_0, \ldots, A_n$ of sets in $\mathcal{F}$ such that $A = A_0 \subset \ldots \subset A_n = B$ and $A_i \setminus A_{i-1} \in \mathcal{F}$.

semi-field = semi-ring and $\Omega \in \mathcal{F}$

ring = $\emptyset \in \mathcal{F}$, finite unions, differences

field = ring and $\Omega \in \mathcal{F}$

additive-class = $\emptyset \in \mathcal{F}$, disjoint finite unions, complements

If $\mathcal{C}$ is a semi-field on Ω then the smallest field containing $\mathcal{C}$ consists of all finite disjoint unions of sets from $\mathcal{C}$ (Theorem 1.1.9(3))

Theorem (1.1.19). *Let $\mathcal{C}$ be a class of subsets of a set Ω, $\mathcal{F}_0$ the smallest additive class on Ω containing $\mathcal{C}$ and $\mathcal{F}_1$ the smallest field on Ω containing $\mathcal{C}$. Suppose $\mathcal{C}$ has the following properties:*

(i) $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{F}_0$.

(ii) $A, B \in \mathcal{C} \Rightarrow A \setminus B \in \mathcal{F}_0$.

Then $\mathcal{F}_0 = \mathcal{F}_1$.

Theorem (1.4.6, Stone Representation Theorem). *Let B be a Boolean algebra. Then there exists a compact Hausdorff totally disconnected space X such that B and the field $\mathcal{F}$ of all clopen subsets of X are isomorphic.*

A Boolean algebra B is said to satisfy the *countable chain condition* if every collection of pairwise disjoint elements in B is at most countable.

Theorem (1.4.8). *Let B be a Boolean σ-algebra satisfying the c.c.c. Then B is a complete Boolean algebra.*

2 Charges

$\mathcal{F}$ - field of subsets of a set Ω. $\mu\colon \mathcal{F} \to \langle -\infty, \infty \rangle$, finitely additive = *charge*.

(8) Using Banach limits we can construct shift-invariant charges.

(9) For every $S\colon \Omega \to \Omega$ such that $S^{-1}(A) \in \mathcal{F}$ whenever $A \in \mathcal{F}$ we can define using Banach limits an S-invariant probability charge μ on $\mathcal{F}$.

(10) Using Banach limits we can construct density charges.

$ba(\Omega, \mathcal{F})$ = the space of all bounded charges on $\mathcal{F}$. With the norm $\|\mu\| = |\mu|(\Omega)$ it is a Banach lattice.

Theorem (2.2.4). *Let $\mu \in ba(\Omega, \mathcal{F})$. Then for any $F \in \mathcal{F}$*

$$|\mu|(F) = \sup \sum_{i=1}^{n} |\mu(F_i)|,$$

where the supremum is taken over all finite partitions $F_1, \ldots, F_n$ of F in $\mathcal{F}$. Further, $|\mu|(\Omega) \leq 2 \sup\{|\mu(F); F \in \mathcal{F}|\}$.

3 Extensions of charges

3.1 Real valued set functions and induced functionals

Let $\mathcal{F}$ be a collection of subsets of a set Ω, let $\mathcal{L}(\mathcal{F}) = \{f \colon \Omega \to R; f = \sum_{i=1}^{n} r_i I_{A_i}$ for some $A_1, \ldots, A_n \in \mathcal{F}$ and $r_1, \ldots, r_n$ rational numbers $\}$. It is obviously a linear space over Q. [1]

If μ is a real valued function on $\mathcal{F}$ then we set $T(\sum_{i=1}^{n} r_i I_{A_i}) = \sum_{i=1}^{n} r_i \mu(A_i)$.

Proposition (3.1.3). *T is well defined if and only if*

$$\sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{m} \mu(B_i)$$

holds for any two finite sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ of not necessarily distinct sets from $\mathcal{F}$ satisfying

$$\sum_{i=1}^{n} I_{A_i} = \sum_{j=1}^{m} I_{B_j}.$$

Lemma. *(3.1.4)*

Theorem. *(3.1.9) Let $\mathcal{F}$ be any collection of subsects of a set Ω and μ a real valued function on $\mathcal{F}$. TODO*

3.2 Real partial charges and their extensions

Let $\mathcal{C}$ be a collection of subsets of Ω. A real valued function μ on $\mathcal{C}$ is called a *real partial charge* if $\sum_{i=1}^{n} \mu(A_i) = \sum_{j=1}^{m} \mu(B_j)$ whenever $\sum_{i=1}^{n} I_{A_i} = \sum_{j=1}^{m} I_{B_j}$ for $A_1, \ldots, A_n, B_1, \ldots, B_m$ in $\mathcal{C}$, i.e. the functional T defined on $\mathcal{L}(\mathcal{C})$ is well defined.

