Schechter: Handbook of analysis and its foundations

Notes from [S].

6 Nets and convergence

Nets are particularly helpful for understanding topologies that are known to be non-metrizable e.g., the weak topology of an infinite-dimensional normed vector space or understanding topologies that are not known to be metrizable. But nets are also occasionally useful in metric spaces; two examples of this are the proof of Caristi's Theorem given in 19.45 and the explanation of Riemann integrals given in 24.7.

directed sets

(7.4) Let $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $(\mathcal{B}, \supseteq)$ is a directed sets if and only if $\mathcal{B}$ is a filterbase on X.

Nets

Definition. (7.7) Let $x\colon \mathbb{D} \to X$ and let $S \subseteq X$. We shall say that

S is a *tail set* of the net if S is of the form $\{x_\delta; \delta \geq \delta_0\}$ for some $\delta_0 \in \mathbb{D}$.

S is an *eventual* (or *residual*) set of the net if S contains some tail set i.e., if there is some $\delta_0 \in \mathbb{D}$ such that $\{x_\delta; \delta \geq \delta_0\} \subseteq S$. In this case we say that $x_\delta \in S$ happens *eventually,* or that $x_\delta \in S$ happens *for all δ sufficiently large.*

S is a *frequent* (or *cofinal*) set of the net if S meets every tail set i.e., if for each $\delta_0 \in \mathbb{D}$ there is some $\delta \geq \delta_0$ such that $x_\delta \in S$. In this case we say that $x_\delta \in S$ happens *frequently,* or that $x_\delta \in S$ happens *for arbitrarily large values of δ.*

S is *infrequent* if it is not frequent.

These terms can also be applied to subsets of a directed set $\mathbb{D}$, by viewing the identity map $i\colon \mathbb{D} \to \mathbb{D}$ as a $\mathbb{D}$-valued net.

(7.9) $\mathcal{B} = \{$tail sets of the net$(x_\delta)\}$ is a filterbase on X; the proper filter that it generates is $\mathcal{F} = \{$eventual sets of the net$(x_\delta)\}$.

filterbase of tails

eventuality filter

(7.11) $\mathcal{B}$ is a filter or filterbase $\mapsto$ the *canonical net* of $\mathcal{B}$ on $\{(x, S) \in X \times \mathcal{B}; x \in S\}$

(7.12) If $\mathcal{B}$ is any filterbase on a set X, then there also exists a net $(x_\alpha : \alpha \in A)$ whose filterbase of tails is $\mathcal{B}$ and such that the directed set A is antisymmetric i.e., it is also a poset. (Consequently, our applications would not be greatly affected if we made antisymmetry a part of our definition of directed set.)

Subnets

(7.14) *Preview and historical remarks*

subsequences $\subseteq$ frequent subnets $\subseteq$ Willard subnets $\subseteq$ Kelley subnets $\subseteq$ AA subnets

Although the three definitions require slightly different proofs of theorems, they yield essentially the same statements of theorems; their near-interchangeability will follow from results in 7.19 and 15.38. The Kelley definition is oldest and is most widely used in the literature, but the other two definitions are simpler. The AA definition is the most general and yields the simplest proofs. For those reasons and other reasons indicated below, this book will use the term "subnet" to mean "AA subnet" except where noted explicitly.

However, the two systems were not easily interchangeable; there was some awkwardness in the translation. Most mathematicians in convergence theory ended up using either nets or filters, but not both.

The difficulty is removed by a more general approach to subnets that has been suggested independently by several mathematicians (Smiley [1957], Aarnes and Andenaes [1972], Murdeshwar [1983], and perhaps others) but which, nevertheless, seems not to be widely known yet. We shall name this approach after Aarnes and Andenæs, because they investigated it in greatest depth.

Definition. (7.15) Let $(x_\alpha : \alpha \in \mathbb{A})$ and $(y_\beta : \beta \in \mathbb{B})$ be nets in a set X, with eventuality filters $\mathcal{F}$ and $\mathcal{G}$, respectively. Then:

a) The following conditions are equivalent. If any (hence all)of them are satisfied, we shall say that (y_β) is a *subset* of (x_α) (or more precisely, an *AA subnet* or a *subnet in the sense of Aarnes and Andenæs*).

   (A) Every (y_β)-frequent subset of X is also (x_α)-frequent.

   (B) Every (x_α)-eventual subset of X is also (y_β)-eventual.

   (C) $\mathcal{G} \supseteq \mathcal{F}$

   (D) Each (x_α)-tail set contains some (y_β)-tail set.

   (E) For each eventual set $\mathbb{S} \subseteq \mathbb{A}$, the set $y^{-1}(x(\mathbb{S}))$ is eventual in $\mathbb{B}$.

b) *Kelley subnet*

c) *Willard subnet*

d)

(7.16) c: *frequent subnet* (or *cofinal subnet*)

(7.17) c: c. Definition. Two nets have the same eventuality filter if and only if each net is a subnet of the other. We shall then say the nets are *AA-equivalent,* or simply *equivalent.*

Lemma (Lemma on Common Subnets). *(7.18) 7.18. Lemma on Common Subnets. Let $(u_a : a \in \mathbb{A})$, $(v_b : b \in \mathbb{B})$, and $(w_c : c \in \mathbb{C})$ be three nets taking values in a set X. Say the nets have eventuality filters $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$, respectively. Then the following conditions are equivalent:*

(A) *$F \cap G \cap H$ is nonempty, for every $F \in \mathcal{F}$, $G \in \mathcal{G}$ and $H \in \mathcal{H}$*

(B) $\mathcal{M} = \{S \subseteq X; S \supseteq F \cap G \cap H$ *for some* $F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H}\}$

(C) *The three filters have a common proper superfilter i.e., there exists a proper filter which contains all three given filters.*

(D) *The three nets have a common AA subnet i.e., there exists a net (p_λ) which is an AA subnet of each of the given nets.*

(E) *The three given nets have a common Willard subnet i.e., there exists a net $(p_\lambda : \lambda \in \mathbb{L})$ which is a Willard subnet of each of the three given nets. (It is understood that three different functions are used for the monotone mappings f from $\mathbb{L}$ into $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$.) Furthermore, that net can be chosen so that it is a* maximal common AA subnet *of the three given nets i.e., so that if (q_μ) is any common AA subnet of the three given nets, then (q_μ) is also an AA subnet of (p_λ).*

Corollary (Corollary on equivalent subnets). *(7.19) If (y_β) is an AA subnet of (x_α), then (y_β) is equivalent (in the sense of 7.17.c) to a Willard subnet of (x_α).*

(7.19) We have seen that every Willard subnet is a Kelley subnet, every Kelley subnet is an AA subnet, and every AA subnet is equivalent to a Willard subnet. Consequently, the three types of subnets can be used interchangeably in many contexts. See especially 15.38. (=definition of cluster points)

(7.17) c. Definition. Two nets have the same eventuality filter if and only if each net is a subnet of the other. We shall then say the nets are AA-equivalent, or simply equivalent.

(17.29) Example of the inadequacy of frequent subnets.

19 Metric and uniform convergence

Banach's fixed point theorem

Theorem (Caristi's Theorem; Browder, Caristi and Kirk). *(19.45) Let (X, d) be a complete metric space, let $v\colon X \to [0, +\infty)$ be some lower semicontinuous function, and let $f\colon X \to X$ be some function such that $d(t, f(t)) \leq v(t) - v(f(t))$ for all $t \in X$. Then f has at least one fixed point.*

Contents

References

[S] Eric Schechter. *Handbook of Analysis and its Foundations.* Academic Press, San Diego, 1997.