Let $\mathcal{C}$ be a collection of subsets of Ω and μ a positive real valued function on $\mathcal{C}$. μ is said to be a *positive real partial charge* if $\sum_{i=1}^{n} \mu(A_i) \leq \sum_{j=1}^{m} \mu(B_j)$ whenever $\sum_{i=1}^{n} I_{A_i} \leq \sum_{j=1}^{m} I_{B_j}$ for $A_1, \ldots, A_n, B_1, \ldots, B_m$ in $\mathcal{C}$.

μ is a real partial charge and $\mu(C) \geq 0 \not\Rightarrow \mu$ is a positive real partial charge

$\mathcal{F}$ - a field. Every restriction of a real charge is a real partial charge. Restriction of a positive bounded charge is a positive real partial charge.

[1] My question: What about reals in place of rationals?

Theorem (3.2.4). *Let μ be a real partial charge on a collection $\mathcal{C}$ of subsets of a set Ω. Let $A \subset \Omega$ be such that $A \notin \mathcal{C}$. Then there exists a real partial charge $\overline{\mu}$ on $\mathcal{C} \cup \{A\}$ which is an extension of μ. ($\mu(A)$ can be arbitrary real number)*

Theorem (3.2.5). *Let μ be a real partial charge on a collection $\mathcal{C}$ of subsets of a set Ω. Let $\mathcal{F}$ be any field on Ω containing $\mathcal{C}$. Then there exists a real charge $\overline{\mu}$ on $\mathcal{F}$ which is an extension of μ.*

$$\mu_i(A) = \sup \frac{\sum_{i=1}^{n} \mu(A_i) - \sum_{j=1}^{m} \mu(B_j)}{k}.$$

The supremum is taken over all finite collections $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots B_m$ of sets from $\mathcal{C}$ and positive integers k such that

$$k\chi_A + \sum_{j=1}^{m} \chi_{B_j} \geq \sum_{i=1}^{n} \chi_{A_i}.$$

Similarly,

$$\mu_e(A) = \inf \frac{\sum_{i=1}^{n} \mu(A_i) - \sum_{j=1}^{m} \mu(B_j)}{k}.$$

The infimum is taken over all finite collections $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots B_m$ and positive integers k such that

$$k\chi_A + \sum_{j=1}^{m} \chi_{B_j} \leq \sum_{i=1}^{n} \chi_{A_i}$$

Proposition (3.2.7). *Let $\mathcal{C}$ be a collection of subsets of a set Ω with $\Omega \in \mathcal{C}$. Let μ be a positive real partial charge on $\mathcal{C}$. Let A be any subset of Ω. If $\overline{\mu}$ is a positive real partial charge on $\mathcal{C} \cup \{A\}$ which is an extension of μ then,*

$$\mu_i(A) \leq \overline{\mu}(A) \leq \mu_e(A).$$

Proposition (3.2.8). *Let $\mathcal{C}$ be a collection of subsets of a set Ω with $\Omega \in \mathcal{C}$. Let μ be a positive real partial charge on $\mathcal{C}$. Then the following statements are true:*

(i) $0 \leq \mu_i(A) \leq \mu_e(A) \leq \mu(\Omega)$

(ii) *If $A \in \mathcal{C}$ or $I_A \in \mathcal{L}(\mathcal{C})$ then $\mu_i(A) = \mu_e(A) = T(I_A)$, where T is the linear functional on $\mathcal{L}(\mathcal{C})$ induced by μ.*

(iii) *If $A, B \subset \Omega$ and $A \cap B = \emptyset$, then*

$$\mu_i(A)+\mu_i(B) \leq \mu_i(A\cup B) \leq \mu_i(A)+\mu_e(B) \leq \mu_e(A\cup B) \leq \mu_e(A)+\mu_e(B).$$

(iv) *If $A \in \mathcal{C}$, $B \in \Omega$ and $A \cap B = \emptyset$ then $\mu_i(A \cup B) = \mu(A) + \mu_i(B)$ and $\mu_e(A \cup B) = \mu(A) + \mu_e(B)$.*

(v) If $A \cap B = \emptyset$ and $A \cup B \in \mathcal{C}$, then

$$\mu(A \cup B) = \mu_i(A) + \mu_e(B).$$

Theorem (3.2.9). *Let $\mathcal{C}$ be any collection of subsets of Ω with $\Omega \in \mathcal{C}$. Let μ be a positive real partial charge on $\mathcal{C}$. Let $A \subset \Omega$, $A \notin \mathcal{C}$. Then there exists a positive real partial charge on $\mathcal{C} \cup \{A\}$ which is an extension of μ. ($\overline{\mu}(A) = d$, $\mu_i(A) \leq d \leq \mu_e(A)$ - arbitrary)*

Theorem (3.2.10). *Let $\mathcal{C}$ be any collection of subsets of Ω with $\Omega \in \mathcal{C}$. Let μ be a positive real partial charge on $\mathcal{C}$. Let $\mathcal{F}$ be any field on Ω containing $\mathcal{C}$. Then there is a positive charge $\overline{\mu}$ on $\mathcal{F}$ which is an extension of μ.*

3.3 Extension procedure of Los and Marczewski

Proposition (3.3.1). *Let $\mathcal{C}$ be a field of subsets of a set Ω. Let μ be a positive bounded charge on $\mathcal{C}$. Then for any subset A of Ω: $\mu_i(A) = \sup\{\mu(B); B \subset A, B \in \mathcal{C}\}$; $\mu_e(A) = \inf\{\mu(C); A \subset C, C \in \mathcal{C}\}$.*

Proposition (3.3.2). *Let $\mathcal{C}$ be a field of subsets of a set Ω. Let μ be a positive bounded charge on $\mathcal{C}$. If A and B are two subsets of Ω satisfying the conditions $A \subset C$, $B \subset D$, $C \cap D = \emptyset$ and $C, D \in \mathcal{C}$, then*

$$\mu_i(A \cup B) = \mu_i(A) + \mu_i(B),$$
$$\mu_e(A \cup B) = \mu_e(A) + \mu_e(B).$$

Theorem 3.3.3 = extension from the field $\mathcal{C}$ to the smallest field containing $\mathcal{C}$ and A; $\mu(A)$ can be chosen between μ_i and μ_e.

Corollary (3.3.4). *Let $\mathcal{C}$ be a field of subsets of a set Ω. Let μ be a positive bounded charge on $\mathcal{C}$. Let $\mathcal{F}$ be a field on Ω containing $\mathcal{C}$. Then there exists a positive bounded charge $\overline{\mu}$ on $\mathcal{F}$ such that $\overline{\mu}$ is an extension of μ from $\mathcal{C}$ to $\mathcal{F}$ and that the range of $\overline{\mu}$ is a subset of the closure of the range of μ on $\mathcal{C}$.*

Corollary (3.3.6). *Let $\mathcal{C}$ be a field of subsets of a set Ω and μ a positive bounded charge on $\mathcal{C}$. Let $\mathcal{F}$ be a field on Ω containing $\mathcal{C}$. Then there exists a bounded charge $\overline{\mu}$ on $\mathcal{F}$ which is an extension of μ.*

3.4 Extension of partial charges in the general case

In this section we examine the situation when $\Omega \notin \mathcal{C}$ and also the extension of partial charges on $\mathcal{C}$ taking infinite values.

3.5 Miscellaneous extensions

Theorem 3.5.1 - existence of extension for semi-ring, semi-field...

3.6 Common extensions

Necessary and sufficient conditions for extension of two measures on two subfields.

4 Integration

4.7 $ba(\Omega, \mathcal{F})$ as a dual space

TODO

Appendix 1: Notes and comments

Chapter 3

The results of Sections 3.1 and 3.2 are due to Tarski (1938) and Horn and Tarski (1948). Our treatment is slightly different from the one given in Horn and Tarski (1948).